Some Methods for Solving Quasi-Equilibrium Problems

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Abstract

Let $X$ be a given nonempty closed convex subset in $\mathbb{R}^n$ and let $f : X \times X \rightarrow \mathbb{R}$ be a given bifunction with $f(x,x) = 0$ for all $x \in X$. We are interested in the following quasi-equilibrium problem

\[
\begin{cases}
\text{Find } x^* \in K(x^*) \text{ such that } \\
f(x^*, y) \geq 0 \quad \forall y \in K(x^*)
\end{cases}
\]

(QEP)

where $K(\cdot)$ is a multivalued mapping $X \rightarrow 2^X$. When the mapping $x \rightarrow K(x)$ is such that $K(x) = X$ for all $x \in X$, (QEP) reduces naturally to the equilibrium problem:

\[
\begin{cases}
\text{Find } x^* \in X \text{ such that } \\
f(x^*, y) \geq 0 \quad \forall y \in X
\end{cases}
\]

(EP)

In the case when $f(x,y) = F(x)^T(y-x)$ in (QEP) with $F : X \rightarrow \mathbb{R}^n$, we obtain an important class of problems, namely the class of quasi-variational inequality problems:

\[
\begin{cases}
\text{Find } x^* \in K(x^*) \text{ such that } \\
F(x^*)^T(y-x^*) \geq 0 \quad \forall y \in K(x^*)
\end{cases}
\]

(QVIP)

Note that the generalized Nash equilibrium problem (GNEP), which is an important model fruitfully used in many different applications, can be reformulated as a (QEP) or (QVIP).

The (EP), (GNEP), (QVIP) and (VIP) have received considerable attention in recent years from theoretical viewpoint, applications and solution methodologies. Compared with the (QEP), the literature on the algorithms for the quasi-equilibrium problems is not as extensive. See, for example, the following recent surveys and the references cited therein:

In this talk we focus on some numerical algorithms for solving quasi-equilibrium problems.\(^1\)

Following are the assumptions we make.

**Assumption (A):**

(i) \(X\) is a nonempty closed convex subset in \(\mathbb{R}^n\).

(ii) \(f : X \times \Lambda \to \mathbb{R}\), where \(\Lambda\) is an open convex set containing \(X\), is a continuous function on \(X \times \Lambda\), \(f(x, x) = 0\), and \(y \mapsto f(x, y)\) is a convex function on \(\Lambda\) for all \(x \in X\).

(iii) The set-valued mapping \(K : X \to 2^X\) is continuous on \(X\), where \(K(x)\) is a nonempty closed convex set of \(X\) for all \(x \in X\).

(iv) \(x \in K(x)\) for all \(x \in X\).

(v) The set \(S^* = \{x \in S | f(x, y) \geq 0 \ \forall y \in T\}\) is nonempty, where \(S = \cap_{x \in X} K(x)\) and \(T = \cup_{x \in X} K(x)\).

(vi) The equilibrium bifunction \(f(\cdot, \cdot)\) is pseudomonotone over \(X\) with respect to \(S^*\), i.e.,

\[\forall x^* \in S^*, \forall y \in X \quad f(y, x^*) \leq 0.\]

Our general method can be expressed as follows.

**General Method (QE):**

**Step 0** Let \(x^0 \in X\), \(\mu \in (0, 1)\), and \(\gamma \in (0, 2)\). Set \(k = 0\).

**Step 1** Compute \(y^k = \arg \min_{y \in K(x^k)} [f(x^k, y) + \frac{1}{2} \|y - x^k\|^2]\). If \(y^k = x^k\) then Stop.

**Step 2** Otherwise, find a direction \(d^k\) such that \(\langle d^k, x^k - x^* \rangle \geq \mu \|x^k - y^k\|^2 > 0\) for every \(x^* \in S^*\).

**Step 3** Compute \(x^k(\beta_k) = P_{K(x^k)}(x^k - \beta_k d^k)\) where \(\beta_k\) is such that

\[\|x^k(\beta_k) - x^*\|^2 \leq \|x^k - x^*\|^2 - \gamma (2 - \gamma) \mu^2 \|x^k - y^k\|^4 / \|d^k\|^2\]

for every \(x^* \in S^*\). Set \(x^{k+1} = x^k(\beta_k), k := k + 1\) and go to Step 1.

Once we have the general method, we detail how to realize concretely the construction of a direction \(d^k\) satisfying the inequality of Step 2 and how to perform a line search along the direction \(-d^k\) to reduce the distance of \(x^k\) to the solution set of the quasi-equilibrium problem. A general convergence theorem applicable to each algorithm is presented.

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We introduce and study variational inequalities in the complex domain, along with some technical tools useful in their study. In particular we discuss and clarify several issues related to the so called "Wirtinger derivatives" of functions of several complex variables.

We then extend to the complex domain some recent developments in the field of the distributed solution of (generalized) Nash equilibrium problems and briefly discuss the problem of selecting a particular solution in monotone games of complex variables when multiple solutions exist.

In order to illustrate our techniques we consider some new MIMO games over vector Gaussian Interference Channels, modeling some distributed resource allocation problems in MIMO Cognitive Radio systems and femtocells. These games are examples of Nash equilibrium problems that can not be handled by current methodologies.
We will provide a global regularity result for the solution map of the generalized equation
\[ 0 \in H(x, p), \]
by assuming openness and Aubin properties with respect to some sets \( U \) and \( V \)
of the map \( H \). As a consequence, we infer the regularity of the solution map of an inclusion
problem that is controlled by the composition of a parametric map with another map which
fulfills suitable regularity assumptions. The result can be applied to investigate a global
condition number for convex vector-valued problems in the Euclidean framework.

(Joint work with G. Kassay and R. Pini.)
LINEAR OPENNESS OF THE COMPOSITION OF SET-VALUED MAPS
AND APPLICATIONS TO VARIATIONAL SYSTEMS

R. Pini

Given the generalized equation $0 \in H(x, p)$, where $H : X \times P \to 2^W$ and $X, W, P$ are metric spaces, our aim is to investigate the regularity properties of the solution map $S = S(p)$. In particular, we are interested in the case $H(x, p) = G(F_1(x, p), F_2(x, p))$, and we infer Lipschitz continuity of the set-valued map $S$ via suitable properties of the maps $F_1, F_2$ and $G$. The main tools are the Nadler fixed point theorem and the Lim lemma. We obtain, as a special case, the well-known result concerning the sum of two set-valued maps.

(Joint work with M. Bianchi and G. Kassay.)
Does convexity arise in optimization naturally?

Fabián Flores-Bazán

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The convex world offers the most desirable properties in optimization, and the lack of that provides an interesting challenge in mathematics. In this lecture we show various instances from mathematical programming to calculus of variations where convexity is present in one way or in another. Among the issues to be described lie: KKT optimality conditions; Dines-type convex theorem and the S-lemma in quadratic programming; optimal value functions; local optimality implies global.

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References


Some applications of variational inequalities to equilibrium problems

Patrizia Daniele – Università degli Studi di Catania

In this talk we show how many problems of topical interest such as: transportation networks, spatially distributed economic markets, supply chain network models, financial equilibrium problems, static electric power supply chain networks, supply chain network with recycled materials, … can be studied in the framework of variational inequalities, which express in a compact and handy form the equilibrium conditions. Moreover, we shall present, in a Hilbert space setting, a general random traffic equilibrium problem, giving a random generalized Wardrop equilibrium condition.
Variational inequalities, bilevel models and the optimal pollution emission price problem

Laura Scrimali

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Abstract

The concern for environmental quality has become one of the most important issue in the last decades. Rapid urbanization, population growth, and industrialization have led to severe environmental problems, especially in developing countries.

Recently, bilevel programming has been recognized to be a successful approach in modeling pollution abatement problems. Inspired by [2], we investigate the complex behavior of manufacturers of a supply chain in the presence of central planning decisions. In particular, we present a continuous-time central planning model in which the government chooses the optimal price of the pollution emission with consideration to manufacturers’ response to the price. On the other hand, the manufacturers choose the optimal quantities of production to maximize their profits, given the price of pollution emission. Such a situation can best be formulated as a bilevel model, where the government’s problem is the leader’s problem with the goal of maximizing the social welfare via taxation, and the manufacturer’s problem defines the follower’s problem.

Using some new recent results on variational inequality and infinite dimensional duality theory, we reformulate the bilevel programming problem into a one level optimization problem. In particular, we give the equilibrium conditions underlying the manufacturer’ problem, in which Lagrange multipliers associated with model constraints are present. Then, we prove the equivalence between the lower-level problem and an opportune variational inequality where Lagrange multipliers do not appear. Finally, we show that the resulting variational inequality is equivalent to the equilibrium conditions; hence the lower-level problem can be
replaced by the equilibrium conditions. We also ensure the existence of Lagrange multipliers, so that the equilibrium conditions are well-defined. Moreover, we investigate the existence of solutions and provide a numerical example.

References


Existence Results for Generalized Vector Equilibrium Problems on Unbounded Sets

E. ALLEVI\(^1\), I.V. KONNOV\(^2\), AND M. ROCCO\(^3\)

Abstract

Many problems of practical interest in optimization, economics and engineering involve equilibrium in their description; this fact has motivated researchers to establish general results on the existence of solutions for equilibrium problems; see e.g. [2, 3]. In fact there is a vast literature on equilibrium problems and their treatment in optimization, variational and quasi-variational inequalities, and complementarity problems. Many authors investigated different equilibrium models, extending scalar equilibrium problems to the vector-valued and set-valued cases; see e.g. [4, 11, 12, 13].

In most of the papers on the existence of solutions of generalized vector equilibrium problems either boundedness of the feasible set or a certain coercivity condition is assumed. The purpose of the present paper is to provide some existence theorems concerning solutions of generalized vector equilibrium problems on an unbounded set with set-valued maps defined on reflexive Banach spaces, exploiting a new coercivity condition, which was introduced in [7] for scalar bifunctions and in [8] for vector-valued functions.

Key words: Generalized vector equilibrium problems, set-valued bifunctions, coercivity condition, existence results.

References


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An existence result for quasiequilibrium problems in separable Banach spaces

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Abstract

A quasiequilibrium problem is an equilibrium problem in which the constraint set is subject to modifications depending on the considered point. This model encompasses many relevant problems as special cases, among which variational and quasivariational inequalities, social (or generalized) Nash equilibrium problems, mixed quasivariational-like inequalities and so on. We furnish an existence result which proof relies on an important selection theorem due to Michael. In our result the usual assumption of closedness of the constraint map is not required and also the finite-dimensionality of the space is dropped.