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We consider two theoretical tools that have been introduced decades ago but whose usage is not widespread in modern literature on matrix pencils. One is *dual pencils*, a pair of pencils with the same regular part and related singular structures. They were introduced by V. Kublanovskaya in the 1980s. The other is *Wong chains*, families of subspaces, associated with (possibly singular) matrix pencils, that generalize Jordan chains. They were introduced by K.T. Wong in the 1970s. Together, dual pencils and Wong chains form a powerful theoretical framework to treat elegantly singular pencils in applications, especially in the context of linearizations of matrix polynomials.

We first give a self-contained introduction to these two concepts, using modern language and extending them to a more general form; we describe the relation between them and show how they act on the Kronecker form of a pencil and on spectral and singular structures (eigenvalues, eigenvectors and minimal bases). Then we present several new applications of these results to more recent topics in matrix pencil theory, including: constraints on the minimal indices of singular Hamiltonian and symplectic pencils, new sufficient conditions under which pencils in $\mathbb{L}_1$, $\mathbb{L}_2$ linearization spaces are strong linearizations, a new perspective on Fiedler pencils, and a link between the Möller-Stetter theorem and some linearizations of matrix polynomials.

Keywords: matrix pencil, Wong chain, linearization, matrix polynomial, singular pencil, Fiedler pencil, pencil duality, Kronecker canonical form

MSC classification: 15A18 15A22

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1 Introduction

Consider a pair of matrices of the same size $R_0, R_1 \in \mathbb{C}^{n\times p}$. In matrix theory, a degree-1 matrix polynomial $R_1x + R_0 \in \mathbb{C}[x]^{n\times p}$ is known as a *matrix pencil* [18, 19].

Two pencils $L(x) := L_1x + L_0 \in \mathbb{C}[x]^{m\times n}$ and $R(x) := R_1x + R_0 \in \mathbb{C}[x]^{n\times p}$ are *dual* if the following two conditions hold:

D1 $L_1R_0 = L_0R_1$,

D2 $\operatorname{rank}\begin{bmatrix} L_1 & L_0 \end{bmatrix} + \operatorname{rank}\begin{bmatrix} R_1 \\ R_0 \end{bmatrix} = 2n.$

In this case, we say that $L(x)$ is a *left dual* of $R(x)$, and, conversely, $R(x)$ is a *right dual* of $L(x)$. Two dual pencils have the same eigenvalues and regular Kronecker structure, while their singular parts (if any) are related in a precise way.

Given a pencil $R(x)$ and four complex numbers $\alpha, \beta, \gamma, \delta$ with $\alpha\delta \neq \beta\gamma$, the *Wong chain* attached to the eigenvalue $\lambda := \frac{\alpha}{\beta} \in \mathbb{C} \cup \{\infty\}$ is the family of nested subspaces $\{0\} = \mathcal{W}_0^{(\lambda)} \subseteq \mathcal{W}_1^{(\lambda)} \subseteq \mathcal{W}_2^{(\lambda)} \subseteq \dots$ defined by the following property: for each $k \geq 0$: $\mathcal{W}_{k+1}^{(\lambda)}$ is the preimage under the map $\alpha R_1 + \beta R_0$ of the space $(\gamma R_1 + \delta R_0)\mathcal{W}_k^{(\lambda)}$. This family depends only on $R(x)$ and λ, as we prove in the following. Wong chains are essentially a generalization of Jordan chains and can also be defined for singular pencils.

Wong chains have been introduced in [40], and only recently reappeared in the study of matrix pencils [4, 5, 6]; the definition that we use here is a generalized version. Dual pencils appear in [26, Section 1.3], where they are given the name of *consistent pencils* and some of their theoretical properties are stated. Moreover, one can recognize the use of duality (of regular pencils only, which is a less interesting case) in the study of doubling and inverse-free methods [2, 9, 31], as well as in the work [3], which gives an elegant algebraic theory of operations on matrix pencils.

Yet, these tools seems to be underused with respect to their potential and we would like to bring them back to the attention of the matrix pencil community. We will argue that they are an elegant device for the theoretical study of matrix pencils, that allows us to obtain new results and revisit old ones, greatly simplifying the treatment of singular cases.

The structure of the paper is the following: in Section 2, we recall some basic definitions and classical results on matrix pencils and matrix polynomials. In Sections 3 and 4 we introduce Wong chains and dual pencils and describe their properties, especially in relation to the Kronecker canonical form, eigenvectors and minimal bases. We then show how they can be used for several tasks in different applications:

- describing the possible Kronecker forms of singular symplectic and Hamiltonian pencils (Section 5);
- revisiting and simplifying proofs about the spectral properties of square (possibly singular) Fiedler pencils (Section 7, after a brief introduction to linearizations in Section 6);

- developing a connection between duality and the vector spaces of linearizations $\mathbb{L}_1$ and $\mathbb{L}_2$ introduced in [29], obtaining new insight for the singular case (Section 8);
- illustrating a connection between the Möller-Stetter theorem and some specific linearizations of a square matrix polynomial (Section 9).

We conclude the paper by describing two different methods that can be used to compute duals, and showing how they can be combined with the theory presented here to derive old and new linearizations (Section 10).

Most of the theory developed in this paper is applicable to any field $\mathbb{F}$ of characteristic other than 2. If the field is not closed, eigenvalues are sought in its algebraic closure. For simplicity, however, our exposition is for $\mathbb{F} = \mathbb{C}$.

2 Preliminaries on matrix pencils and polynomials

In this section, we recall some classical definitions and results on matrix pencils and polynomials. Throughout the paper, the ring of scalar polynomials with coefficients in $\mathbb{C}$ is denoted by $\mathbb{C}[x]$, and the set of those with degree not larger than d by $\mathbb{C}[x]_d$. We denote by $\mathfrak{R}^{m\times n}$ the set of $m \times n$ matrices with coefficients in a ring $\mathfrak{R}$. The dimensions m and n are allowed to be zero, following the convention described for instance in [7, page 83].

We denote by $\operatorname{diag}(M_1, M_2, \dots, M_m)$ the block diagonal matrix formed by concatenating diagonally the (not necessarily square) blocks $M_1, M_2, \dots, M_m$. We introduce the notation

$$K_{k,k+1}(x) := \begin{bmatrix} 0_{k\times 1} & I_k \end{bmatrix} x - \begin{bmatrix} I_k & 0_{k\times 1} \end{bmatrix}, \quad K_{k+1,k}(x) := K_{k,k+1}(x)^T,$$

where k is allowed to be zero (giving 0×1 and 1×0 blocks, respectively). Moreover, we let $J_k^{(\lambda)}$ denote a Jordan block with size k and eigenvalue λ.

The following result about matrix pencils is classical [19, Chapter 12], and reduces to the Jordan canonical form of a matrix when one considers monic square pencils.

Theorem 2.1 (Kronecker canonical form)**.** *For every matrix pencil $R(x) \in \mathbb{C}[x]_1^{n\times p}$, there exist nonsingular matrices $V \in \mathbb{C}^{n\times n}, W \in \mathbb{C}^{p\times p}$ such that $B(x) = VR(x)W$ has the form $B(x) = \operatorname{diag}(B^{(1)}(x), B^{(2)}(x), \dots, B^{(t)}(x))$, where each block $B^{(i)}(x)$ is one among:*

1. *$xI - J_{k_i}^{(\lambda)}$ (Jordan block of size k_i),*
2. *$xJ_{k_i}^{(0)} - I$ (Jordan block at infinity of size k_i),*
3. *$K_{k_i,k_i+1}(x)$ (right singular block of size $k_i \times (k_i + 1)$),*
4. *$K_{k_i+1,k_i}(x)$ (left singular block of size $(k_i + 1) \times k_i$).*

The pencil $B(x)$ is unique up to a permutation of the diagonal blocks. Therefore, the number of blocks of each kind, size and eigenvalue is an invariant of the pencil $R(x)$.

For the sake of brevity, throughout the paper we will often use the acronym KCF.

It is straightforward to generalize the concept of a matrix pencil to polynomials of higher degree. This leads to the definition of *matrix polynomials* [22], $A(x) := \sum_{i=0}^{d} A_i x^i \in \mathbb{C}[x]_d^{n\times p}$. A matrix polynomial is called *regular* if it is square and $\det A(x)$ is not the zero polynomial, and *singular* otherwise.

Sometimes, in the theory of matrix polynomials it is convenient to allow for a zero leading coefficient (see, e.g., [30]). For this reason, in our exposition we will not exclude this possibility. When we write about a matrix polynomial $A(x) = \sum_{i=0}^{d} A_i x^i$, we agree that the leading factor could be the zero matrix. The natural number d is therefore an arbitrarily fixed *grade*, equal to or larger than the degree, which is attached artificially to the polynomial [30]. However, in most applications the leading coefficient is nonzero: a reader uncomfortable with the concept of grade may simply think of d as the degree.

A finite eigenvalue of a matrix polynomial $A(x)$ is defined as a complex number λ such that the rank of $A(\lambda)$ as a matrix over the field $\mathbb{C}$ is lower than the rank of $A(x)$ as a matrix over the field of rational functions $\mathbb{C}(x)$. Infinite eigenvalues can be defined as zero eigenvalues of $\operatorname{rev} A(x)$, where the operator rev is defined by

$$\operatorname{rev} \sum_{i=0}^{d} A_i x^i := \sum_{i=0}^{d} A_{d-i} x^i.$$

Furthermore, the Jordan invariants can be extended to the polynomial case, resulting in the concepts of *elementary divisors* and *partial multiplicities*; we refer the reader to the classic books [18, 22] and to the recent work [15] for their definitions, which are not needed in detail here.

If $A(x)$ is an $n \times p$ matrix polynomial, then $\ker_{\mathbb{C}(x)} A(x)$ is a subspace of $\mathbb{C}(x)^p$, and it always has a polynomial basis, i.e., a basis composed by vectors $v^{(k)} \in \mathbb{C}[x]^p$. The *degree* of a vector polynomial $v(x) = [v_i(x)] \in \mathbb{C}[x]^p$ is defined as $\max_{i\in\{1,2,\ldots,p\}} \deg v_i(x)$. A *minimal basis* of $A(x)$ [17] is a basis for the subspace $\ker_{\mathbb{C}(x)} A(x)$ composed entirely of vector polynomials such that the sum of the degrees of its column vectors, known as the *order* of the basis, is minimal among all possible polynomial bases. The degrees of the vectors that form a minimal basis, known as *(right) minimal indices*, are uniquely defined independently of the choice of the basis. In the rest of this paper, with a slight abuse of notation, we say that a matrix is a *basis* of a certain subspace to mean that its columns are a basis of the subspace. More generally, we may speak of a minimal basis for a subspace $\mathcal{U} \subseteq \mathbb{C}(x)^p$ (not necessarily seen as the kernel of some matrix polynomial).

It is known that minimal bases transform well under multiplication by invertible constant matrices; we give an explicit statement below.

Lemma 2.2 ([11, 17])**.** *Let $A(x) \in \mathbb{C}[x]^{n\times p}$, $V \in GL_n(\mathbb{C})$ and $W \in GL_p(\mathbb{C})$. If $M(x)$ is a minimal basis for $VA(x)W$, then $WM(x)$ is a minimal basis for $A(x)$, and has the same minimal indices.*

A simple consequence of Lemma 2.2 is that one can determine a minimal basis of a pencil from its KCF, as can be shown by direct verification.

Lemma 2.3. *Let $R(x) \in \mathbb{C}[x]_1^{n\times p}$ be a matrix pencil, and suppose that $R(x)$ has KCF $B(x) := VR(x)W = \operatorname{diag}(B^{(1)}(x), B^{(2)}(x), \dots, B^{(t)}(x))$. Let for each $i = 1, 2, \dots, t$*

$$M^{(i)}(x) := \begin{cases} \begin{bmatrix} x^{k_i} & x^{k_i-1} & \cdots & x & 1 \end{bmatrix}^T & \text{if } B^{(i)}(x) \text{ is of the form } K_{k_i,k_i+1}(x) \\ \text{the empty vector in } \mathbb{C}^{k_i\times 0} & \text{otherwise.} \end{cases} \tag{1}$$

Then, $W \operatorname{diag}(M^{(1)}(x), M^{(2)}(x), \dots, M^{(t)}(x))$ is a minimal basis for $R(x)$; in particular, the right minimal indices of $R(x)$ coincide with the row sizes k_i of the right singular blocks in its Kronecker canonical form.

Similarly, one can define the *left minimal indices* as the degrees of a minimal polynomial basis for the left kernel of $A(x)$ (i.e., the space $\{v(x) \in \mathbb{C}[x]^n : v(x)^T A(x) = 0\}$), and for a pencil they coincide with the column sizes k_i of the $K_{k_i+1,k_i}(x)$ Kronecker blocks.

Let us define the *block transpose* $A^{\mathcal{B}}$ of a matrix A partitioned in blocks A_{ij} as the block matrix whose blocks are A_{ji}. Clearly, this definition depends on the choice of the block sizes, which should be clear from the context. Moreover, given the matrix polynomial $A(x) = \sum_{i=0}^d A_i x^i$, we set

$$\operatorname{row}(A) = \begin{bmatrix} A_d & A_{d-1} & \cdots & A_0 \end{bmatrix}; \quad \operatorname{col}(A) = (\operatorname{row}(A))^{\mathcal{B}} = \begin{bmatrix} A_d \\ A_{d-1} \\ \vdots \\ A_0 \end{bmatrix}.$$

If $\operatorname{row}(A)$ has full row rank (or, equivalently, there is no nonzero $w \in \mathbb{C}^n$ such that $w^T A(x) = 0$), we say that $A(x)$ is *row-minimal*. If $\operatorname{col}(A)$ has full column rank (or, equivalently, there is no nonzero $v \in \mathbb{C}^p$ such that $A(x)v = 0$), we say that $A(x)$ is *column-minimal*. In particular, a regular pencil is row- and column-minimal.

Finally, we define for each $n \in \mathbb{N}$ the special matrix

$$J_n := \begin{bmatrix} 0_{n\times n} & I_n \\ -I_n & 0_{n\times n} \end{bmatrix}.$$

3 Dual pencils and Kronecker forms

In this section, we derive some basic results concerning the Kronecker form of dual pencils. These facts will be central in the rest of the paper. Much of the content of this section has appeared in some form in the existing work on dual pencils by Kublanovskaya, Simonova and collaborators: see [25, 26, 37] and the references therein. Here we give a self-contained exposition, following a more modern approach.

With the definitions stated in Section 2, the two conditions that define duality can be rewritten as

D1 $\operatorname{row}(L) J_n \operatorname{col}(R) = 0$,

D2 $\operatorname{rank}\operatorname{row}(L) + \operatorname{rank}\operatorname{col}(R) = 2n$.

This formulation highlights the special role played by the two matrices $\operatorname{col}(R)$ and $\operatorname{row}(L)$, and suggests a way to compute explicitly duals of given pencils.

Lemma 3.1. *Let $L(x) \in \mathbb{C}[x]_1^{m\times n}$ be given, and let $\begin{bmatrix} R_1 \\ R_0 \end{bmatrix} \in \mathbb{C}^{2n\times p}$ be any basis of the kernel of* $\operatorname{row}(L)J_n$. *Then,*

1. *$R(x) := R_1 x + R_0$ is a column-minimal right dual of $L(x)$.*
2. *For any other right dual $\hat{R}(x)$ of $L(x)$, we have $\hat{R}(x) = R(x)N$ for a matrix $N \in \mathbb{C}^{p\times q}$ with full row rank.*

Similarly, let $R(x) \in \mathbb{C}[x]_1^{n\times p}$ be given, and let $\begin{bmatrix} L_1 & L_0 \end{bmatrix} \in \mathbb{C}^{m\times 2n}$ be any matrix whose rows are a basis of the left kernel of $J_n \operatorname{col}(R)$. Then,

1. *$L(x) := L_1 x + L_0$ is a row-minimal left dual of $R(x)$.*
2. *For any other left dual $\hat{L}(x)$ of $R(x)$, we have $\hat{L}(x) = ML(x)$ for a matrix $M \in \mathbb{C}^{k\times m}$ with full column rank.*

Proof. We prove only the first two statements; the second two are analogous. Since $\operatorname{col}(R)$ is a basis of $\ker \operatorname{row}(L)J_n$, we have $\operatorname{row}(L)J_n \operatorname{col}(R) = 0$ and $\operatorname{rank} \operatorname{col}(R) = 2n - \operatorname{rank} \operatorname{row}(L)J_n = 2n - \operatorname{rank} \operatorname{row}(L)$. Furthermore, being a basis, $\operatorname{col}(R)$ has full column rank, i.e., $R(x)$ is column-minimal. Given any dual $\hat{R}(x)$ of $L(x)$, clearly the columns of $\operatorname{col}(\hat{R})$ belong to the kernel of $\operatorname{row}(L)J_n$. Hence they are linear combinations of the columns of $\operatorname{col}(R)$. This is equivalent to saying that there is a matrix N such that $\operatorname{col}(\hat{R}) = \operatorname{col}(R)N$, which implies $\hat{R}(x) = R(x)N$. Finally, if N had rank strictly smaller than p, then

$$2n - \operatorname{rank} \operatorname{row}(L) = \operatorname{rank} \operatorname{col}(\hat{R}) = \operatorname{rank} \operatorname{col}(R)N < p = \operatorname{rank} \operatorname{col}(R) = 2n - \operatorname{rank} \operatorname{row}(L),$$

which is absurd. □

Moreover, we can compute explicitly the duals of Kronecker blocks.

Lemma 3.2.
1. *Let $B(x)$ be any nonsingular Kronecker block ($\lambda \in \mathbb{C}$ or $\lambda = \infty$). Then, $B(x)$ is a row- and column-minimal left and right dual of itself.*
2. *A column-minimal right dual of $K_{k,k+1}(x)$ is $K_{k+1,k+2}(x)$. A row-minimal left dual of $K_{k,k+1}(x)$ is $K_{k-1,k}(x)$ if $k > 0$, and the 0×0 empty matrix if $k = 0$.*
3. *A row-minimal left dual of $K_{k+1,k}(x)$ is $K_{k+2,k+1}(x)$. A column-minimal right dual of $K_{k+1,k}(x)$ is $K_{k,k-1}(x)$ if $k > 0$, and the 0×0 empty matrix if $k = 0$.*

Proof.
1. It is easy to check that $B_1 B_0 = B_0 B_1$, where $B(x) = B_1 x + B_0$, since one of the coefficients B_1 or B_0 is $\pm I$; so **D1** holds. Again because of this identity block, $\operatorname{row}(B)$ and $\operatorname{col}(B)$ have full rank, so **D2** holds as well.

2. Verifying **D1** reduces to the computation

$$\begin{bmatrix} 0 & I_k \end{bmatrix} \begin{bmatrix} -I_{k+1} & 0 \end{bmatrix} = \begin{bmatrix} 0 & -I_k & 0 \end{bmatrix} = \begin{bmatrix} -I_k & 0 \end{bmatrix} \begin{bmatrix} 0 & I_{k+1} \end{bmatrix}. \tag{2}$$

   Moreover, $\operatorname{rank}\operatorname{row}(K_{k,k+1}(x)) = k$ and $\operatorname{rank}\operatorname{col}(K_{k+1,k+2}(x)) = k+2$ are checked easily. Therefore **D2** holds too.

3. It is analogous to 2.

□

The following results characterize completely the KCF of dual pencils; it has (implicitly) appeared in [26, Section 1.3.2]: here we give a direct proof.

Theorem 3.3. *Suppose that a pencil $A(x)$ has Kronecker canonical form $VA(x)W = B(x) = \operatorname{diag}(B^{(1)}(x), B^{(2)}(x), \dots, B^{(t)}(x))$. Then,*

1. *a row-minimal left dual of $A(x)$ is $L(x) = S_l(x)V$, where*

$$S_l(x) = \operatorname{diag}(L^{(1)}(x), L^{(2)}(x), \dots, L^{(t)}(x)),$$

   and

$$L^{(i)}(x) = \begin{cases} B^{(i)}(x) & \textit{if } B^{(i)}(x) \textit{ is any nonsingular Kronecker block,} \\ K_{k-1,k}(x) & \textit{if } B^{(i)}(x) = K_{k,k+1}(x) \textit{ with } k > 0, \\ K_{k+2,k+1}(x) & \textit{if } B^{(i)}(x) = K_{k+1,k}(x), \\ \textit{the } 0 \times 0 \textit{ empty matrix} & \textit{if } B^{(i)}(x) = K_{0,1}(x). \end{cases} \tag{3}$$

2. *All left duals of $A(x)$ have KCF $\operatorname{diag}(S_l(x), K_{1,0}(x), K_{1,0}(x), \dots, K_{1,0}(x))$, where we allow an arbitrary number of $K_{1,0}$ blocks. In addition, a left dual of $A(x)$ is row-minimal if and only if its KCF is $S_l(x)$.*

3. *a column-minimal right dual of $A(x)$ is $R(x) = WS_r(x)$, where*

$$S_r(x) = \operatorname{diag}(R^{(1)}(x), R^{(2)}(x), \dots, R^{(t)}(x)),$$

   and

$$R^{(i)}(x) = \begin{cases} B^{(i)}(x) & \textit{if } B^{(i)}(x) \textit{ is any nonsingular Kronecker block,} \\ K_{k+1,k+2}(x) & \textit{if } B^{(i)}(x) = K_{k,k+1}(x), \\ K_{k,k-1}(x) & \textit{if } B^{(i)}(x) = K_{k+1,k}(x) \textit{ with } k > 0, \\ \textit{the } 0 \times 0 \textit{ empty matrix} & \textit{if } B^{(i)}(x) = K_{1,0}(x). \end{cases} \tag{4}$$

4. *All right duals of $A(x)$ have KCF $\operatorname{diag}(S_r(x), K_{0,1}(x), K_{0,1}(x), \dots, K_{0,1}(x))$, where we allow an arbitrary number of $K_{0,1}$ blocks. In addition, a right dual of $A(x)$ is column-minimal if and only if its KCF is $S_r(x)$.*

Figure 1: The matrices in Example 3.4.

$$L(x) = \begin{bmatrix} x-3.5 & -1 & & & \\ 0 & x-3.5 & & & \\ & & -1 & x & \\ & & & & -1 \\ & & & & x \\ & & & & \end{bmatrix}, \quad R(x) = \begin{bmatrix} x-3.5 & -1 & & & & & \\ 0 & x-3.5 & & & & & \\ & & -1 & x & & & \\ & & & -1 & x & & \\ & & & & & & \end{bmatrix}$$

Proof. 1 The equality $L_1^{(i)} B_0^{(i)} = L_0^{(i)} B_1^{(i)}$ holds for each i because of Lemma 3.2, since these are duals block by block; joining all these blocks diagonally, one has $(S_l)_1 B_0 = (S_l)_0 B_1$. Hence $(S_l)_1 V A_0 W = (S_l)_0 V A_1 W$, and since W is invertible we get **D1**.

It remains to verify **D2**. Let $m_i \times n_i$ be the dimension of $L^{(i)}(x)$. By the row minimality properties proved in Lemma 3.2, $\operatorname{rank} \operatorname{row}(L^{(i)}) = m_i$, hence $\operatorname{rank} \operatorname{col}(B^{(i)}) = 2n_i - m_i$. Now we have $n = \sum_i n_i$, $m = \sum_i m_i$, $\operatorname{rank} \operatorname{row}(L) = \operatorname{rank} \operatorname{row}(S_l) = \sum_i \operatorname{rank} \operatorname{row}(L^{(i)}) = \sum_i m_i = m$, and $\operatorname{rank} \operatorname{col}(B) = \sum_i \operatorname{rank} \operatorname{col}(B^{(i)}) = \sum_i (2n_i - m_i) = 2n - m$.

2 By Lemma 3.1, a generic left dual of $A(x)$ can be written as $\hat{L}(x) = M S_l(x) U$, for a full-column-rank M, since $S_l(x)U$ is a minimal dual. If we complete M to a square invertible matrix as $\begin{bmatrix} M & M' \end{bmatrix}$, then we have

$$\hat{L}(x) = \begin{bmatrix} M & M' \end{bmatrix} \begin{bmatrix} S_l(x) \\ 0 \end{bmatrix} U,$$

which is a KCF for $\hat{L}(x)$ with the required structure.

We omit a proof of Parts 3 and 4 as they are analogous to Parts 1 and 2. $\square$

In other words, Theorem 3.3 states that, when taking a left dual, the regular part of the KCF is unchanged, the right minimal indices decrease by 1, and the left minimal indices increase by 1; the converse holds for right duals.

Example 3.4. The matrices $L(x) = \operatorname{diag}(xI - J_2^{(3.5)}, K_{1,2}(x), K_{2,1}(x), K_{1,0}(x))$ and $R(x) = \operatorname{diag}(xI - J_2^{(3.5)}, K_{2,3}(x), K_{1,0}(x), K_{0,1}(x), K_{0,1}(x))$, shown in Figure 1, are dual. The Kronecker structure of $L(x)$ is determined by that of $R(x)$, except from the number of $K_{1,0}(x)$ blocks, which is arbitrary. *Vice versa*, the blocks of $R(x)$ determine those of $L(x)$ apart from the number of $K_{0,1}(x)$ blocks.

Corollary 3.5. *(Simultaneous Kronecker canonical form) For any pair of dual pencils $L(x), R(x)$, there are nonsingular U, V, W such that $UL(x)V^{-1}$ and $VR(x)W$ are both in KCF.*

It is convenient at this point to state a technical lemma on the interaction of change of basis matrices and minimal bases. We say that a minimal basis is *ordered* if the degrees of its columns are a nondecreasing sequence.

Lemma 3.6. *Suppose that $M(x) \in \mathbb{C}[x]^{p\times q}$ is an ordered minimal basis of $\mathcal{V} \subseteq \mathbb{C}(x)^p$, and for each $d \in \mathbb{N}$ let $n_d(M)$ be the number of minimal indices equal to d in $M(x)$. (Note that $\sum_{d\in\mathbb{N}} n_d(M) = q$, implying in particular that $n_d(M) = 0$ for infinitely many values of d).*

Let $\hat{M}(x) = M(x)T(x)$ for some $T(x) \in \mathbb{C}^{q\times q}$. Then $\hat{M}(x)$ is an ordered minimal basis of $\mathcal{V}$ if and only if $T(x)$ satisfies the following two conditions:

- *$T(x)$ is block upper triangular, with constant invertible diagonal blocks of sizes $n_d(M) \times n_d(M)$, for $d = 0, 1, 2, \dots$;*
- *$\deg T_{ij}(x) \leq \deg M_j(x) - \deg M_i(x)$, where $M_i(x)$ denotes the ith column of $M(x)$ and $T_{ij}(x)$ denotes the (i, j)th element of $T(x)$.*

Proof. Suppose that $\hat{M}(x)$ is an ordered minimal basis. Then it must have the same minimal indices as $M(x)$, i.e., $\deg M_j(x) = \deg \hat{M}_j(x)\ \forall\ j$. By [17, Main Theorem, part 4b] we have that $\deg M_j(x) = \deg \hat{M}_j(x) = \max_{1\leq i\leq q}(\deg M_i(x) + \deg T_{ij}(x))$. (Here the usual convention $\deg 0 = -\infty$ is used). Therefore, $\deg T_{ij}(x) \leq \deg M_j(x) - \deg M_i(x)$. In particular, this implies that $T(x)$ is block upper triangular with constant diagonal blocks of size $n_d(M) \times n_d(M)$. Finally, since $T(x)$ is invertible, so must be its diagonal blocks.

Conversely, suppose that $T(x)$ is of the form above. Then $\hat{M}(x)$ is a polynomial basis for $\mathcal{V}$, and the degree of each of its columns is, at most, equal to those of $M(x)$. But $M(x)$ is a minimal basis, and hence, we must have exact equality. Thus, $\hat{M}(x)$ is an ordered minimal basis for $\mathcal{V}$. □

Remark 3.7. With the notation of Lemma 3.6, it is convenient to observe for future reference the following simple consequence of Theorem 3.3. If $L(x)$ and $R(x)$ are a pair of dual pencils and if $N(x)$ (resp. $M(x)$) is a minimal basis of $L(x)$ (resp. $R(x)$) then for all $d \geq 1$ it holds $n_d(M) = n_{d-1}(N)$.

The following theorem shows how minimal bases change under duality. A similar result (for the special case where $(\gamma, \delta) = (1, 0)$ and $R(x)$ is column-minimal) is stated in [26, Equation (1.3.4)].

Theorem 3.8. *Let $L(x) = L_1x + L_0$ and $R(x) = R_1x + R_0$ be a pair of dual pencils. Suppose that $M(x)$ is a right minimal basis for $R(x)$. For any $(\gamma, \delta) \in \mathbb{C}^2 \setminus \{(0, 0)\}$, the nonzero columns of $N(x) := \frac{\gamma R_1+\delta R_0}{\gamma-\delta x} M(x)$ form a right minimal basis of $L(x)$.*

Proof. Let $UL(x)V^{-1} = \operatorname{diag}(L^{(1)}(x), L^{(2)}(x), \dots, L^{(t)}(x), K_{1,0}(x), K_{1,0}(x), \dots, K_{1,0}(x))$, $VR(x)W = \operatorname{diag}(B^{(1)}(x), B^{(2)}(x), \dots, B^{(t)}(x)) =: B(x)$, with $L^{(i)}(x)$ defined as in (3), be a simultaneous canonical form for $L(x)$ and $R(x)$, and let $\hat{M}(x) := W \operatorname{diag}(M^{(1)}(x), M^{(2)}(x), \dots, M^{(t)}(x))$ be the special minimal basis for $R(x)$ constructed as shown in Lemma 2.3. We first prove the result for the minimal basis $\hat{M}(x)$.

Let us define for $i = 1, 2, \ldots, t$

$$N^{(i)}(x) := \frac{\gamma B_1^{(i)} + \delta B_0^{(i)}}{\gamma - \delta x} M^{(i)}(x).$$

One can verify directly that

$$N^{(i)}(x) = \begin{cases} \begin{bmatrix} x^{k_i-1} & x^{k_i-2} & \cdots & x & 1 \end{bmatrix}^T & \text{if } B^{(i)}(x) \text{ is of the form } K_{k_i,k_i+1}(x) \text{ for } k_i > 0, \\ \text{the empty vector of size } 0 \times 1 & \text{if } B^{(i)}(x) \text{ is of the form } K_{0,1}(x), \\ \text{the empty vector of size } (k_i+1) \times 0 & \text{if } B^{(i)}(x) \text{ is of the form } K_{k_i+1,k_i(x)}, \\ \text{the empty vector of size } k_i \times 0 & \text{otherwise.} \end{cases} \tag{5}$$

Hence

$$\begin{aligned} \frac{\gamma R_1 + \delta R_0}{\gamma - \delta x} \hat{M}(x) &= V^{-1} \frac{\gamma B_1 + \delta B_0}{\gamma - \delta x} \operatorname{diag}(M^{(1)}(x), M^{(2)}(x), \ldots, M^{(t)}(x))(x) \\ &= V^{-1} \operatorname{diag}(N^{(1)}(x), N^{(2)}(x), \ldots, N^{(t)}(x)) =: \hat{N}(x). \end{aligned}$$

This matrix contains some zero columns (those corresponding to the second case in (5)), and if one removes them, what is left is precisely the special minimal basis for $L(x)$ obtained by applying Lemma 2.3 to it. This proves the theorem for the special case of $\hat{M}(x)$ as a minimal basis.

Let us now take a generic minimal basis $M(x) \in \mathbb{C}^{p \times q}$. Up to permutation of columns, we may assume that $M(x)$ and $\hat{M}(x)$ are both ordered minimal bases. In particular, they have the same number of degree-0 columns, so we can partition them consistently as

$$M(x) =: \begin{bmatrix} M_c & M_n(x) \end{bmatrix}, \quad \hat{M}(x) =: \begin{bmatrix} \hat{M}_c & \hat{M}_n(x) \end{bmatrix}, \tag{6}$$

M_c and $\hat{M}_c$ being two bases of $\ker \operatorname{col}(R)$. Now let $T(x) \in \mathbb{C}(x)^{q \times q}$ be the change of basis matrix such that $M(x) = \hat{M}(x)T(x)$. Note that $T(x)$ is block upper triangular by Lemma 3.6, and we can partition it as

$$T(x) =: \begin{bmatrix} T_{cc} & T_{cn}(x) \\ 0 & T_{nn}(x) \end{bmatrix},$$

where T_{cc} and $T_{nn}(x)$ are square with sizes conformable with (6). Now we can write

$$N(x) = \frac{\gamma R_1 + \delta R_0}{\gamma - \delta x} M(x) = \begin{bmatrix} 0 & \frac{\gamma R_1 + \delta R_0}{\gamma - \delta x} \left(\hat{M}_c T_{cn}(x) + \hat{M}_n(x) T_{nn}(x) \right) \end{bmatrix} = \hat{N}(x) T_{nn}(x).$$

Lemma 3.6 shows that $T_{nn}(x)$ must be block upper triangular with constant invertible diagonal blocks whose sizes are prescribed by the nonzero minimal indices of $M(x)$. Hence, by the fact that $\hat{N}(x)$ is an ordered minimal basis, and by Lemma 3.6 and Remark 3.7, $N(x)$ is also an ordered minimal basis.

□

A converse result of Theorem 3.8 can be also stated. It appeared in [26, equation (1.3.4)], for the special case of $\mathrm{col}(R)$ having orthonormal columns.

Theorem 3.9. *Let $L(x) \in \mathbb{C}[x]_1^{m\times n}$ have minimal basis $N(x)$, and suppose that $R(x) \in \mathbb{C}[x]_1^{n\times p}$ is a column-minimal right dual of $L(x)$. Suppose moreover that $Q(x) = Q_1x + Q_0 \in \mathbb{C}[x]_1^{p\times n}$ is such that $Q_0R_1 - Q_1R_0 = I_p$. Then $M(x) = Q(x)N(x)$ is a minimal basis for $R(x)$.*

Proof. Note first that the existence of $Q(x)$ is guaranteed by the column-minimality of $R(x)$, since one can complete $\mathrm{col}(R)$ to an invertible matrix.

Let first $\hat{M}(x)$ and $\hat{N}(x)$ be the special minimal bases constructed in Lemma 2.3, as in the proof of Theorem 3.8. Note that for each $(\gamma, \delta) \neq (0, 0)$ we have $\hat{N}(x) = \frac{\gamma R_1+\delta R_0}{\gamma-\delta x}\hat{M}(x)$, without spurious zero columns, thanks to the hypothesis that $R(x)$ is column-minimal. In particular, $\hat{N}(x) = R_1\hat{M}(x)$ and $x\hat{N}(x) = -R_0\hat{M}(x)$, by choosing $(\gamma, \delta) = (1, 0)$ and $(0, 1)$, respectively. Hence $Q(x)\hat{N}(x) = (Q_0R_1 - Q_1R_0)\hat{M}(x) = \hat{M}(x)$, which proves the statement for this special choice of bases.

We may assume without loss of generality that $\hat{M}(x), \hat{N}(x)$ and $N(x)$ are ordered minimal bases. By Lemma 3.6, $N(x) = \hat{N}(x)T(x)$ for a block-triangular $T(x)$ with the degree properties stated in the theorem, and hence $\hat{M}(x)T(x) = Q(x)\hat{N}(x)T(x) = M(x)$. Thanks again to Lemma 3.6 again, $M(x)$ is a minimal basis.

□

Remark 3.10. The hypothesis of column-minimality of $R(x)$ in Theorem 3.9 can be relaxed: if $R(x)$ is not column-minimal, $Q(x)$ is constructed analogously starting from a basis of $\mathrm{col}(R)$, and the minimal basis is given by $M(x) = \begin{bmatrix}\ker\mathrm{col}(R) & Q(x)N(x)\end{bmatrix}$.

One can formulate variants of Theorems 3.8 and 3.9 for left minimal bases, obtaining analogous results. We omit the details.

4 Wong chains, Kronecker forms and duals

In this section, we agree to the convention that $\frac{1}{0} =: \infty$, useful to analyze infinite eigenvalues. Moreover, we will need the following lemma.

Lemma 4.1. *Let $L(x)$ and $R(x)$ be a pair of dual pencils. Then the following identity holds for all $\alpha, \beta, \gamma, \delta \in \mathbb{C}$:*

$$(\alpha L_1 + \beta L_0)(\gamma R_1 + \delta R_0) = (\gamma L_1 + \delta L_0)(\alpha R_1 + \beta R_0). \tag{7}$$

Proof. If we expand the products, the identity reduces to $(\alpha\delta-\beta\gamma)L_1R_0 = (\alpha\delta-\beta\gamma)L_0R_1$, which holds because $L_1R_0 = L_0R_1$. □

If $L(x)$ and $R(x)$ are square regular pencils, and v is an eigenvector of $R(x)$ with eigenvalue $\lambda = \frac{\alpha}{\beta} \in \mathbb{C} \cup \{\infty\}$, then Lemma 4.1 implies

$$0 = (\gamma L_1 + \delta L_0)(\alpha R_1 + \beta R_0)v = (\alpha L_1 + \beta L_0)(\gamma R_1 + \delta R_0)v,$$

and thus, for each γ, δ such that $\frac{\gamma}{\delta} \neq \frac{\alpha}{\beta}$, $w = (\gamma R_1 + \delta R_0)v$ is an eigenvector of $L(x)$ with the same eigenvalue $\frac{\alpha}{\beta}$.

It looks natural to try to generalize this relation to singular pencils. However, an additional difficulty appears in defining the needed quantities. Consider the simplest case of an eigenvalues of geometric multiplicity 1. In the regular case, the eigenvector is then uniquely defined up to a scalar nonzero constant, but it is easy to verify that this is not the case if $\dim \ker_{\mathbb{C}(x)} R(x) > 0$.

Even with regular monic pencils, i.e., matrices, the Jordan canonical form is unique, but Jordan chains are not. For instance, the matrix $\left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right]$ has a well-defined (up to scalar multiples) eigenvector $\left[\begin{smallmatrix} 1 & 0 \end{smallmatrix}\right]^T$, but the second vector of the Jordan chain can be freely chosen as $\left[\begin{smallmatrix} \alpha & 1 \end{smallmatrix}\right]^T$ for any $\alpha \in \mathbb{C}$, so it is *not* uniquely defined up to scalar multiples. The situation is even more involved in presence of multiple Kronecker blocks with the same eigenvalue, or of singular blocks. In other words, while the KCF $K(x)$ of a pencil is unique, the transformation matrices U and V are not, and therefore cannot be used to introduce uniquely defined quantities.

Wong chains are useful to solve these issues. The key idea is to overcome the ill-posedness of the definition by looking at a chain of subspaces rather than vectors. Wong chains can be canonically defined for any (possibly singular) pencil, and without the need for arbitrary basis choices.

Let $R(x) = R_1 x + R_0 \in \mathbb{C}[x]_1^{m\times n}$ be a pencil, $\lambda = \frac{\alpha}{\beta} \in \mathbb{C} \cup \{\infty\}$ and $\mu = \frac{\gamma}{\delta} \in \mathbb{C} \cup \{\infty\}$ such that $\alpha\delta \neq \beta\gamma$. For each $k \in \mathbb{N}$, we define the vector subspaces $\mathcal{W}_k^{(\lambda,\mu)} \subseteq \mathbb{C}^n$ through the following recurrence:

$$\begin{aligned} \mathcal{W}_0^{(\lambda,\mu)} &= \{0\}, \\ \mathcal{W}_{k+1}^{(\lambda,\mu)} &= (\alpha R_1 + \beta R_0)^{\leftarrow} (\gamma R_1 + \delta R_0) \mathcal{W}_k^{(\lambda,\mu)}. \end{aligned} \tag{8}$$

We have used the following notations to denote how a matrix $M \in \mathbb{C}^{m\times n}$ acts on a vector subspace $\mathcal{V}$: $M\mathcal{V} := \{Mv \in \mathbb{C}^m \mid v \in \mathcal{V}\}$ (image of $\mathcal{V} \subseteq \mathbb{C}^n$ via M), and $M^{\leftarrow}\mathcal{V} := \{w \in \mathbb{C}^n \mid Mw \in \mathcal{V}\}$ (preimage of $\mathcal{V} \subseteq \mathbb{C}^m$ under M).

The following properties hold.

Lemma 4.2. *$\mathcal{W}_k^{(\lambda,\mu)} \subseteq \mathcal{W}_{k+1}^{(\lambda,\mu)}$ for all k.*

Proof. The relation is obvious for $k = 0$. Suppose that $w \in \mathcal{W}_k^{(\lambda,\mu)}$, and assume $\mathcal{W}_{k-1}^{(\lambda,\mu)} \subseteq \mathcal{W}_k^{(\lambda,\mu)}$. Then there exists $w' \in \mathcal{W}_{k-1}^{(\lambda,\mu)}$ such that $(\alpha R_1 + \beta R_0)w = (\gamma R_1 + \delta R_0)w'$; but since $w' \in \mathcal{W}_k^{(\lambda,\mu)}$, $w \in \mathcal{W}_{k+1}^{(\lambda,\mu)}$, which proves the lemma by induction. $\square$

Corollary 4.3. *If $\mathcal{W}_{k_0}^{(\lambda,\mu)} = \mathcal{W}_{k_0-1}^{(\lambda,\mu)}$ then $\bigcup_k \mathcal{W}_k^{(\lambda,\mu)} = \mathcal{W}_{k_0}^{(\lambda,\mu)}$.*

Proof. If $\mathcal{W}_{k_0}^{(\lambda,\mu)} = \mathcal{W}_{k_0-1}^{(\lambda,\mu)}$ then by the definition $\mathcal{W}_k^{(\lambda,\mu)} = \mathcal{W}_{k_0-1}^{(\lambda,\mu)}$ for all $k \geq k_0 - 1$. The statement then follows by Lemma 4.2. $\square$

In fact, the sequences of subspaces that we have defined do not depend on the particular choice of (γ, δ), as we prove in the following theorem.

Theorem 4.4. *The subspaces $\mathcal{W}_k^{(\lambda,\mu)}$ defined in* (8) *do not depend on μ.*

Proof. Clearly, $\mathcal{W}_0^{(\lambda,\mu)} = \{0\}$ and $\mathcal{W}_1^{(\lambda,\mu)} = \ker(\alpha R_1 + \beta R_0)$ do not depend on γ, δ. Now by induction on k suppose that $\mathcal{W}_k^{(\lambda,\mu)}$ is independent of the choice of μ. Let $\mu_0 = \frac{\gamma_0}{\delta_0}, \mu_1 = \frac{\gamma_1}{\delta_1}$ be such that $\alpha\delta_i \neq \beta\gamma_i$ for $i = 0, 1$. Let $v_1 \in \mathcal{W}_{k+1}^{(\lambda,\mu_1)}$ and let $x, y \in \mathbb{C}$ be such that $\left[\begin{smallmatrix} \alpha & \gamma_0 \\ \beta & \delta_0 \end{smallmatrix}\right]\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] = \left[\begin{smallmatrix} \gamma_1 \\ \delta_1 \end{smallmatrix}\right]$. Then there exists $w \in \mathcal{W}_k^{(\lambda,\mu_1)}$ such that $(\alpha R_1 + \beta R_0)v_1 = (\gamma_1 R_1 + \delta_1 R_0)w = (\alpha R_1 + \beta R_0)xw + (\gamma_0 R_1 + \delta_0 R_0)yw = (\gamma_0 R_1 + \delta_0 R_0)(xw' + yw)$ for some $w' \in \mathcal{W}_{k-1}^{(\lambda,\mu_0)} \subseteq \mathcal{W}_k^{(\lambda,\mu_0)} = \mathcal{W}_k^{(\lambda,\mu_1)}$, by Lemma 4.2 and the inductive hypothesis. Hence, $\mathcal{W}_{k+1}^{(\lambda,\mu_1)} \subseteq \mathcal{W}_{k+1}^{(\lambda,\mu_0)}$. Switching the roles of μ_0 and μ_1 yields the reverse inclusion, implying that equality holds. □

Hence we can omit the superscript μ and give a formal definition.

Definition 4.5. Let $R(x) = R_1 x + R_0 \in \mathbb{C}[x]_1^{m\times n}$ be a pencil, and $\lambda = \frac{\alpha}{\beta} \in \mathbb{C} \cup \{\infty\}$. The *Wong chain* for $R(x)$ attached to λ is the sequence of subspaces $(\mathcal{W}_k^{(\lambda)})_{k\in\mathbb{N}}$ given by $\mathcal{W}_k^{(\lambda)} = \mathcal{W}_k^{(\lambda,\mu)}$ (defined in (8)) for each $\mu \neq \lambda$.

In this definition, the chain for a projective point λ can be constructed using any other point $\mu \neq \lambda$. In [4, 5, 40] only the special cases $\lambda = \infty, \mu = 0$ and $\lambda = 0, \mu = \infty$ appear, while in [6, 36] the authors allow also $\lambda \in \mathbb{C}, \mu = \infty$. Theorem 4.4 implies that it is possible to change the second base point μ without altering the corresponding subspace chain. As far as we know, this observation appears here for the first time.

To help the reader build a clearer picture of what Wong chains are, we give a more explicit characterization in terms of the Kronecker chains. This characterization seems to be implicit in results such as [4, Section 3] and [36, Section 2], but it helps to state it in full form with proofs.

The following simple result is needed in the following.

Lemma 4.6. *Let $A \in \mathbb{C}^{m\times n}$, and $v \in \mathbb{C}^m$. If there exists $w \in \mathbb{C}^n$ such that $Aw = v$, then $A^{\leftarrow}\operatorname{span}(v) = \operatorname{span}(w) + \ker A$; otherwise $A^{\leftarrow}\operatorname{span}(v) = \ker A$.*

Proof. In the first case, let $w' \in \operatorname{span}(w) + \ker A$. Then $w' = \alpha w + w''$ for some $\alpha \in \mathbb{C}$ and some $w'' \in \ker A$. Hence, $Aw' = \alpha v \in \operatorname{span}(v)$. Conversely, suppose that $Aw' = \alpha v$ for some $\alpha \in \mathbb{C}$; then $Aw' - A(\alpha w) = \alpha v - \alpha v = 0$, so $w' - \alpha w \in \ker A$.

In the second case, the equation $Aw' = \alpha v$ has a solution $w' \in \mathbb{C}^n$ only for $\alpha = 0$, hence $A^{\leftarrow}\operatorname{span}(v) = A^{\leftarrow}\{0\} = \ker A$. □

We start from a lemma characterizing the Wong chains of single Kronecker blocks.

Lemma 4.7. *1. Let $\lambda \in \mathbb{C} \cup \{\infty\}$ and either $B(x) = xI_k - J_k^{(\nu)}$ for some $\nu \in \mathbb{C}$, or $B(x) = xJ^{(0)} - I$ and $\nu := \infty$ (regular Kronecker blocks). Then the Wong chains of $B(x)$ are given by:*

a) if $\lambda = \nu$, $\mathcal{W}_j^{(\lambda)} = \operatorname{span}(e_1, e_2, \ldots, e_{\min(j,k)})$, where e_i is the ith column of I_k.

b) if $\lambda \neq \nu$, $\mathcal{W}_j^{(\lambda)} = \{0\}$ for each $j \in \mathbb{N}$.

2. *Let* $B(x) = K_{k+1,k}(x)$*. Then,* $\mathcal{W}_j^{(\lambda)} = \{0\}$ *for each* $j \in \mathbb{N}$ *and any* $\lambda \in \mathbb{C} \cup \{\infty\}$*.*

3. *Let* $B(x) = K_{k,k+1}(x)$ *and* $\lambda \in \mathbb{C}$*. Then,* $\mathcal{W}_j^{(\lambda)} = \operatorname{span}(v_1, v_2, \dots, v_{\min(j,k+1)})$*, where the ith component of the vector* $v_j \in \mathbb{C}^{k+1}$ *is given by*

$$(v_j)_i = \begin{cases} \binom{k+1-i}{j-1}\lambda^{k+2-i-j} & \text{if } i \leq k+2-j, \\ 0 & \text{otherwise.} \end{cases} \tag{9}$$

4. *Let* $B(x) = K_{k,k+1}(x)$ *and* $\lambda = \infty$*. Then,* $\mathcal{W}_j^{(\lambda)} = \operatorname{span}(e_1, e_2, \dots, e_{\min(j,k+1)})$*, where* e_i *is the ith column of* I_{k+1}*.*

For an example that clarifies the structure in (9), see (10) in the following. Note in particular that the vectors v_j are linearly independent since they are the columns of an anti-triangular matrix with 1 on the anti-diagonal.

Proof. Note that, by Theorem 4.4, within the proof we are allowed to pick a particular choice of γ and δ, as long as $\alpha\delta \neq \beta\gamma$. Moreover, throughout the proof it is convenient to label the coefficients of the pencil $B(x) =: B_1 x + B_0$ (the specific forms of B_1, B_0 vary according to what item in the statement is being considered).

To prove items 1b and 2, it suffices to observe that, since in both cases the matrix $\alpha B_1 + \beta B_0$ has full column rank, the preimage of $\{0\}$ is $\{0\}$, and hence all the subspaces in the chain are $\{0\}$.

We now prove Item 1a. We set $\alpha = -\lambda, \beta = -1, \gamma = 1, \delta = 0$ if $\nu = \lambda \in \mathbb{C}$ and $\alpha = 1, \beta = 0, \gamma = 0, \delta = -1$ if $\lambda = \infty$: in both cases, $\alpha B_1 + \beta B_0 = J_k^{(0)}$ and $\gamma B_1 + \delta B_0 = I$. The statement follows by noting that

$$(J_k^{(0)})^{\leftarrow} \operatorname{span}(e_1, e_2, \dots, e_j) = \operatorname{span}(e_1, e_2, \dots, e_{j+1}) \quad \text{for } j < k.$$

For Item 3., we set $\alpha = \lambda, \beta = 1, \gamma = 1, \delta = 0$, and let for brevity $N := \lambda B_1 + B_0$. It is easy to verify that $\operatorname{rank} N = k$ and $\ker N = \operatorname{span}(v_1)$, Moreover, we have that $N v_{j+1} = -B_1 v_j$ for each $j = 1, 2, \dots, k$: indeed, expanding the definitions, this reduces to the identity

$$\lambda \binom{k-i}{j} \lambda^{k-i-j} - \binom{k+1-i}{j} \lambda^{k+1-i-j} = -\binom{k-i}{j-1} \lambda^{k+1-i-j}.$$

Hence, by Lemma 4.6, $N^{\leftarrow} B_1 \operatorname{span}(v_j) = \operatorname{span}(v_1, v_{j+1})$, for each $j = 1, 2, \dots, k$.

Now, we can prove the statement $\mathcal{W}_j^{(\lambda)} = \operatorname{span}(v_1, v_2, \dots, v_j)$ for $1 \leq j \leq k+1$ by induction on j: the base case is $N^{\leftarrow} B_1 \{0\} = \ker N = \operatorname{span}(v_1)$, and the inductive step is

$$N^{\leftarrow} B_1 \mathcal{W}_j^{\lambda} = N^{\leftarrow} B_1 \operatorname{span}(v_1) + N^{\leftarrow} B_1 \operatorname{span}(v_2) + \dots + N^{\leftarrow} B_1 \operatorname{span}(v_j) = \operatorname{span}(v_1, v_2, \dots, v_{j+1}).$$

Finally, the proof of Item 4 is analogous to the one of Item 3: we set $\alpha = 1, \beta = 0, \gamma = 0, \delta = 1$; then it is easy to check that $\operatorname{rank} B_1 = k$, $\ker B_1 = \operatorname{span}(e_1)$, $B_1 e_{j+1} = -B_0 e_j$, and one can argue as above. □

Now we can put together the results on the Kronecker blocks.

Theorem 4.8 ([36, Lemma 2.7])**.** *Let $R(x) \in \mathbb{C}[x]_1^{n\times p}$ and $\lambda \in \mathbb{C}\cup\{\infty\}$ be given. Suppose that $VR(x)W = B(x) = \operatorname{diag}(B^{(1)}(x), B^{(2)}(x), \dots, B^{(t)}(x))$, and that $(\operatorname{Im} U_j^{(i)})_{j\in\mathbb{N}}$ is the Wong chain of $B^{(i)}(x)$ attached to λ, for each $i = 1, 2, \dots, t$. Then, the Wong chain of $R(x)$ attached to λ is given by*

$$\mathcal{W}_j^{(\lambda)} = \operatorname{Im}\left(W \operatorname{diag}(U_j^{(1)}, U_j^{(2)}, \dots, U_j^{(t)})\right), \quad \textit{for each } j \in \mathbb{N}.$$

Proof. We prove the result by induction; the base step comes for free, since for $j = 0$ all Wong chains are trivial, and the inductive step is

$$\begin{aligned}
\mathcal{W}_{j+1}^{(\lambda)} &= (\alpha R_1 + \beta R_0)^{\leftarrow}(\gamma R_1 + \delta R_0)\mathcal{W}_j^{(\lambda)} \\
&= \left(V^{-1}(\alpha B_1 + \beta B_0)W^{-1}\right)^{\leftarrow}(V^{-1}(\gamma B_1 + \delta B_0)W^{-1})\mathcal{W}_j^{(\lambda)} \\
&= W(\alpha B_1 + \beta B_0)^{\leftarrow}(\gamma B_1 + \delta B_0) \operatorname{Im}\operatorname{diag}(U_j^{(1)}, U_j^{(2)}, \dots, U_j^{(t)}) \\
&= \operatorname{Im} W \operatorname{diag}(U_{j+1}^{(1)}, U_{j+1}^{(2)}, \dots, U_{j+1}^{(t)}).
\end{aligned}$$

□

Example 4.9. Consider the pencil $R(x)$ in Example 3.4 (and Figure 1), which is already in KCF, and $\lambda = 3.5$. By Lemma 4.7,

- the subspaces in the Wong chain for $B^{(1)}(x) = xI - J_2^{(\lambda)}$ are the images of $U_1^{(1)} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $U_2^{(1)} = U_3^{(1)} = \cdots = I_2$;
- the subspaces in the Wong chain for $B^{(2)}(x) = K_{2,3}(x)$ are the images of
$$U_1^{(2)} = \begin{bmatrix} \lambda^2 \\ \lambda \\ 1 \end{bmatrix}, \quad U_2^{(2)} = \begin{bmatrix} \lambda^2 & 2\lambda \\ \lambda & 1 \\ 1 & 0 \end{bmatrix}, \quad U_3^{(2)} = U_4^{(2)} = \cdots = \begin{bmatrix} \lambda^2 & 2\lambda & 1 \\ \lambda & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}; \tag{10}$$
- the Wong chain for $B^{(3)}(x) = K_{1,0}(x)$ is trivial (image of the empty matrix $U_j^{(3)} \in \mathbb{C}^{0\times 0}$ for each $j \in \mathbb{N}$);
- the subspaces in the Wong chain for $B^{(4)}(x) = B^{(5)}(x) = K_{0,1}(x)$ are the image of $U_j^{(4)} = U_j^{(5)} = \begin{bmatrix} 1 \end{bmatrix}$ for each $j \geq 1$.

Hence, by Theorem 4.8, the Wong chain of $R(x)$ is $\mathcal{W}_j^{(\lambda)} = \operatorname{Im}\operatorname{diag}(U_j^{(1)}, U_j^{(2)}, U_j^{(3)}, U_j^{(4)}, U_j^{(5)})$ for each j; thus,

$$\mathcal{W}_0^{(\lambda)} = \{0\}, \quad \mathcal{W}_1^{(\lambda)} = \operatorname{Im}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \lambda^2 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \mathcal{W}_2^{(\lambda)} = \operatorname{Im}\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda^2 & 2\lambda & 0 & 0 \\ 0 & 0 & \lambda & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

and $\mathcal{W}_k^{(\lambda)} = \mathbb{C}^7$ for $k \geq 3$.

We now state two corollaries that are consequences of Theorem 4.8. The first corollary illustrates that Wong chains extend the concept of Jordan chains. Following [22, Ch. 1 and 7], we define a canonical set of Jordan chains at a finite eigenvalue λ of a regular pencil $R_1 x + R_0$ with geometric multiplicity g as a set of vectors $w_{11}, \dots, w_{1\ell_1}, \dots, w_{g1}, \dots, w_{g\ell_g}$ such that (i) for all $i = 1, \dots, g$ and $j = 1, \dots, \ell_i$, we have $(\lambda R_1 + R_0) w_{i1} = 0$ and $(\lambda R_1 + R_0) w_{ij} + R_1 w_{i,j-1} = 0$; (ii) $w_{11}, w_{21}, \dots, w_{g1}$ are linearly independent and span $\ker(\lambda R_1 + R_0)$; (iii) for all $i = 1, \dots, g$, the Jordan chain $w_{i1}, \dots, w_{i,\ell_i}$ has maximal length among all Jordan chains whose eigenvector w_{i1} is such that $w_{11}, w_{21}, \dots, w_{i1}$ are linearly independent.

Corollary 4.10. *Let $R(x)$ be a regular matrix pencil let $\lambda \in \mathbb{C}$ be given, and let $w_{11}, w_{12}, \dots, w_{1\ell_1}$, $w_{21}, \dots, w_{g\ell_g}$ be a canonical set of Jordan chains for $R(x)$ at the eigenvalue λ. Then, $\mathcal{W}_k^{(\lambda)} = \operatorname{span}\{w_{ij} \mid j \leq k\}$.*

Proof. Thanks to (i), (iii) and Lemma 4.6,

$$(R_1\lambda + R_0)^{\leftarrow} R_1 \operatorname{span}(w_{ij}) = \begin{cases} \operatorname{span}(w_{i,j+1}) + \ker(\lambda R_1 + R_0) & \text{if } j < \ell_i, \\ \ker(\lambda R_1 + R_0) & \text{if } j = \ell_i. \end{cases} \tag{11}$$

We can now prove by induction that the spaces $\operatorname{span}\{w_{ij} \mid j \leq k\}$ satisfy the conditions that define Wong chains. For $k = 1$, (ii) implies that $\operatorname{span}\{w_{i1}\} = \ker(\lambda R_1 + R_0) = \mathcal{W}_1^{(\lambda)}$. For $k > 1$, using (11) we have

$$(R_1\lambda + R_0)^{\leftarrow} R_1 \mathcal{W}_{k-1} = (R_1\lambda + R_0)^{\leftarrow} R_1 \operatorname{span}\{w_{ij} \mid j \leq k-1\} = \operatorname{span}\{w_{ij} \mid j \leq k\}.$$

□

For the second corollary, we first need a definition and a lemma: the *columnwise reversal* of a matrix polynomial $A(x)$ whose columns are $[A_1(x), \dots, A_n(x)]$ is $\operatorname{cwRev} A(x) := [\operatorname{rev} A_1(x), \dots, \operatorname{rev} A_n(x)]$, where each reversal is taken with grade equal to the degree of the respective column. It holds

Lemma 4.11. *[12, Theorem 3.2],[35, Theorem 4.1] If $M(x)$ is a minimal basis for a pencil $R(x)$, then* $\operatorname{cwRev} M(x)$ *is a minimal basis for* $\operatorname{rev} R(x)$.

Corollary 4.12. *Suppose that $\lambda \in \mathbb{C} \cup \{\infty\}$ is not an eigenvalue of the pencil $R(x)$, and suppose that $M(x)$ is a minimal basis of $R(x)$, with $d = \deg M(x)$. If $\lambda \in \mathbb{C}$, then let*

$$M(x) = \sum_{i=0}^{d} M_i (x - \lambda)^i$$

be a Taylor expansion of $M(x)$ around λ, while if $\lambda = \infty$ let

$$\operatorname{cwRev} M(x) = \sum_{i=0}^{d} M_i x^i.$$

Then, it holds $\mathcal{W}_j^{(\lambda)} = \operatorname{Im} M_0 + \operatorname{Im} M_1 + \cdots + \operatorname{Im} M_{\min(j-1,d)}$.

Proof. Assume first $\lambda \in \mathbb{C}$. If the minimal basis is the special one $\hat{M}(x)$ introduced in Lemma 2.3, then the statement follows from Lemma 4.7 and Theorem 4.8. Hence we simply have to prove that the subspaces $\operatorname{Im} M_0 + \operatorname{Im} M_1 + \cdots + \operatorname{Im} M_{\min(j-1,d)}$ do not depend on the choice of the minimal basis. Assume, with no loss of generality, that $M(x)$ is ordered. Expanding in a Taylor series around λ the change of basis equation $M(x) = \hat{M}(x)T(x)$, we obtain

$$\begin{bmatrix} M_0 & \cdots & M_k \end{bmatrix} = \begin{bmatrix} \hat{M}_0 & \cdots & \hat{M}_k \end{bmatrix} \begin{bmatrix} T_0 & T_1 & \cdots & T_k \\ & T_0 & \cdots & T_{k-1} \\ & & \ddots & \\ & & & T_0 \end{bmatrix}, \tag{12}$$

with $T(x) = \sum_i T_i(x-\lambda)^i$. Since $T_0 = T(\lambda)$ is invertible by Lemma 3.6, Equation (12) implies that $\begin{bmatrix} M_0 & \cdots & M_k \end{bmatrix}$ and $\begin{bmatrix} \hat{M}_0 & \cdots & \hat{M}_k \end{bmatrix}$ have the same column space.

For $\lambda = \infty$, the result follows by using Lemma 4.11 and reducing to the case $\lambda = 0$ for the reversed pencil. □

The characterization of Wong chains given here provides an immediate explanation of the following fact, already showed in [6]: the right minimal indices, which are well defined, can be determined from the fact that, for any λ which is not an eigenvalue of $R(x)$, $\dim \mathcal{W}_{k+1}^{(\lambda)} - \dim \mathcal{W}_k^{(\lambda)}$ is the number of right Kronecker blocks of size k or greater.

One can define *left Wong chains* as the Wong chains of $R(x)^T$, and, using them, reconstruct left minimal indices and generalize left eigenvectors and Jordan chains. We omit the details.

It is natural to investigate how Wong chains change under duality, which is shown by our next result.

Theorem 4.13. *Let $L(x)$ and $R(x)$ be a pair of dual pencils. Let $(\mathcal{V}_k^{(\lambda)})_k$ and $(\mathcal{W}_k^{(\lambda)})_k$ be the Wong chains of $L(x)$ and $R(x)$, respectively, attached to $\lambda = \frac{\alpha}{\beta} \in \mathbb{C} \cup \{\infty\}$. The relation*

$$\mathcal{V}_k^{(\lambda)} = (\gamma R_1 + \delta R_0)\mathcal{W}_k^{(\lambda)}$$

holds for all k, γ, δ provided that $\alpha\delta \neq \beta\gamma$.

Proof. We give a proof by induction on k. For $k = 0$ the relation is obvious since $\{0\} = (\gamma R_1 + \delta R_0)\{0\}$.

Suppose now that the thesis is true for $\mathcal{V}_{k-1}^{(\lambda)}$, $\mathcal{W}_{k-1}^{(\lambda)}$. Assume that $w \in \mathcal{W}_k^{(\lambda)}$, i.e., $\exists w' \in \mathcal{W}_{k-1}^{(\lambda)}$ such that $(\alpha R_1 + \beta R_0)w = (\gamma R_1 + \delta R_0)w'$. Then for $v = (\gamma R_1 + \delta R_0)w$ Lemma 4.1 implies $(\alpha L_1 + \beta L_0)v = (\gamma L_1 + \delta L_0)(\alpha R_1 + \beta R_0)w = (\gamma L_1 + \delta L_0)(\gamma R_1 + \delta R_0)w'$, and $(\gamma R_1 + \delta R_0)w' \in \mathcal{V}_{k-1}^{(\lambda)}$ by the inductive hypothesis: thus, $v \in \mathcal{V}_k^{(\lambda)}$ and therefore $(\gamma R_1 + \delta R_0)\mathcal{W}_k^{(\lambda)} \subseteq \mathcal{V}_k^{(\lambda)}$.

Conversely, let $v \in \mathcal{V}_k^{(\lambda)}$. Then, $(\alpha L_1 + \beta L_0)v = (\gamma L_1 + \delta L_0)v'$ for some $v' \in \mathcal{V}_{k-1}^{(\lambda)}$. This implies $\operatorname{row}(L)J_n \left[\begin{smallmatrix} \beta v - \delta v' \\ \gamma v' - \alpha v \end{smallmatrix}\right] = 0$. By Lemma 3.1, $\operatorname{col}(R)$ spans $\ker \operatorname{row}(L)J_n$, so there exists a vector w such that $\left[\begin{smallmatrix} \beta v - \delta v' \\ \gamma v' - \alpha v \end{smallmatrix}\right] = \operatorname{col}(R)w$, that is, $R_1 w = \beta v - \delta v'$, $R_0 w = \gamma v' - \alpha v$.

Observe that $(\alpha R_1 + \beta R_0)w = (\beta\gamma - \alpha\delta)v' = (\alpha\delta - \beta\gamma)(\gamma R_1 + \delta R_0)w'$ for some $w' \in \mathcal{W}_{k-1}^{(\lambda)}$, by the inductive hypothesis. Thus, $w \in \mathcal{W}_k^{(\lambda)}$. Moreover, $(\gamma R_1 + \delta R_0)w = (\beta\gamma - \alpha\delta)v$. Therefore, $\mathcal{V}_k^{(\lambda)} \subseteq (\gamma R_1 + \delta R_0)\mathcal{W}_k^{(\lambda)}$, which concludes the proof. $\square$

One may wonder if the hypothesis $\alpha\delta \neq \beta\gamma$ is necessary in Theorem 4.13; the following example shows that it is indeed the case.

Example 4.14. Let $L(x) = \begin{bmatrix} 0 & x-1 \end{bmatrix}$ and $R(x) = \begin{bmatrix} x & 1 & 0 \\ 0 & 0 & x-1 \end{bmatrix}$, which are dual, and $\lambda = 1$.

The Wong chain for $R(x)$ is $\mathcal{W}_0^{(1)} = \{0\}$, $\mathcal{W}_1^{(1)} = \ker(R_1+R_0)$, which is the column space of the matrix $\begin{bmatrix} 0 & 1 \\ 0 & -1 \\ 1 & 0 \end{bmatrix}$, and $\mathbb{C}^3 = \mathcal{W}_2^{(1)} = \mathcal{W}_3^{(1)} = \dots$. On the other hand, the Wong chain for $L(x)$ is $\mathcal{V}_0^{(1)} = \{0\}$, $\mathbb{C}^2 = \mathcal{V}_1^{(1)} = \mathcal{V}_2^{(1)} = \dots$. Notice that $(R_1 + R_0)\mathcal{W}_1^{(1)} = \{0\} \neq \mathcal{V}_1^{(1)}$.

5 Structured pencils

The theoretical tools that we have presented can be used to derive new results on the minimal indices of some structured pencils. First of all, let us define the conjugate-transpose of matrix polynomials in the following sense (term-by-term): if $A(x) = \sum_{i=0}^{d} A_i x^i$, then we set $A^*(x) := \sum_{i=0}^{d} A_i^* x^i$.

We consider here the following structures. A pencil $S(x) = S_1 x + S_0 \in \mathbb{C}[x]_1^{2n\times 2n}$ is called *symplectic* if it is row-minimal and $S_0 J_n S_0^* = S_1 J_n S_1^*$. A pencil $H(x) = H_1 x + H_0 \in \mathbb{C}[x]_1^{2n\times 2n}$ is called *Hamiltonian* if it is row-minimal and $H_0 J_n H_1^* + H_1 J_n H_0^* = 0$.

These definitions generalize those given in [28] for regular pencils; however, the row-minimality condition is not present there, since it is automatically satisfied in the regular case. We discuss in Section 5.4 why it has been added here.

5.1 Symplectic pencils

As a first step, we pick out a particular right dual of a symplectic pencil.

Proposition 5.1. *Let $S(x) = S_1 x + S_0$ be a symplectic pencil. Then $T(x) = T_1 x + T_0 := J_n \operatorname{rev} S^*(x) = J_n S_0^* x + J_n S_1^*$ is a column-minimal right dual of $S(x)$.*

Proof. Symplecticity implies that $S_1 T_0 = S_0 T_1$, which is **D1**. Condition **D2** and column-minimality of $T(x)$ follow from row-minimality of $S(x)$, since

$$\operatorname{rank} \operatorname{col}(T) = \operatorname{rank} \begin{bmatrix} 0 & J_n \\ J_n & 0 \end{bmatrix} (\operatorname{row}(S))^* = \operatorname{rank} (\operatorname{row}(S))^* = 2n.$$

$\square$

We can use this result to determine a relation between the left and right minimal bases and indices of $S(x)$. Recall now the definition of a columnwise reversal (see Section 4) and Lemma 4.11.

Proposition 5.2. *Let $S(x) \in \mathbb{C}[x]_1^{2n\times 2n}$ be a symplectic pencil.*

1. *Let $N(x) \in \mathbb{C}[x]^{2n\times k}$ be a matrix polynomial such that the rows of $N^*(x)$ are a left minimal basis for $S(x)$. Then $J_n S_0^* \operatorname{cwRev} N(x)$ is a right minimal basis for $S(x)$.*

2. *Let $\nu_1, \nu_2, \ldots, \nu_s$ be the left minimal indices of $S(x)$. Then, $\nu_i \geq 1$ for each i, and the right minimal indices of $S(x)$ are $\nu_1 - 1, \nu_2 - 1, \ldots, \nu_s - 1$.*

Proof. 1. If the rows of $N^*(x)$ are a left minimal basis for $S(x)$, then the columns of $N(x)$ are a right minimal basis of $S^*(x)$. Therefore, $\operatorname{cwRev} N(x)$ is a right minimal basis for $\operatorname{rev} S^*(x)$ and, thus, for $T(x) = J_n \operatorname{rev} S^*(x)$. By Theorem 3.8, this implies that $J_n S_0^* \operatorname{cwRev} N(x)$ is a right minimal basis for $S(x)$.

2. First note that $\nu_i > 0$ for each i because we are taking $S(x)$ to be row-minimal. Let $\mu_1, \mu_2, \ldots, \mu_s$ be the right singular indices of $S(x)$. The pencil $S^*(x)$ has left minimal indices $\mu_1, \ldots, \mu_s$ and right minimal indices $\nu_1, \ldots, \nu_s$; since neither premultiplying by an invertible matrix nor applying rev changes the minimal indices, the same holds for $T(x)$. But $S(x)$ and $T(x)$ are a dual pair, hence by Theorem 3.3 the left minimal indices of $T(x)$ are equal to the left minimal indices of $S(x)$ decreased by 1 each, i.e., $\mu_s = \nu_s - 1$ (upon reordering the μ_i).

□

5.2 Hamiltonian pencils

We proceed with the same strategy for the Hamiltonian case.

Proposition 5.3. *Let $H(x) = H_1 x + H_0$ be Hamiltonian. Then $G(x) = G_1 x + G_0 := J_n H^*(-x) = -J_n H_1^* x + J_n H_0^*$ is a column-minimal right dual of $H(x)$.*

Proof. Hamiltonianity implies that $H_1 G_0 = H_0 G_1$, and

$$\operatorname{rank} \operatorname{col}(G) = \operatorname{rank} \begin{bmatrix} -J_n & 0 \\ 0 & J_n \end{bmatrix} (\operatorname{row}(H))^* = \operatorname{rank} (\operatorname{row}(H))^* = 2n.$$

□

This implies analogous results on the minimal bases and indices of Hamiltonian pencils. The proof is essentially the same.

Proposition 5.4. *Let $H(x) \in \mathbb{C}[x]_1^{2n\times 2n}$ be a Hamiltonian pencil.*

1. *Let $N(x) \in \mathbb{C}[x]^{2n\times k}$ be a matrix polynomial such that the rows of $N^*(x)$ are a left minimal basis for $H(x)$. Then $J_n H_1^* N(-x)$ is a right minimal basis for $H(x)$.*

2. *Let $\nu_1, \nu_2, \ldots, \nu_s$ be the left minimal indices of $H(x)$. Then, $\nu_i \geq 1$ for each i, and the right minimal indices of $H(x)$ are $\nu_1 - 1, \nu_2 - 1, \ldots, \nu_s - 1$.*

5.3 Explicit constructions

One may wonder if there are further restrictions on the possible minimal indices of symplectic and Hamiltonian pencils. We prove here that the answer is no: for any sequence of positive integers $\nu_1, \nu_2, \ldots, \nu_s$, there exist a symplectic and a Hamiltonian pencil with exactly the ν_i as left minimal indices (and hence, by Propositions 5.2 and 5.4, $\nu_1 - 1, \nu_2 - 1, \ldots, \nu_s - 1$ as right minimal indices). First of all, we need the following basic building block.

Let $\mathcal{S}_n$ be the $n \times n$ anticyclic up-shift matrix, i.e., the $n \times n$ matrix such that

$$(\mathcal{S}_n)_{ij} = \begin{cases} 1 & j - i = 1, \\ -1 & i = n, j = 1, \\ 0 & \text{otherwise.} \end{cases}$$

For each $n \geq 1$, the $2n \times 2n$ pencil

$$H(x) = \operatorname{diag}(-\mathcal{S}_{n+1} K_{n+1,n}(-x), K_{n-1,n}(x)) \tag{13}$$

is Hamiltonian and has a left minimal index n, a right minimal index $n-1$ and no regular part.

Example 5.5. The smallest such examples are

$$\begin{bmatrix} x & 0 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} x & 1 & 0 & 0 \\ 0 & x & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & x \end{bmatrix}.$$

Then, one needs a method to build direct sums of Hamiltonian pencils.

Lemma 5.6. *Let the pencils*

$$\begin{bmatrix} A^{(i)}(x) & B^{(i)}(x) \\ C^{(i)}(x) & D^{(i)}(x) \end{bmatrix}, \quad A^{(i)}, B^{(i)}, C^{(i)}, D^{(i)} \in \mathbb{C}[x]^{n_i \times n_i}$$

be Hamiltonian, for $i = 1, 2, \ldots, m$*. Let* $A(x) = \operatorname{diag}(A^{(1)}(x), A^{(2)}(x), \ldots, A^{(m)}(x))$*, and define* $B(x)$*,* $C(x)$*, and* $D(x)$ *analogously. Then,*

$$\begin{bmatrix} A(x) & B(x) \\ C(x) & D(x) \end{bmatrix}$$

is Hamiltonian.

By taking direct sums of blocks of the form (13), one can achieve all the possible combinations of minimal indices allowed by Proposition 5.4.

As for symplectic pencils, one can show that, given a Hamiltonian example $H(x) = H_1 x + H_0$, the pencil $S(x) = (H_1 - H_0)x + (H_1 + H_0)$ is symplectic and has the same minimal indices as $H(x)$ (for a proof of the last assertion, see for instance [35]).

5.4 Further remarks on symplectic and Hamiltonian pencils

The assumption that symplectic and Hamiltonian pencils must be row-minimal is not classical. The reason is that existing theory focuses on the regular case only [28], for which it is automatically satisfied.

If this assumption is relaxed, the results on the minimal indices do not hold anymore, in general. Indeed, consider the following examples.

Example 5.7. Let

$$L^{(1)}(x) = \left[\begin{smallmatrix} 0 & 0 \\ 1 & x \end{smallmatrix}\right], \qquad L^{(2)}(x) = \begin{bmatrix} 0&0&0&0\\0&0&0&0\\1&0&1&0\\x&0&x&0 \end{bmatrix}, \qquad L^{(3)}(x) = \begin{bmatrix} 1&0&0&0\\x&1&0&0\\0&x&0&0\\0&0&0&0 \end{bmatrix}.$$

Each one of these pencils is both symplectic and Hamiltonian. $L^{(1)}(x)$ has left minimal index 0 and right minimal index 1; $L^{(2)}$ has left minimal indices $\{0, 0, 1\}$ and right minimal indices $\{0, 0, 0\}$; $L^{(3)}(x)$ has left minimal indices $\{0, 2\}$ and right minimal indices $\{0, 0\}$.

We do not see a pattern that could lead to a similar result for this larger class of matrices.

Moreover, the recent work [33] presents a procedure to extract in a stable way a Hamiltonian pencil from the so-called extended pencils appearing in many control theory applications; in that setting, one always obtains row-minimal pencils, a condition that appears naturally in the development of the algorithm. In view of these observations, we consider the row-minimality requirement to be the most natural one in the study of singular structured pencils.

In principle, one might extend symplectic and Hamiltonian structure to matrix polynomials of higher grade. This can be done by imposing, respectively, the condition

$$P(x)J_n[P(x)]^* = \operatorname{rev}(P(x))J_n[\operatorname{rev}(P(x))]^* \quad \text{or} \quad P(x)J_n[P(x)]^* = P(-x)J_n[P(-x)]^*.$$

Characterizing the minimal indices of these structured matrix polynomials is an open problem that we leave for future research.

As a final note, we point out that all the results stated in this section continue to hold if we replace all the conjugate transposes with transposes.

6 Linearizations of matrix polynomials

In many applications, the study of spectral properties of matrix polynomials is a central topic [8, 21, 22, 24, 34, 38]. A common technique to find the eigenvalues of a matrix polynomial is converting to a linear problem using the following method. Given a matrix polynomial $A(x) \in \mathbb{C}[x]_d^{m\times n}$, a pencil $L(x) \in \mathbb{C}[x]_1^{(m+p)\times(n+p)}$ is called a *linearization* of $A(x)$ if there are $E(x) \in \mathbb{C}[x]^{(m+p)\times(m+p)}$, $F(x) \in \mathbb{C}[x]^{(n+p)\times(n+p)}$ such that

$$L(x) = E(x)\operatorname{diag}(A(x), I_p)F(x), \tag{14}$$

where $p \geq 0$ and $\det E(x), \det F(x)$ are nonzero constants. When $m = n$, the most natural (and common) choice is $p = n(d-1)$. Linearizations have the same finite elementary divisors (hence the same eigenvalues) as the starting matrix polynomial [22]. Not every linearization preserves the partial multiplicities of the eigenvalue ∞ [20, 27], however. Linearizations that do are called *strong linearizations*, and they satisfy the additional property that $\operatorname{rev} L(x)$ is a linearization for $\operatorname{rev} A(x)$.

Several different methods to construct linearizations of $A(x) = \sum_{i=0}^{d} A_i x^i$ have been studied; we recall some of the most common ones.

Companion forms [22, Chapter 1] The well-known *companion matrix* of a scalar polynomial generalizes easily to matrix polynomials. The pencil $C(x) := C_1 x + C_0$, where

$$C_1 = \operatorname{diag}(A_d, -I_{(d-1)n}), \quad C_0 = \begin{bmatrix} A_{d-1} & A_{d-2} & \cdots & A_0 \\ I_n & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & I_n & 0 \end{bmatrix} \tag{15}$$

is known as *first companion form*[1], while $\hat{C}(x) := \hat{C}_1 x + \hat{C}_0$ with

$$\hat{C}_1 = \operatorname{diag}(A_d, -I_{(d-1)m}), \quad \hat{C}_0 = \begin{bmatrix} A_{d-1} & I_m & \dots & 0 \\ A_{d-2} & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & I_m \\ A_0 & 0 & \dots & 0 \end{bmatrix} \tag{16}$$

as *second companion form*. Notice that $\hat{C}(x)$ is the block transpose of $C(x)$ in the square case.

Vector spaces of linearizations [29] A large family of linearizations for a square matrix polynomial $A(x) \in \mathbb{C}[x]_d^{n \times n}$ is found inside the vector space $\mathbb{L}_1(A)$ of pencils $L_1 x + L_0$ that satisfy

$$L_1 \begin{bmatrix} I_{dn} & 0_{dn \times n} \end{bmatrix} + L_0 \begin{bmatrix} 0_{dn \times n} & I_{dn} \end{bmatrix} = (v \otimes I_n) \operatorname{row}(A) \tag{17}$$

for some $v \in \mathbb{C}^d$, where $\otimes$ denotes the Kronecker product. The operation on the left-hand side is called *column-shifted sum* in [29]. A second vector space $\mathbb{L}_2(A)$ is defined as the set of the block transposes of all pencils $L(x) \in \mathbb{L}_1(A)$, or, equivalently, those which satisfy a similar relation, given by block-transposing everything in (17). Note that the first companion form $C(x)$ is in $\mathbb{L}_1(A)$ and the second companion form $\hat{C}(x)$ is in $\mathbb{L}_2(A)$.

The pencils in the intersection $\mathbb{DL}(A) := \mathbb{L}_1(A) \cap \mathbb{L}_2(A)$ have many interesting properties; for any regular matrix polynomial, almost all of them are strong linearizations.

[1] There is no agreement in the literature on the position of the minus signs in several special pencils, including (15) and (18) below. Our choice is not standard, but minimizes the number of minus signs of which we have to keep track along the computations.

Fiedler pencils [1] Define the matrices $F_0 := \operatorname{diag}(I_{(d-1)n}, A_0)$, $F_i := \operatorname{diag}(I_{(d-i-1)n}, G_i, I_{n(i-1)})$, for $i = 1, 2, \ldots, d-1$, where

$$G_i := \begin{bmatrix} A_i & I_n \\ I_n & 0_{n\times n} \end{bmatrix}, \tag{18}$$

and $F_d := C_1$.

For each permutation σ of $(0, 1, \ldots, d-1)$, the pencil $F_d x + \prod_{i=0}^{d-1} F_{\sigma(i)}$ is a linearization; in particular, the two permutations $(d-1, d-2, \ldots, 0)$ and $(0, 1, 2, \ldots, d-1)$ yield the two companion forms. Another interesting special case is the block pentadiagonal pencil which corresponds to the permutation $(0, 2, \ldots, 1, 3, \ldots)$. It is remarkable that, after expanding the products, the constant terms of all Fiedler linearizations can be written explicitly by using only 0, I and $A_0, A_1, \ldots A_{d-1}$ as blocks. Several additional generalizations of Fiedler pencils exist [1, 39].

The following result characterizes the Kronecker form of all strong linearizations [12, 15].

Theorem 6.1. *Let $A(x)$ be a matrix polynomial. A pencil $L(x)$ is a strong linearization of $A(x)$ if and only if:*

1. *The eigenvalues of $L(x)$ and of $A(x)$ coincide.*
2. *For each eigenvalue $\lambda \in \mathbb{C} \cup \{\infty\}$, the sizes of the regular blocks of eigenvalue λ in the KCF of $L(x)$ coincide with the partial multiplicities associated to λ in $A(x)$.*
3. *The numbers of right and left singular blocks in the KCF of $L(x)$ are equal to $\dim \ker_{\mathbb{C}(x)} A(x)$ and $\dim \ker_{\mathbb{C}(x)} A(x)^*$, respectively.*

Note that there is no constraint on the minimal indices, i.e., the sizes of the singular left and right blocks of $L(x)$. Those may indeed vary for different linearizations. This behaviour fits well in our framework, since duality can be used to ensure that the needed quantities are preserved. In particular, the following result can be established.

Theorem 6.2. *Let $A(x)$ be a matrix polynomial and $L(x)$ be a strong linearization for $A(x)$. Then,*

1. *If $L(x)$ is row-minimal, then any column-minimal right dual $R(x)$ of $L(x)$ is also a strong linearization of $A(x)$.*
2. *If $L(x)$ is column-minimal, then any row-minimal left dual $F(x)$ of $L(x)$ is also a strong linearization of $A(x)$.*
3. *If $R(x)$ is a right dual of $L(x)$, and $\widetilde{L}(x)$ is another left dual of $R(x)$ with the same number of rows as $L(x)$, then $\widetilde{L}(x)$ is also a strong linearization of $A(x)$.*
4. *If $F(x)$ is a left dual of $L(x)$, and $\widetilde{L}(x)$ is another right dual of $F(x)$ with the same number of columns as $L(x)$, then $\widetilde{L}(x)$ is also a strong linearization of $A(x)$.*

Proof. 1. By Theorem 3.3, the eigenvalues, regular block sizes, and number of singular blocks in the KCFs of $L(x)$ and $R(x)$ are the same; indeed, the only possibilities for them to change would be additional $K_{1,0}(x)$ blocks in $L(x)$ or $K_{0,1}(x)$ ones in $R(x)$, but they are ruled out by the minimality assumptions. Hence, if $L(x)$ satisfies the conditions in Theorem 6.1, then so does $R(x)$.

3. By Theorem 3.3, the eigenvalues, regular block sizes, and number of right singular blocks in the KCFs of $L(x)$, $\widetilde{L}(x)$ and $R(x)$ are the same, apart from (possibly) additional $K_{1,0}(x)$ blocks in $L(x)$ or $\widetilde{L}(x)$ and additional $K_{0,1}(x)$ blocks in $R(x)$. Since $L(x)$ and $\widetilde{L}(x)$ have the same number of rows, they must have the same number of these additional $K_{1,0}(x)$ blocks. Hence, if $L(x)$ satisfies the conditions in Theorem 6.1, then so does $\widetilde{L}(x)$.

2.,4. They are analogous to 1. and 3.

□

7 Fiedler linearizations as duals

In this section, we use the theory developed above to revisit some of the known results for Fiedler pencils. We focus on square matrix polynomials only; this leaves out the more involved rectangular case, treated in [14].

We first need some results on the first companion form, whose proof is readily obtained with a few algebraic manipulations (see also [12, Lemma 5.1]).

Lemma 7.1. *1. The first companion form $C(x)$ is a strong linearization for any (regular or singular) $A(x)$.*

2. If $A(x)$ is regular and v is an eigenvector of $A(x)$ with eigenvalue $\lambda = \frac{\alpha}{\beta}$, then

$$\begin{bmatrix}\alpha^{d-1}I & \alpha^{d-2}\beta^1 I & \cdots & \alpha\beta^{d-2}I & \beta^{d-1}I\end{bmatrix}^{\mathcal{B}} v \tag{19}$$

is a right eigenvector of $C(x)$ with eigenvalue λ. If $\lambda \neq \infty$, we can use the slightly simpler formula

$$\begin{bmatrix}\lambda^{d-1}I & \lambda^{d-2}I & \cdots & \lambda I & I\end{bmatrix}^{\mathcal{B}} v. \tag{20}$$

3. If $M(x)$ is a right minimal basis for $A(x)$, then

$$\begin{bmatrix}x^{d-1}I & x^{d-2}I & \cdots & xI & I\end{bmatrix}^{\mathcal{B}} M(x)$$

is a right minimal basis for $C(x)$. In particular, the right minimal indices of $C(x)$ are obtained increasing by $d-1$ the right minimal indices of $A(x)$.

Another useful definition is the following [13]: given a permutation σ of $(0, 1, \dots, d-1)$, we say that σ has a *consecution* at i if $\sigma(i) < \sigma(i+1)$, and an *inversion* at i otherwise.

As a first result, we give a new proof of the fact that all square Fiedler pencils are linearizations. Our argument follows closely the original proof of [1] for regular pencils, apart from some minor differences in notation. Indeed, [1, Lemma 2.2] is a very special case of our Theorem 3.3, but the argument there works only for regular pencils. The fact that Fiedler pencils are linearizations even in the singular case was first proved in [13], six years after the regular case and with a completely different technique based on keeping track of a large number of unimodular transformations. Duality allows us to reuse almost verbatim the proof for the regular case, instead.

Theorem 7.2. *For a (possibly singular) matrix polynomial $A(x) \in \mathbb{C}[x]_d^{n\times n}$, all Fiedler pencils are strong linearizations.*

Proof. For any $j > i$, we set $F_{j:i} = F_{j-1}F_{j-2}\cdots F_i$ for short. Since F_i and F_j commute for any i, j with $|i - j| > 1$, we can always rearrange the product $\prod_{i=0}^{d-1} F_{\sigma(i)}$ in the form

$$F_{c_1:0}F_{c_2:c_1}F_{c_3:c_2}\cdots F_{c_\Gamma:c_{\Gamma-1}}F_{d:c_\Gamma}, \tag{21}$$

for a suitable sequence $0 = c_0 < c_1 < \cdots < c_\Gamma < d$, with the only operation of changing the order of pairs of commuting matrices. One can see that $c_1, \ldots, c_\Gamma$ are exactly the indices i such that σ has an inversion at $i - 1$.

We prove the following result by induction on Γ: all Fiedler pencils with Γ consecutions are strong linearizations for $A(x)$, and each of their right singular indices is greater than or equal to $d - 1 - \Gamma$.

If $\Gamma = 0$, then we have $F_dx + F_{d-1}F_{d-2}\ldots F_0 = C(x)$, the first companion form, so the result follows from Lemma 7.1.

Now, assuming that we have proved the result for a sequence $c_2, c_3, \ldots, c_\Gamma$, with $\Gamma < d - 1$, we prepend an extra element c_1 and prove it for the sequence $c_1, c_2, c_3, \ldots, c_\Gamma$. Let $P = F_{c_1:0}$ and $Q = F_{c_2:c_1}F_{c_3:c_2}F_{c_4:c_3}\cdots F_{d:c_\Gamma}$; the latter is nonsingular since all F_i for $i \notin \{0, d\}$ are nonsingular. Note that P commutes with all terms in Q apart from $F_{c_2:c_1}$. Since $F_{c_2:c_1}F_{c_1:0} = F_{c_2:0}$, the Fiedler pencil $F_dx + QP$ is the one associated to $c_2, c_3, \ldots, c_\Gamma$, which is a strong linearization by the inductive hypothesis. Moreover, also from the inductive hypothesis, it is column-minimal, since all its right minimal indices are greater than 0.

We premultiply this pencil by the nonsingular matrix Q^{-1}, to obtain $R(x) := Q^{-1}F_dx + P$, which is still a column-minimal strong linearization for $A(x)$. Now we claim that $L(x) = F_dx + PQ$ is a row-minimal left dual of $R(x)$. Condition **D1** is verified since F_d and P commute, so we only need to check that $\operatorname{rank}\operatorname{row}(L) = dn$. Due to the structure of F_d, for this to hold it is enough to prove that PQ has a $n \times n$ identity block somewhere in its first block row. Due to the structure of the involved matrices, $F_{c_1:0}F_{c_2:c_1}F_{c_3:c_2}\cdots F_{c_\Gamma:c_{\Gamma-1}}$ has $\begin{bmatrix} I & 0 & \ldots & 0\end{bmatrix}$ as its first block row, while $F_{d:c_\Gamma}$ has an identity in the block in position $(1, d - c_\Gamma + 1)$.

We have proved that $L(x)$ is a row-minimal left dual of the column-minimal $R(x)$, hence by Theorem 6.2 it is a strong linearization of $A(x)$, too. □

Corollary 7.3 ([13])**.** *Let $r_1, r_2, \ldots, r_h$ be the right minimal indices of a matrix polynomial $A(x) \in \mathbb{C}[x]_d^{n\times n}$, and consider a Fiedler pencil $F(x)$ with Γ consecutions associated* *with $A(x)$. Then, the right minimal indices of $F(x)$ are $r_1 + (d-1-\Gamma), r_2 + (d-1-\Gamma), \ldots, r_h + (d-1-\Gamma)$.*

Proof. In the proof of Theorem 7.2, to construct $F(x)$ we start from $C(x)$, which has right minimal indices $r_1 + d - 1, r_2 + d - 1, \ldots, r_h + d - 1$, and obtain $F(x)$ after taking a left dual $\Gamma \leq d-1$ times. Each of these times the right minimal indices are decreased by 1, hence by keeping track of their values we get the above result. $\square$

Note that $d - 1 - \Gamma$ is the number of inversions of σ.

We can expand on the proof of Theorem 7.2 to find a minimal basis for each Fiedler pencil explicitly.

Theorem 7.4. *Consider the Fiedler pencil $F(x)$ associated to a given permutation with consecutions $c_1, c_2, \ldots, c_\Gamma$, and let $T = F_{d:c_\Gamma}^{-1} F_d F_{d:c_{\Gamma-1}}^{-1} F_d \cdots F_{d:c_1}^{-1} F_d$. If $M_C(x)$ is a right minimal basis for the first companion form $C(x)$, then $M_F(x) = T M_C(x)$ is a right minimal basis for $F(x)$.*

Proof. Our plan is following the proof of Theorem 7.2 and showing how right minimal bases change along the needed duality operations. The result is obvious if $\Gamma = 0$; now, let us suppose that it holds for a permutation with consecutions $c_2 < c_3 < \cdots < c_\Gamma$ and prove it for the same sequence with an additional consecution $c_1 < c_2 < \cdots < c_\Gamma$. Applying Theorem 3.8 with $(\gamma, \delta) = (1, 0)$, we obtain that a right minimal basis for the new permutation is

$$M_F(x) = (Q^{-1} F_d) F_{d:c_\Gamma}^{-1} F_d F_{d:c_{\Gamma-1}}^{-1} F_d \cdots F_{d:c_2}^{-1} F_d M_C(x),$$

with $Q = F_{c_2:c_1} F_{c_3:c_2} F_{c_4:c_3} \cdots F_{d:c_\Gamma}$ as in the proof of Theorem 7.2. As the Fiedler pencils that we use in duality operations are column-minimal, there are no spurious zero columns in $M_F(x)$. Since each term $F_{c_i:c_{i-1}}^{-1}$ appearing in Q^{-1} commutes with F_d and with all the terms $F_{d:c_j}^{-1}$ for $j > i$, we can reorder the factors to obtain

$$\begin{aligned} M_F(x) &= F_{d:c_\Gamma}^{-1} F_d (F_{c_\Gamma:c_{\Gamma-1}}^{-1} F_{d:c_\Gamma}^{-1}) F_d (F_{c_{\Gamma-1}:c_{\Gamma-2}}^{-1} F_{d:c_{\Gamma-1}}^{-1}) F_d \cdots (F_{c_2:c_1}^{-1} F_{d:c_2}^{-1}) F_d M_C(x) \\ &= F_{d:c_\Gamma}^{-1} F_d F_{d:c_{\Gamma-1}}^{-1} F_d F_{d:c_{\Gamma-2}}^{-1} F_d \cdots F_{d:c_1}^{-1} F_d M_C(x) = T. \end{aligned}$$

$\square$

By combining the result in Theorem 7.4 with Lemma 7.1(3), we get the following corollary.

Corollary 7.5. *If $M_A(x)$ is a minimal basis for $A(x)$, then a right minimal basis for $F(x)$ is $M_F(x) = T(x) M_A(x)$, where*

$$T(x) = T \begin{bmatrix} x^{d-1} I & x^{d-2} I & \cdots & xI & I \end{bmatrix}^{\mathcal{B}}.$$

Using the fact that

$$F_{d:c_i}^{-1} F_d = \operatorname{diag}\left(\begin{bmatrix} 0 & 0 & \cdots & A_d \\ -I & 0 & \cdots & A_{d-1} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & -I & A_{c_i} \end{bmatrix}, I \right),$$

one can compute the product one factor at a time, starting from the right, and verify that this means that the i-th block of $(-1)^{\Gamma}T(x)$ contains $-x^{\mathcal{I}_i}(A_d x^{i-1} + A_{d-1}x^{i-2} + \cdots + A_{d-i+1})$ if i is a consecution and $x^{\mathcal{I}_i}I$ if i is an inversion, where $\mathcal{I}_i$ is the number of inversions j with $j \leq i$ — see for instance Example 7.9 below. This is in agreement with the equivalent expression found in [13], where the polynomials $P_i(x) = A_d x^{i-1} + A_{d-1}x^{i-2} + \cdots + A_{d-i+1}$ are called *Horner shifts* of $A(x)$.

A similar result holds for Wong chains.

Theorem 7.6. *Let $(\mathcal{W}_i^{(\lambda)})_i$ be the Wong chain attached to $\lambda \neq \infty$ for the first companion form $C(x)$, and $(\mathcal{V}_i^{(\lambda)})_i$ be the Wong chain for a Fiedler pencil $F(x)$ attached to the same eigenvalue λ. Then, $\mathcal{V}_i^{(\lambda)} = T\mathcal{W}_i^{(\lambda)}$, with T defined as in Theorem 7.4.*

Proof. Once again, we follow the proof of Theorem 7.2 and show how the Wong chains change along the duality operations. Since we suppose $\lambda \neq \infty$, we can use $(\gamma, \delta) = (1, 0)$ in Theorem 4.13, so the Wong chains are multiplied by $Q^{-1}F_d$ at each duality, exactly like minimal bases. The same argument as in the proof of Theorem 7.4 applies, and yields the same matrix T. □

For a regular pencil, $\mathcal{W}_1^{(\lambda)}$ is the subspace of all eigenvectors with eigenvalue λ. Hence, using (20), we get the following result.

Corollary 7.7. *If v is an eigenvector of eigenvalue $\lambda \neq \infty$ for a regular matrix polynomial $A(x)$, then the corresponding eigenvector of $F(x)$ is $T(\lambda)v$.*

Once again, this agrees with the expressions in [13]. Our results on Wong chains, however, are more general and can be applied to defective eigenvalues and singular pencils as well.

With another choice of (γ, δ), we can obtain results with an excluded eigenvalue at 0 rather than ∞.

Theorem 7.8. *Let $(\mathcal{W}_i^{(\lambda)})_i$ be the Wong chain attached to $\lambda \neq 0$ for the first companion form $C(x)$, and $(\mathcal{V}_i^{(\lambda)})_i$ be the Wong chain for a Fiedler pencil $F(x)$ attached to the same eigenvalue λ. Let $\widetilde{T} := F_{c_1:0}F_{c_2:0}\dots F_{c_\Gamma:0}$. Then, $\mathcal{V}_i^{(\lambda)} = \widetilde{T}\mathcal{W}_i^{(\lambda)}$.*

Proof. We use $(\gamma, \delta) = (0, 1)$, and proceed as in the previous cases. At each step, we premultiply $F_{c_2:0}\dots F_{c_\Gamma:0}$ by $P = F_{c_1:0}$, so the result is even easier to obtain. □

Example 7.9. Consider the Fiedler linearization $F_6x + F_0F_1F_3F_5F_2F_4$ [13, Examples 3.5 and subsequent ones]. It has three consecutions in $c_1 = 1, c_2 = 2, c_3 = 4$. Therefore, its expression as in (21) is $F_6x + F_0F_1F_3F_2F_5F_4 = F_6x + F_{1:0}F_{2:1}F_{4:2}F_{6:4}$. By Corollary 7.5, a right minimal basis for it is given by $T(x)M_A(x)$, where $M_A(x)$ is a minimal basis for $A(x)$ and

$$T(x) = \begin{bmatrix} x^2I & xI & -xP_2(x) & I & -P_4(x) & -P_5(x) \end{bmatrix}^{\mathcal{B}}. \tag{22}$$

Similarly, all Wong chains with $\lambda \neq \infty$ are obtained from those of $A(x)$ by left multiplication by $T(\lambda)$. Obtaining an analogous result that includes $\lambda = \infty$ with Theorem 7.8 is not more complicated, but we have to move to the projective version with formula (19). It turns out that all eigenvectors with $\lambda = \frac{\alpha}{\beta} \neq 0$ are recovered by left multiplication by $\widetilde{T}(\alpha,\beta)$, with

$$\widetilde{T}(\alpha,\beta) = \begin{bmatrix} \alpha^5 I & \alpha^4\beta I & \widetilde{P}_2(\alpha,\beta) & \alpha^3\beta^2 I & \widetilde{P}_4(\alpha,\beta) & \widetilde{P}_5(\alpha,\beta) \end{bmatrix}.$$

and $\widetilde{P}_i(\alpha,\beta) = \sum_{j=0}^{d-1-i} A_i \alpha^i \beta^{d-1-i}$.

One can obtain a completely analogous set of results for the left eigenvectors and minimal bases, by starting with the second companion form and performing repeatedly right dualities, one for each inversion in the associated permutation.

8 Duality and the $\mathbb{L}_1(A)$, $\mathbb{L}_2(A)$ linearization spaces

Let us consider a square $n \times n$ matrix polynomial $A(x)$ of grade d. In this section we study the space $\mathbb{L}_1(A)$ and its connection with duality. An important pencil belonging to $\mathbb{L}_1(A)$ is the first companion form $C(x)$, defined as in (15). Throughout this section, we set $\mu = n - \operatorname{rank}\operatorname{row}(A)$, and let $B \in \mathbb{C}^{(dn+n)\times(dn+\mu)}$ be any matrix whose columns form a basis of $\ker \operatorname{row}(A)$.

The following result holds (see also [26, Section 1.4.3]).

Proposition 8.1. *Let* $M(x) = \begin{bmatrix} 0_{dn\times n} & -I_{dn} \end{bmatrix} x + \begin{bmatrix} I_{dn} & 0_{dn\times n} \end{bmatrix}$. *The* $dn \times (dn+\mu)$ *pencil* $D(x) = M(x)B$ *is a column-minimal right dual of* $C(x)$.

Proof. It is straightforward to check that the structure of the first companion form gives $\operatorname{rank}\operatorname{row}(C) = dn - \mu$. The matrix $\operatorname{col}(M)$ has full column rank, implying $\operatorname{rank}\operatorname{col}(D) = \operatorname{rank} B = dn + \mu$, hence **D2** holds; it remains to check **D1**:

$$\operatorname{row}(C) J_{dn} \operatorname{col}(D) = \operatorname{row}(C) J_{dn} \operatorname{col}(M) B = \begin{bmatrix} \operatorname{row}(A) \\ 0 \end{bmatrix} B = 0.$$

□

Using the first part of Theorem 6.2, we get the following corollary.

Corollary 8.2. *If* $A(x)$ *is row-minimal, then* $D(x)$ *is a strong linearization for it.*

We now give a sufficient condition for a pencil belonging to $\mathbb{L}_1(A)$ to be a strong linearization.

Theorem 8.3. *Let* $L(x) \in \mathbb{L}_1(A)$ *be such that* $\operatorname{rank}\operatorname{row}(L) = dn - \mu$. *Then,* $L(x)$ *is a left dual of* $D(x)$ *and a strong linearization of* $A(x)$.

Proof. By definition, $L(x)$ must satisfy (17), which can be rewritten as $\operatorname{row}(L) J_{dn} \operatorname{col}(M) = (v \otimes I_n) \operatorname{row}(A)$, using the notation of Proposition 8.1. It follows immediately that $\operatorname{row}(L) J_{dn} \operatorname{col}(D) = \operatorname{row}(L) J_{dn} \operatorname{col}(M) B = 0$, which is **D1**. The rank condition in the hypothesis yields **D2**, so $L(x)$ is a left dual of $D(x)$. The second statement follows from Part 3 of Theorem 6.2. □

A stronger condition for a pencil in $\mathbb{L}_1(A)$ to be a strong linearization, called *having full Z-rank*, appears in [12, Definition 4.3]. The following inclusions hold for pencils in $\mathbb{L}_1(A)$:

$$\{\text{pencils with full Z-rank}\} \subset \{\text{left duals of } D(x)\} \subset \{\text{strong linearizations of } A(x)\}. \tag{23}$$

Both inclusions are strict: the pencil in [12, Example 2] is an example of left dual of $D(x)$ which has not full Z-rank, and the following example shows that the second inclusion is strict as well.

Example 8.4. Consider the matrix polynomial $A(x) = \begin{bmatrix} 1 & 0 & 0 \\ x & 0 & 0 \\ x^2 & 0 & 0 \end{bmatrix} =: A_0 + A_1 x + A_2 x^2$. $A(x)$ has no elementary divisors, its left minimal indices are $1, 1$, and its right minimal indices are $0, 0$.

The 6×6 pencil

$$L(x) = \begin{bmatrix} A_2 x - A_1 & 2A_1 x + A_0 \\ H & -Hx \end{bmatrix}, \quad \text{with } H = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

belongs to $\mathbb{L}_1(A)$. It is a strong linearization of $A(x)$ by Theorem 6.1: indeed, it has no elementary divisors and the dimension of its left and right kernels are 2.

The first companion form of $A(x)$ has left minimal indices $\{1, 1\}$; therefore, by Theorem 3.3, $D(x)$ has left minimal indices $\{0, 0\}$, and all its left duals have left minimal indices $\{1, 1\}$, together with possibly some extra zeros. However, $L(x)$ has left minimal indices $\{0, 2\}$, thus it cannot be a left dual of $D(x)$.

In the case of a regular $A(x)$, all the inclusions in (23) become equalities:

Theorem 8.5. *Let $A(x) \in \mathbb{C}[x]_d^{n\times n}$ be a regular matrix polynomial, and $L(x) \in \mathbb{L}_1(A)$. Then, the following are equivalent:*

1. *$L(x)$ is a strong linearization of $A(x)$,*
2. *$L(x)$ has full Z-rank,*
3. *$L(x)$ is regular,*
4. *$L(x)$ is a left dual of $D(x)$,*
5. *$L(x)$ is row-minimal.*

The first three equivalent conditions of the theorem appear already in [29]; here we add (4) and (5). The last condition (5) seems the simplest one to check in practice.

Proof. As stated above, the equivalence between (1), (2) and (3) is proved in [29]. The arrows (5) $\implies$ (4) $\implies$ (1) are given by Theorem 8.3. The implication (3) $\implies$ (5) is clear: every regular pencil is row-minimal. □

Analogous results concerning the link between $\mathbb{L}_2(A)$ and the right duals of the left duals of the second companion form can be easily obtained by adapting the arguments used in this section.

9 Linearizations and Möller-Stetter theorem

The linearization $D(x)$ described in Section 8 can be introduced in an alternative way as the generalization of a construction that is used in commutative computational algebra to find the solution to (scalar) polynomial systems. While the theory developed so far is self-consistent, it is still interesting to investigate the link between the dual linearization $D(x)$ and the Möller-Stetter theorem in algebraic geometry.

In commutative algebra, a polynomial ideal $\mathcal{I} = (f_1, \dots, f_k) \subseteq \mathbb{C}[x_1, \dots, x_m]$ is called *zero-dimensional* if the system

$$\begin{cases} f_1(x_1, \dots, x_m) = 0 \\ \qquad\vdots \\ f_k(x_1, \dots, x_m) = 0 \end{cases}$$

has only a finite number of solutions, or, equivalently [10, Finiteness Theorem, Section 2.5], if the quotient space $\mathbb{C}[x_1, \dots, x_m]/\mathcal{I}$ is finite-dimensional. The elements of the quotient space are usually denoted by the notation $[p]_{\mathcal{I}} := \{q \in \mathbb{C}[x_1, \dots, x_m] \mid q = p + r,\ r \in \mathcal{I}\}$.

When this holds, we have the following result, which we present for simplicity in the case of univariate polynomials ($m = 1$; see [10, Section 2.4] for the most general version).

Theorem 9.1 (Möller–Stetter)**.** *Let $f, g \in \mathbb{C}[x]$, and denote by $\mathcal{I}$ the (principal) ideal generated by f. Consider the linear multiplication map $\mathcal{M}_g : \mathbb{C}[x]/\mathcal{I} \to \mathbb{C}[x]/\mathcal{I}$ defined as $[p]_{\mathcal{I}} \mapsto [pg]_{\mathcal{I}}$. When a basis of $\mathbb{C}[x]/\mathcal{I}$ is chosen, this map is represented by a matrix M_g. Its eigenvalues (counted with multiplicity) are the values of $g(x_i)$, where x_i are the solutions (counted with multiplicity) x_i of the polynomial equation $f = 0$.*

Even though this stronger condition is usually not of interest in the commutative algebra applications, it is possible to prove that M_x is a linearization of the single polynomial equation $f = 0$ that generates the ideal (which is essentially unique, as $\mathbb{C}[x]$ is a principal ideal domain).

Proposition 9.2. *Let f be a nonzero polynomial in $\mathbb{C}[x]$. Then $M_x - Ix$, where M_x is the matrix that represents the multiplication map $\mathcal{M}_x$ defined in Theorem 9.1, is a strong linearization of f.*

Proof. Clearly, $\mathbb{C}[x]/\mathcal{I}$ is a vector space of dimension $\deg f$ over $\mathbb{C}$. If the monomial basis $(1, x, \dots, x^{\deg f - 1})$ is chosen, then the multiplication operator is represented by the companion matrix of f. Then the result follows by the properties of the companion linearization. Any other basis can be reconducted to the monomial basis by a change of basis. $\square$

We aim to generalize this result to matrix polynomials in order to produce linearizations. Let us consider the space of all row vector polynomials of grade d

$$\mathcal{W} := \mathbb{C}[x]_d^{1\times n} = \left\{ \sum_{i=0}^{d} v_i x^i \mid v_i \in \mathbb{C}^{1\times n} \text{ for all } i = 0, 1, \dots, d \right\}.$$

This space is isomorphic to $\mathbb{C}^{(d+1)n}$, via $\mathcal{R} : v(x) \mapsto \operatorname{row}(v)$. For any row-minimal grade-d matrix polynomial $A(x) \in \mathbb{C}[x]_d^{n\times n}$, the rows of the matrix $\operatorname{row}(A)$ span an n-dimensional subspace $\mathcal{A}$ of $\mathbb{C}^{(d+1)n}$. We consider its image under $\mathcal{R}^{-1}$,

$$\mathcal{I}_A := \mathcal{R}^{-1}(\mathcal{A}) = \{r^T A(x) \mid r \in \mathbb{C}^n\} \subseteq \mathcal{W}.$$

The notation $\mathcal{I}_A$ suggests that it will play the role of the "ideal generated by $A(x)$" in our generalization of the Möller-Stetter theorem. More formally, consider the quotient space $\mathcal{Q} := \mathcal{W}/\mathcal{I}_A$. The elements of $\mathcal{Q}$ are the equivalence classes $[w(x)]_{\mathcal{I}_A} := \{a(x) \in \mathcal{W} \mid a(x) = w(x) + r(x),\ r(x) \in \mathcal{I}_A\}$. Acting with $\mathcal{R}$ we immediately obtain the corresponding quotient space $\mathcal{R}(\mathcal{Q})$, i.e., the set of the equivalence classes defined as $[\operatorname{row}(w(x))]_{\mathcal{A}} := \{a \in \mathbb{C}^{(d+1)n} \mid a^T = \operatorname{row}(w(x)) + r^T \operatorname{row}(A),\ r \in \mathbb{C}^n\}$.

We can prove the following result, which can be seen as a generalization of Theorem 9.1 (for $g(x) = x$) to matrix polynomials.

Theorem 9.3. *Let $A(x) \in \mathbb{C}[x]_d^{n\times n}$ be a row-minimal matrix polynomial of grade d. Let $\mathcal{V} := \mathbb{C}[x]_{d-1}^{1\times n}$, $\mathcal{W} := \mathbb{C}[x]_d^{1\times n}$, and $\mathcal{Q} = \mathcal{W}/\mathcal{I}_A$. Let $\overline{\mathcal{M}}_x$ and $\overline{\mathcal{M}}_1$ be the maps "multiplication by x" and "multiplication by 1" between the spaces $\mathcal{V}$ and $\mathcal{Q}$, i.e.,*

$$\begin{aligned} \overline{\mathcal{M}}_x : \mathcal{V} \to \mathcal{Q} \quad & v(x) \mapsto [xv(x)]_{\mathcal{I}_A}, \\ \overline{\mathcal{M}}_1 : \mathcal{V} \to \mathcal{Q} \quad & v(x) \mapsto [v(x)]_{\mathcal{I}_A}. \end{aligned}$$

When bases of $\mathcal{V}$ and $\mathcal{Q}$ are chosen, these maps are represented by matrices $\overline{M}_x$, $\overline{M}_1$. The pencil $\overline{M}_x - \overline{M}_1 x$ is a strong linearization of $A(x)$.

Proof. It is sufficient to prove the result for a specific choice of the bases; we shall prove that a suitable choice produces the strong linearization $D(x) = M(x)B$ defined in Proposition 8.1, which is a strong linearization by Corollary 8.2.

The maps $\overline{\mathcal{M}}_x$ and $\overline{\mathcal{M}}_1$ can be written as $\pi \circ \mathcal{M}_x$ and $\pi \circ \mathcal{M}_1$, where

$$\begin{aligned} \mathcal{M}_x : V \to W \quad & v(x) \mapsto xv(x), \\ \mathcal{M}_1 : V \to W \quad & v(x) \mapsto v(x), \end{aligned}$$

and $\pi : v(x) \mapsto [v(x)]_{\mathcal{I}_A}$ is the projection onto the quotient space $\mathcal{Q}$. We use as bases of $\mathcal{V}$ and $\mathcal{W}$ the images of the canonical bases of $\mathbb{C}^{dn}$ and $\mathbb{C}^{(d+1)n}$ via the isomorphism $\mathcal{R}$; in these bases, $\mathcal{M}_x$ and $\mathcal{M}_1$ are represented by right multiplication of row vectors by $\begin{bmatrix} I_{dn} & 0_{dn\times n} \end{bmatrix}$ and $\begin{bmatrix} 0_{dn\times n} & I_{dn} \end{bmatrix}$, respectively.

Thanks to the definition of B, the map $v \mapsto vB$ (i.e., the action of B on row vectors) has kernel equal to $\mathcal{A}$. Hence, by the first isomorphism theorem, its image is isomorphic to

$\mathbb{C}^{(d+1)n}/\mathcal{A}$, which is itself isomorphic via $\mathcal{R}$ to $\mathcal{W}/\mathcal{I}_A = \mathcal{Q}$. Therefore, the map $v \mapsto vB$ passes to the quotient and becomes a projection onto $\mathcal{Q}$.

Composing maps, we obtain that $\overline{\mathcal{M}}_x$ and $\overline{\mathcal{M}}_1$ are represented by $\overline{M}_x = \begin{bmatrix} I_{dn} & 0_{dn\times n} \end{bmatrix} B$ and $\overline{M}_1 = \begin{bmatrix} 0_{dn\times n} & I_{dn} \end{bmatrix} B$, respectively. So $\overline{M}_x - \overline{M}_1 x = D(x)$. □

10 Constructing duals

The relation $L_1 R_0 = L_0 R_1$ has been studied extensively for regular pencils in the context of pencil arithmetic and inverse-free matrix iterative algorithms [2, 3, 9, 16, 31]. Two main techniques exist for constructing L_0, L_1 starting from R_0, R_1 (or vice versa).

QR factorization [26, Section 1.5.4.7], [2, 3, 31] Construct the QR factorization

$$\operatorname{col}(R) = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix} \begin{bmatrix} U \\ 0 \end{bmatrix},$$

and take $L_0 = Q_{12}^*$, $L_1 = -Q_{22}^*$. In practice, a QR factorization with column pivoting [23, Section 5.5.6] should be used, since $\operatorname{col}(R)$ being close-to-rank-deficient could be a concern here.

Enforcing an identity block [9, 32] Suppose that the identity matrix is a submatrix of $\operatorname{col}(R)$, for a pencil $R \in \mathbb{C}[x]_1^{n\times p}$. Then, we can select a permutation matrix $\Pi \in \mathbb{C}^{2n\times 2n}$ and $X \in \mathbb{C}^{(2n-p)\times p}$ such that

$$\operatorname{col}(R) = \Pi \begin{bmatrix} I_p \\ X \end{bmatrix}.$$

Then, the identity

$$0 = \left(\begin{bmatrix} -X & I_{2n-p} \end{bmatrix} \Pi^T \right) \Pi \begin{bmatrix} I_p \\ X \end{bmatrix}$$

holds, and thus we can choose

$$\begin{bmatrix} L_0 & -L_1 \end{bmatrix} = \begin{bmatrix} -X & I_{2n-p} \end{bmatrix} \Pi^T.$$

Slightly generalizing, if a $p \times p$ submatrix Y of $\operatorname{col}(R)$ is known to be nonsingular, we have

$$\operatorname{col}(R) = \Pi \begin{bmatrix} Y \\ Z \end{bmatrix} = \Pi \begin{bmatrix} I \\ ZY^{-1} \end{bmatrix} Y$$

and thus

$$\begin{bmatrix} L_0 & -L_1 \end{bmatrix} = \begin{bmatrix} -ZY^{-1} & I \end{bmatrix} \Pi^T.$$

Example 10.1. Let us start from the first companion form of a square matrix polynomial $A(x)$ with grade $d = 3$, for which

$$\operatorname{row}(C) = \begin{bmatrix} A_3 & & \\ & -I & \\ & & -I \\ A_2 & A_1 & A_0 \\ I & & \\ & I & \end{bmatrix}.$$

The $3n \times 3n$ matrix formed by the block rows number 2, 3 and 5 is nonsingular. Therefore, we choose a permutation Π that rearranges the block in the new order $(5, 2, 3, 1, 4, 6)$. In this way,

$$X = ZY^{-1} = \begin{bmatrix} A_3 & & \\ A_2 & A_1 & A_0 \\ & I & \end{bmatrix} \begin{bmatrix} I & & \\ & -I & \\ & & -I \end{bmatrix}^{-1},$$

and

$$\begin{bmatrix} L_0 & -L_1 \end{bmatrix} = \begin{bmatrix} -X & I \end{bmatrix} \Pi^{-1} = \begin{bmatrix} I & & & & -A_3 & \\ & A_1 & A_0 & I & -A_2 & \\ & I & & & & I \end{bmatrix}.$$

Hence the pencil $L(x) = L_1 x + L_0$ is a row-minimal left dual of $C(x)$, and, by Theorem 6.2, a strong linearization of $A(x)$. Indeed, $L(x)$ constructed here is a pencil belonging to a generalized Fiedler family [1, Example 2.5].

11 Conclusions

In this paper, we brought some attention on the duality of matrix pencils and on Wong chains, while generalizing the latter. We have given a self-consistent introduction, and shown explicitly how they interact among themselves and with Kronecker canonical forms. We have given several examples from the study of matrix pencils where these ideas give a more manageable framework. They have allowed us to derive new results and to simplify proofs of, and shed more light into, known properties.

Moreover, we find the connection to the Möller-Stetter theorem a promising new point of view to look at linearizations.

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