Two Families of Languages Related to ALGOL*

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Introduction

A serious drawback in the application of modern data processing systems is the cost and time consumed in programming these complexes. The user's problems and their solutions are described in a natural language such as English. To utilize the services of a data processor, it is necessary to convert this language description into machine language, to wit, program steps. Recently, attempts have arisen to bridge the gap between these two languages. The method has been to construct languages (called problem oriented languages, or POL) that are

(i) rich enough to allow a description of a set of problems and their solutions;

(ii) reasonably close to the user's ordinary language of description and solution; and

(iii) formal enough to permit a mechanical translation into machine language.

COBOL and ALGOL are two examples of POL.

The purpose of this investigation is to gain some insight into the syntax of POL, in particular ALGOL [1]. Specifically, the method of defining constituent parts of ALGOL 60 is abstracted, this giving rise to a family of sets of strings; and mathematical facts about the resulting family deduced. Now an ALGOL-like definable language (we hesitate to use the inclusive term "POL") may be viewed either as one of these sets (the set of sentences); or else, as a finite collection of these sets, one of which is the set of sentences, and the remaining, the constituent parts of the language used to construct the sentences. This is in line with one current view of natural languages [4, 5, 6]. The defining scheme for ALGOL turns out to be equivalent to one of the several schemes described by Chomsky [6] in his attempt to analyze the syntax of natural languages. Of course, POL, as special kinds of languages, should fit into a general theory of language. However, it is reasonable to expect that POL, as artificial languages contrived so as to be capable of being mechanically translated into machine language, should have a syntax simpler than that of the natural languages.

The technical results achieved in this paper are as follows. Two families of sets (of strings), the family of definable sets and the family of sequentially definable sets, are described. Definable sets are obtained from a system of simultaneous equations, all the equations being of a certain form. This system, essentially parallel in nature, is an abstraction of the ALGOL method of description. Definable sets turn out to be identical to the type 2 languages (with identity) introduced by Chomsky [6]. Sequentially definable sets are obtained from a system

* Received February, 1961; revised August, 1961.

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of equations which are solved sequentially, that is, one at a time. The second system is a special case of the first, namely, when elimination of variables is possible. The two families of sets are not identical (Section 4). It is known that if α is a definable set and β is a regular set, then $\alpha-\beta$ is also definable. This result is shown not to be true if α is sequentially definable. However, if α is sequentially definable and β is finite, then $\alpha-\beta$ is sequentially definable (Theorem 3). Finally, if in a system of equations deriving a definable set, each equation is linear on the right or if the coefficients in all the equations commute, then the derived set is sequentially definable (Section 5).

1. *Definable and Sequentially Definable Sets*

Consider the following three equations from ALGOL:

$$\begin{aligned}&\langle\text{identifier}\rangle ::= \langle\text{letter}\rangle|\langle\text{identifier}\rangle\ \langle\text{letter}\rangle|\langle\text{identifier}\rangle\ \langle\text{digit}\rangle\\ &\langle\text{unsigned integer}\rangle ::= \langle\text{digit}\rangle|\langle\text{unsigned integer}\rangle\ \langle\text{digit}\rangle\\ &\langle\text{label}\rangle ::= \langle\text{identifier}\rangle|\langle\text{unsigned integer}\rangle.\end{aligned} \tag{1}$$

These three equations are typical of the equations used to define the various constituent parts comprising ALGOL. Rewrite (1) as follows. Let I_1 be the set of identifiers, i.e., $I_1 = \langle\text{identifier}\rangle$, let L_1 be the set of letters (finite), D_1 the set of digits (finite), U_1 the set of unsigned integers, and L_5 the set of labels. Replace $::=$ by $=$ and $|$ by $+$. Then (1) assumes the more compact form:

$$\begin{aligned}I_1 &= L_1 + I_1L_1 + I_1D_1\\ U_1 &= D_1 + U_1D_1\\ L_5 &= I_1 + U_1\,.\end{aligned} \tag{2}$$

Implied in (1), and thus in (2), is that I_1, U_1, and L_5 are generated by L_1 and D_1. Ignoring the finite sets, (2) may be represented in functional form by either

$$\begin{aligned}I_1 &= f_1(I_1, U_1, L_5)\\ U_1 &= f_2(I_1, U_1, L_5)\\ L_5 &= f_3(I_1, U_1, L_5)\end{aligned} \tag{3}$$

or

$$\begin{aligned}I_1 &= f_4(I_1)\\ U_1 &= f_5(I_1, U_1)\\ L_5 &= f_6(I_1, U_1, L_5).\end{aligned} \tag{4}$$

Now there are systems of equations in ALGOL which have the form (3) but not (4), such as that used in defining the arithmetic expressions. In this paper we shall consider both types of systems, (3) because of its generality and (4) because of its inherently simple form.

The remainder of this section is devoted to formalizing the above concepts in order to subject them to mathematical analysis.

Notation. Let $\Sigma = \{a, b, \cdots\}$ be a finite nonempty alphabet, i.e., a finite set of primitive symbols or letters. Let $\theta(\Sigma)$, or θ for short, denote the set of all words, i.e., strings, formed from the letters in Σ, including the empty word ϵ.

Consider functions $f(\xi^{(1)}, \cdots, \xi^{(n)})$ which are constructed from a finite number of set variables $\xi^{(1)}, \cdots, \xi^{(n)}$, each $\xi^{(i)}$ ranging over all subsets of θ, and a finite number of subsets of θ (called *coefficients*); using the operations of "$+$" (addition or set union) and "$\cdot$" (multiplication or complex product[1]) a finite number of times. Since multiplication is distributive over addition, each of these functions may be regarded as in polynomial form, i.e., $f = \sum_{i=1}^{s} \Pi_i$, where each Π_i is a product of set variables and constants.

Each of the functions described in the preceding paragraph is increasing, that is, if[2] $(\xi^{(1)}, \cdots, \xi^{(n)}) \subseteq (\nu^{(1)}, \cdots, \nu^{(n)})$ then $f(\xi^{(1)}, \cdots, \xi^{(n)}) \subseteq f(\nu^{(1)}, \cdots, \nu^{(n)})$. More generally, suppose that $f_1, \cdots, f_n$ is a sequence of functions of $\xi^{(1)}, \cdots, \xi^{(n)}$ each of the type described above. Let 2^θ be the family of all subsets of θ. Let $f = (f_1, \cdots, f_n)$ be the mapping f of $(2^\theta)^n$ (Cartesian product of 2^θ taken n times) into $(2^\theta)^n$ defined by

$$f(\xi^{(1)}, \cdots, \xi^{(n)}) = (f_1(\xi^{(1)}, \cdots, \xi^{(n)}), \cdots, f_n(\xi^{(1)}, \cdots \xi^{(n)})).$$

Each function $f = (f_1, \cdots, f_n)$ is an increasing function in the sense that if $\xi = (\xi^{(1)}, \cdots, \xi^{(n)}) \subseteq \nu = (\nu^{(1)}, \cdots, \nu^{(n)})$, then $f(\xi) \subseteq f(\nu)$.

Now the functions in (2), and more generally, the defining functions in ALGOL have an additional restriction, namely that all coefficients are finite sets.

Definition. A function f (of the type described above) is said to be a *standard* function if each coefficient is a finite set.

For our purposes, a system of functions $f_1, \cdots, f_n$ such as appearing in (2) may be considered as the single function $f = (f_1, \cdots, f_n)$.

Definition. Let $f_1, \cdots, f_n$ be a sequence of n standard functions of $\xi = (\xi^{(1)}, \cdots, \xi^{(n)})$ each. Then $f(\xi) = (f_1, \cdots, f_n)$ is called a *n-tuple standard* function.

We are now ready to discuss the solution to a system of equations such as (2). Theorem 1 below shows that the formal definition next to be given coincides with the meaning intended for these systems in ALGOL.

Definition. A subset γ of θ is said to be *definable* if there exists an n-tuple stand-

[1] Let $A_1, \cdots, A_m$ be a sequence of sets of words. The (complex) *product* $A_1 \cdot A_2 \cdot \cdots \cdot A_m$, or $A_1 \cdots A_m$ for short, is the set of words $\{x_1 \cdots x_m \mid \text{each } x_i \text{ in } A_i\}$, $x_1 \cdots x_m$ being the word formed from the concatenation of the words x_i in the given order. If one or more of the A_i, say $A_{j(1)}, \cdots, A_{j(r)}$ consist of just a single word, say $a_{j(1)}, \cdots, a_{j(r)}$ respectively, then $a_{j(i)}$ is written instead of $A_{j(i)}$ at each occurrence. For example, Ab is written instead of $A\{b\}$ and ϵ is written instead of $\{\epsilon\}$.

[2] By $(\xi^{(1)}, \cdots, \xi^{(n)})$ is meant the set of n-tuples $\{(x_1, \cdots, x_n) \mid x_i \text{ in } \xi^{(i)}, 1 \leqq i \leqq n\}$. $(\xi^{(1)}, \cdots, \xi^{(n)})$ will be referred to as an n-tuple of sets. Each $\xi^{(i)}$ will be called a *coordinate*. Observe that $(\xi^{(1)}, \cdots, \xi^{(n)}) \subseteq (\nu^{(1)}, \cdots, \nu^{(n)})$ if and only if $\xi^{(i)} \subseteq \nu^{(i)}$ for each i.

ard function f such that one of the coordinates of the minimal fixed point of f is[3] γ.

Occasionally, as when specifying an equation for illustrative purposes, we shall write $f(\xi^{(1)}, \cdots, \xi^{(n)}) = (\xi^{(1)}, \cdots, \xi^{(n)})$ as the system of n equations

$$\begin{aligned} \xi^{(1)} &= f_1(\xi^{(1)}, \cdots, \xi^{(n)}) \\ \xi^{(2)} &= f_2(\xi^{(1)}, \cdots, \xi^{(n)}) \\ &\vdots \\ \xi^{(n)} &= f_n(\xi^{(1)}, \cdots, \xi^{(n)}) \end{aligned}$$

The previous discussion dealt with systems of the type (3). It is now a simple matter to discuss systems of the type (4).

Definition. The n-tuple standard function $f(\xi^{(1)}, \cdots, \xi^{(n)}) = (f_1, \cdots, f_n)$ is said to be an *n-tuple sequentially standard function* if $f_i = f_i(\xi^{(1)}, \cdots, \xi^{(i)})$ for $1 \leqq i \leqq n$. A set is said to be *sequentially definable* if it is definable by an n-tuple sequentially standard function.

Let T_0 be the finite subsets of θ. For $n \geqq 0$ let T_{n+1} be the family of sets which are a minimal fixed point to at least one polynomial in one set variable with coefficients in T_n. A set γ is sequentially definable if and only if γ is in one of the families[4] T_n.

Having specified two families of sets, namely, the family of definable sets and the family of sequentially definable sets, it is natural to inquire as to whether or not the two families are equivalent. It would be helpful if the two families were the same since the structure of sets in the latter family intuitively appears simpler than that in the former. However, as remarked in the Introduction, it is shown in Section 4 that there exist definable sets which are not sequentially definable.

It is a well-known mathematical result that each n-tuple (sequentially) standard function has a minimal fixed point [10]. The minimal fixed point of an n-tuple (sequentially) standard function will now be found by the recursive procedure for calculating the variables in systems of equations having the form of (3) or (4) that is indicated in ALGOL [1, p. 301]. This shows that n-tuple standard functions and their minimal fixed points serve as a model for the defining systems of equations in ALGOL.

Notation. Let φ denote the empty set.

THEOREM 1. *For $1 \leqq i \leqq n$ let f_i be a polynomial in the variables $\xi^{(1)}, \cdots, \xi^{(n)}$. Let $f(\xi^{(1)}, \cdots, \xi^{(n)}) = (f_1, \cdots, f_n)$. Let $\alpha_0 = (\alpha_0^{(1)}, \cdots, \alpha_0^{(n)}) = f(\varphi, \cdots, \varphi)$, and let each $\alpha_{i+1} = (\alpha_{i+1}^{(1)}, \cdots, \alpha_{i+1}^{(n)}) = f(\alpha_i)$. Let $\alpha = (\alpha^{(1)}, \cdots, \alpha^{(n)})$ be the minimal fixed point of f. Then $\alpha^{(j)} = \bigcup_{k=0}^{\infty} \alpha_k^{(j)}$ for each j and $\alpha = \bigcup_{k=0}^{\infty} \alpha_k$.*

PROOF. By induction it is easily seen that $\alpha_k \subseteq \alpha_{k+1}$ for each k. Let $\beta =$

[3] $\alpha = (\alpha^{(1)}, \cdots, \alpha^{(n)})$ is said to be a *fixed point* of $f(\xi^{(1)}, \cdots, \xi^{(n)}) = (f_1, \cdots, f_n)$ if $f(\alpha) = \alpha$. In addition, if $\alpha \subseteq \beta$ for each n-tuple of sets $\beta = (\beta^{(1)}, \cdots, \beta^{(n)})$ such that $f(\beta) = \beta$, then α is said to be a *minimal* fixed point. Clearly there is at most one minimal fixed point.

[4] See Lemma 6 of Section 4.

$(\beta^{(1)}, \cdots, \beta^{(n)})$, where $\beta^{(j)} = \bigcup_{k=0}^{\infty} \alpha_k^{(j)}$ for each j. It will be shown that $\alpha = \beta = \bigcup_k \alpha_k$.

Let $x = (x_1, \cdots, x_n)$ be an element of β. For each integer i there is an integer $k(i)$ such that x_k is in $\alpha_{k(i)}^{i}$. Let $k = \max \{k(i)/i\}$. Then x is an α_k. Since $\alpha_k \subseteq \alpha_{k+1} = f(\alpha_k) \subseteq f(\beta)$, x is in $f(\beta)$, i.e., $\beta \subseteq f(\beta)$. Also, $\beta \subseteq \bigcup_k \alpha_k$. Since $\alpha_k \subseteq \beta$ for each k, $\bigcup_k \alpha_k \subseteq \beta$. Therefore $\beta = \bigcup_k \alpha_k \subseteq f(\beta)$.

Let $x = (x_1, \cdots, x_n)$ be an element of $f(\beta)$. Let $\xi = (\xi^{(1)}, \cdots, \xi^{(n)})$. For each integer i let $f_i(\xi) = \sum_{r=1}^{s_i} \Pi_{i,r}(\xi)$, where $\Pi_{i,r}(\xi)$ is the product of coefficients and set variables $\xi^{(j)}$. Since x_i is in the set $\sum_r \Pi_{i,r}(\beta)$, x_i is in $\Pi_{i,r_i}(\beta)$ for some integer r_i. Let $\Pi_{i,r_i}(\xi) = \gamma_{i,1} \cdots \gamma_{i,t(i)}$, where each $\gamma_{i,u}(\xi)$ is either a coefficient set or one of the variables $\xi^{(j)}$. Then there exist elements $y_{i,u}$ in $\gamma_{i,u}(\beta)$ such that $x_i = y_{i,1} \cdots y_{i,t(i)}$. Hence x_i is in $\Pi_{i,r_i}(\alpha_{k(i)})$ for some integer $k(i)$ sufficiently large. Then x is in $f(\alpha_k)$, where $k = \max \{k(i)/i\}$. As $f(\alpha_k) = \alpha_{k+1} \subseteq \beta$ it follows that $f(\beta) \subseteq \beta$, whence $f(\beta) = \beta$. Therefore β is a fixed point of f. It remains to show that β is the minimal fixed point of f.

To this end observe that $\alpha_0 = f(\varphi, \cdots, \varphi) \subseteq f(\alpha) = \alpha$. Continuing by induction, suppose that $\alpha_k \subseteq \alpha$. Then $\alpha_{k+1} = f(\alpha_k) \subseteq f(\alpha) = \alpha$. Therefore $\beta = \bigcup_{\alpha_k} \subseteq \alpha$. Due to the minimality property of α, $\beta = \alpha$. Q.E.D.

Notation. Let $f(\xi^{(1)}, \cdots, \xi^{(n)}) = (f_1, \cdots, f_n)$, where each f_i is a polynomial in the variables $\xi^{(1)}, \cdots, \xi^{(n)}$; and let $\alpha = (\alpha^{(1)}, \cdots, \alpha^{(n)})$ be the minimal fixed point of f. This is abbreviated by "Let $f(\xi) = (f_1, \cdots, f_n)$, each f_i a polynomial in the variables $\xi^{(1)}, \cdots, \xi^{(n)}$; and let α be its minimal fixed point." Whenever this occurs in the sequel it will be understood that $\alpha_0 = (\alpha_0^{(1)}, \cdots, \alpha_0^{(n)}) = f(\varphi, \cdots, \varphi)$, and for each $k \geqq 0$, $\alpha_{k+1} = (\alpha_k^{(1)}, \cdots, \alpha_k^{(n)}) = f(\alpha_k)$.

Except in the simplest of examples, the sets $\alpha_k^{(j)}$ become complicated rather quickly. To illustrate, consider the system of two equations:

$$\xi^{(1)} = \xi^{(1)}c + \xi^{(2)}d + a$$
$$\xi^{(2)} = \xi^{(1)}d + \xi^{(2)}c + b.$$

Here $\alpha_0^{(1)} = \{a\}$, $\alpha_0^{(2)} = \{b\}$, $\alpha_1^{(1)} = \{a, ac, bd\}$, $\alpha_1^{(2)} = \{b, ad, bc\}$, $\alpha_2^{(1)} = \{a, ac, bd, ac^2, bdc, ad^2, bcd\}$, etc.

As mentioned in the Introduction, the nth coordinate $\alpha^{(n)}$ of the minimal fixed point of the n-tuple standard function $f(\xi) = (f_1, \cdots, f_n)$ may be considered a language, $\alpha^{(n)}$ being the set of "sentences" and the other coordinates the constituents of the language. The constituents correspond to verbs, noun-phrases, etc. This conforms with one current view of language [4, 5, 6]. In fact, it is shown in Theorem 2 that definable sets is equivalent to the family of type 2 languages with ϵ described in [6].

Following the exposition in [2], a *simple phrase structure system* is an ordered couple (V, P), where V is a finite alphabet, and P is a finite set of *productions* of the form

$$X \rightarrow x \ (X \text{ in } V, \quad x \text{ in } \theta(V)).$$

Write $y \Rightarrow z$ if $y = uXv$, $z = uxv$, and $X \rightarrow x$ is a production of P. Write $y \twoheadrightarrow z$

if either $y = z$ or if there exists a sequence of elements $z_0, \cdots, z_r$ such that $y = z_0, z_r = z$, and $z_i \Rightarrow z_{i+1}$ for each i. A *type 2 grammar with* ϵ is an ordered 4-tuple (V, P, Σ, S), where (V, P) is a simple phrase structure system; Σ is a subset of V, none of whose elements occur on the left side of a production of P; and S is a distinguished element of $V - \Sigma$. If $G = (V, P, \Sigma, S)$ is a type 2 grammar with ϵ then the set

$$L(G) = \{x \mid x \text{ in } \theta(\Sigma),\ S \twoheadrightarrow x\}$$

is called a *type 2 language with* ϵ, or a *simple phrase structure language*.

THEOREM 2. *The family of definable sets is identical with the family of simple phrase structure languages with* ϵ.

PROOF. Let $f(\xi) = (f_1, \cdots, f_n)$ be an n-tuple standard function and α its minimal fixed point. Let $V = \Sigma \cup \{\xi^{(1)}, \cdots, \xi^{(n)}\}$. Since multiplication is distributive over addition, each standard function f_i may be written as $f_i = \sum_j \Pi_{i,j}(\xi)$, where $\Pi_{i,j}(\xi) = \gamma^i_{j,1} \cdots \gamma^i_{j,m(i,j)}$ each $\gamma^i_{j,k}$ being either a variable or a word. For each i and j associate the production $\xi^{(i)} \rightarrow \Pi_{i,j}(\xi)$ and let P be the set of all these productions. Let $G = (V, P, \Sigma, \xi^{(n)})$. Suppose that $\xi^{(i)} \twoheadrightarrow w$ for each i and each word w in $\alpha_k^{(i)}$. This is certainly true for $k = 0$. Let w be a word in $\alpha_{k+1}^{(i)}$. Then w is in $f_i(\alpha_k)$, thus in $\Pi_{i,j}(\alpha_k)$ for some j. Thus $w = x_1 \cdots x_{m(i,j)}$ where, for each a, $x_a = \gamma^i_{j,a}$ if $\gamma^i_{j,a}$ is a word and x_a is in $\alpha_k^{(\nu(a))}$ if $\gamma^i_{j,a}$ is a variable $\xi^{(\nu(a))}$. In the latter case, $\xi^{(\nu(a))} \twoheadrightarrow x_a$ by induction. Since $\xi^{(i)} \rightarrow \Pi_{i,j}(\xi)$ is a production in P, it follows that $\xi^{(i)} \twoheadrightarrow x_1 \cdots x_{m(i,j)} = w$. Thus $\xi^{(i)} \twoheadrightarrow w$ for each word w in $\alpha^{(i)}$. Letting $i = n$, it results that $L(G) \supseteq \alpha^{(n)}$. To see that $L(G) \subseteq \alpha^{(n)}$, whence $L(G) = \alpha^{(n)}$, it suffices to demonstrate that

$$\begin{aligned} &\text{if } x_1 \text{ and } x_2 \text{ are any two words in } \theta(V) \text{ and } x_1 \twoheadrightarrow x_2, \\ &\text{then } x_1(\alpha) = x_2(\alpha) + g(\alpha) \text{ for some set } g(\alpha). \end{aligned} \tag{5}$$

For if w is any word in $L(G)$ then $\xi^{(n)} \twoheadrightarrow w$. Then $\alpha^{(n)} = w + g(\alpha)$, so that w is $\alpha^{(n)}$. Consider those words x_1 and x_2 in $\theta(V)$ for which $x_1 \twoheadrightarrow x_2$. For each such two words x_1 and x_2 let $z_0, \cdots, z_r$ be a sequence such that $x_1 = z_0$, $x_2 = z_r$, and $z_i \Rightarrow z_{i+1}$ for each i. Suppose that $r = 1$. Then for x_1 and x_2 there exist words w_1, w_2, w_3, and w_4 in $\theta(V)$ such that $x_1 = w_1w_3w_2$, $x_2 = w_1w_4w_2$, and $w_3 \rightarrow w_4$ is in P. Since $w_3 \rightarrow w_4$ is in P, $w_3(\alpha) = w_4(\alpha) + g_1(\alpha)$. Then $x_1(\alpha) = w_1(\alpha)w_3(\alpha)w_2(\alpha) = w_1(\alpha)w_4(\alpha)w_2(\alpha) + w_1(\alpha)g_1(\alpha)w_2(\alpha) = x_2(\alpha) + g(\alpha)$, where $g(\alpha) = w_1(\alpha)g_1(\alpha)w_2(\alpha)$. Continuing by induction, suppose that (5) holds fore all words x_1 and x_2 when $r \leqq k$. Let x_1 and x_2 be two words for which $r = k + 1$. Then $x_1 = z_0 \twoheadrightarrow z_k$ and $z_k \twoheadrightarrow z_{k+1} = x_2$. By the induction hypothesis $z_k(\alpha) = x_1(\alpha) + g_1(\alpha)$ and $x_2(\alpha) = z_k(\alpha) + g_2(\alpha)$. Then $x_2(\alpha) = x_1(\alpha) + g(\alpha)$, where $g(\alpha) = g_1(\alpha) + g_2(\alpha)$. Thus (5) holds in general, and $L(G) = \alpha^{(n)}$. Therefore each definable set is a simple phrase structure language.

Finally, let $G = (V, P, \Sigma, S)$ by a type 2 grammar with ϵ. Label the elements of $V - \Sigma$ by $\xi^{(1)}, \cdots, \xi^{(n)}$, with $\xi^{(n)} = S$. For each $\xi^{(i)}$, let $\xi^{(i)} \rightarrow w_1^i, \cdots, \xi^{(i)} \rightarrow w_{t(i)}^i$ be the productions in P for which $\xi^{(i)}$ occurs on the left. Consider the n-tuple standard function $f = (f_1, \cdots, f_n)$, where $f_i = \sum_{j=1}^{t(i)} w_j^i$. The type 2 grammar with ϵ generated by f, by the procedure of the preceding paragraph, is $G = (V, P, \Sigma, S)$. Thus $L(G) = \alpha^{(n)}$, where $(\alpha^{(1)}, \cdots, \alpha^{(n)})$ is the minimal fixed point of f, that is, each simple phrase structure language is a definable set. This completes the proof of Theorem 2.

Let $G = (V, P, \Sigma, \xi^{(n)})$ be a type 2 grammar with ϵ. Let $V - \Sigma = \{\xi^{(i)} \mid 1 \leqq i \leqq n\}$. For each $\xi^{(i)}$ and each production $\xi^{(i)} \to w$, suppose that each variable $\xi^{(j)}$ appearing in w is such that $j \leqq i$. Then $L(G)$ is a sequentially definable set. Furthermore, every sequentially definable set may be generated in such a way.

Because of Theorem 2 and the preceding paragraph, in dealing with definable and sequentially definable sets either the equation or the production point of view may be used, whichever is the more convenient.

2. *Parallel Results*

A number of known facts about definable sets are also true for sequentially definable sets. Several of these are now presented.

It is known that the family of definable sets is closed under $+$, $\cdot$, and * [2].[5] The family of sequentially definable sets is closed under the same operations. For suppose that γ and δ are sequentially definable sets, occurring as the mth and nth coordinates of the minimal fixed points of the m-tuple and n-tuple sequentially standard functions $f(\xi) = (f_1, \cdots, f_m)$ and $g(\nu^{(1)}, \cdots, \nu^{(n)}) = (g_1, \cdots, g_n)$ respectively. Then $\gamma\delta$, $\gamma + \delta$, and γ^* are the $(m + n + 1)$-th coordinates in the minimal fixed points of $(f_1, \cdots, f_m, g_1, \cdots, g_n, g_{n+1})$, $(f_1, \cdots, f_m, g_1, \cdots, g_n, g_{n+2})$, and $(f_1, \cdots, f_m, g_1, \cdots, g_n, g_{n+3})$ respectively, where $g_{n+1} = \xi^{(m)}\nu^{(n)}$, $g_{n+2} = \xi^{(m)} + \nu^{(n)}$, and $g_{n+3} = \xi^{(n+3)}\xi^{(m)} + \epsilon$.

Two important families of sets which have been extensively studied are the regular sets,[6] associated with finite automata[7], and the "recursively enumerable" sets [7], associated with "Turing machines" [7]. Chomsky [6] has observed that the definable sets properly include the regular sets and are properly included in the recursively enumerable sets. The family of sequentially definable sets satisfies the same inclusion. The sequential definability of the regular sets follows from the closure properties of the family. On the other hand, the set $\{a^n ca^n / n \geqq 0\}$ is sequentially definable, being the minimal fixed point of $a\xi a + c$, and is known

[5] The operation "$*$" is defined by $A^* = \bigcup_{i=0}^{\infty} A^i$ for each set A, where $A^0 = \epsilon$ and $A^{i+1} = A^i A$ for $i \geqq 0$.

[6] The family of regular sets is the smallest family of sets which contains the finite sets and which is closed under the operations of $+$, $\cdot$, and $*$ [8].

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[7] An *automaton* is a 5-tuple $(K, \Sigma, \delta, q_0, F)$, where K is a finite nonempty set (of "states"), Σ is a finite nonempty set (of "inputs"), δ is a ("next state") function of $K \times \Sigma$ into K, q_0 is a ("start") state in K, and F is a subset (the "final states") of K [8]. A word (= sequence of inputs) is said to be *accepted* by the automaton if the word takes (by successive application of δ) the automaton from q_0 to one of the states in F. It is known that a set is regular if and only if it is the set of words accepted by some automaton [8].

not to be regular [8]. Another example is from ALGOL. Let P_1 be the variable for the set of proper strings and O_2 the variable for the set of open strings. Then

$$P_1 = P_1H + H$$

and

$$O_2 = P_1 + \text{`}O_2\text{'} + O_2O_2 ,$$

where H is a finite set containing neither of the two symbols 'and'. By a method similar to that given in [8] it can be shown that O_2' is not regular, where (P_1', O_2') is the minimal fixed point. This implies that O_2' is not the set of words accepted by some automaton, that is, an automaton cannot be found which discerns when an arbitrary word in θ is a word in O_2'.

It is known that the family of definable sets is not closed under set intersection (thus not under set complementation) [2, 9]. The family of sequentially definable sets is also not closed under set intersection (thus not under set complementation). In fact, the same example used in [9] is valid here. Let $\alpha_1 = \{a^n/n \geqq 1\}$ and $\alpha_2 = \{b^na^n/n \geqq 1\}$. α_1 and α_2 are the minimal fixed points of $a\xi + a$ and $b\xi a + ba$, respectively. Therefore α_1 and α_2 are sequentially definable sets. The rest of the argument is the same as in [9].

The following result, in slightly different form, is known [9, Lemma 3] for definable sets. The same proof is valid for sequentially definable sets.

THEOREM A. *If α' is an infinite sequentially definable set, then there exist sequentially definable sets δ, β, γ, μ, and ν such that*

(1) δ is infinite;

(2) either μ or ν is not the unit set ϵ;

(3) $\mu\delta\nu \subseteq \delta$;

and (4) $\beta\delta\gamma \subseteq \alpha'$.

One more known result about definable sets carries over, with the same proof, to sequentially definable sets. This result, appearing as Theorem 3.3 of [2] for definable sets, is the following:

THEOREM B. *Let α be a sequentially definable set. If $h(x)$ is a sequentially definable set for each element x in Σ, then $\bigcup_{(w \text{ in } \alpha)(w=x_1 \cdots x_r)} h(x_1) \cdots h(x_r)$ is a sequentially definable set.*

In view of Theorem 2 an equivalent formulation of Theorem B is

THEOREM C. *Let $f(\xi^{(1)}, \cdots, \xi^{(n)}) = (f_1, \cdots, f_n)$, where each f_i is a polynomial in the variables $\xi^{(1)}, \cdots, \xi^{(n)}$ and each coefficient is sequentially definable. Then each coordinate $\alpha^{(i)}$ in the minimal fixed point $\alpha = (\alpha^{(1)}, \cdots, \alpha^{(n)})$ is sequentially definable. Furthermore, let the different coefficients be $A_1, \cdots, A_s$. Let $a_1, \cdots, a_s$ be s abstract symbols and let $h(a_j) = A_j$ for each j. Denote by $g = (g_1, \cdots, g_n)$ the n-tuple sequentially standard function obtained by replacing each A_j by a_j. Let $\beta = (\beta^{(1)}, \cdots, \beta^{(n)})$ be the minimal fixed point of g. Then for each i, $\alpha^{(i)} = \bigcup_{(w \text{ in } \beta^{(i)})(w=x_1 \cdots x_r)} h(x_1) \cdots h(x_r)$.*

3. *Subtraction*

Section 2 dealt with results which were true for both definable and sequentially definable sets. We now show that the parallelism does not carry over to subtraction.

In Theorem 8.1 of [2] it is shown that if a set α' is definable and B is regular then $\alpha' - B$ is also definable. This result is not true for sequentially definable sets.

Example. Let η' be the set consisting of all words having the symmetric form

$$a^{n_{2k-1}}db^{n_{2k-2}}da^{n_{2k-3}}d \cdots db^{n_2}da^{n_1}ca^{n_1}db^{n_2}d \cdots da^{n_{2k-1}},$$

where $k, n_1, \cdots, n_{k-1}$ are positive integers. It is shown in Section 4 that η' is definable but not sequentially definable. Let $M = e\eta'e$. Using Theorem B it is readily verified that M is not sequentially definable (although it is definable). Let α' be the set of those words having the symmetric form

$$ex_k^{n_k} \cdots x_2^{n_2}x_1^{n_1}cx_1^{n_1}x_2^{n_2} \cdots x_k^{n_k}e$$

where $k, n_1, \cdots, n_k$ are positive integers, $x_1 = x_k = a$, $x_i = a, b$, or d, and $x_{i+1} \neq x_i$. Let $f(\xi^{(1)},\xi^{(2)},\xi^{(3)}) = (f_1, f_2, f_3)$, where $f_1 = a\xi^{(1)}a + b\xi^{(1)}b + d\xi^{(1)}d + aca$, $f_2 = a\xi^{(1)}a + a\xi^{(2)}a$, and $f_3 = e\xi^{(2)}e$. The set α' is the third coordinate in the minimal fixed point of f. Thus α' is sequentially definable. Let $A = (K,\Sigma,\delta,p_1,\{p_1\})$ be the automaton[8] defined thusly. $K = \{p_i \mid 1 \leqq i \leqq 6\}$ and $\Sigma = \{a,b,c,d,e\}$. The function δ is defined by $\delta(p_1,e) = p_2$, $\delta(p_2,b) = p_6$, $\delta(p_2,d) = p_3$, $\delta(p_2,e) = p_1$, $\delta(p_3,a) = \delta(p_3,d) = p_6$, $\delta(p_3,b) = p_4$, $\delta(p_4,a) = p_6$, $\delta(p_4,d) = p_5$, $\delta(p_5,a) = p_2$, $\delta(p_5,b) = \delta(p_5,d) = p_6$, and $\delta(p,I) = p$ for all other p and I. The set H of words accepted by A is a regular set.[8] It is readily seen that for each word X in α', $\delta(p_1,X) = p_1$ or $\delta(p_1,X) = p_6$ according as X does or does not belong to M. Thus $\alpha' \cap H = M$. Let B be the complement of H. As H is a regular set, so is B. Now $\alpha' - B = M$. Thus α' is sequentially definable, B is regular, and $\alpha' - B$ is not sequentially definable.

In Theorem 3 below, we present some conditions on B which guarantee that $\alpha' - B$ is sequentially definable when α' is. First we need two lemmas.

LEMMA 1. *Let A be a set such that for each standard function $g(\xi^{(1)}, \cdots, \xi^{(n)})$ and each sequence $\nu^{(1)}, \cdots, \nu^{(n)}$ of sequentially definable sets; all of the sets $(\nu^{(i)} \cap A)$ and $g(\nu^{(1)} \cap A, \cdots, \nu^{(n)} \cap A) - (\nu^{(i)} \cap A)$ are sequentially definable. In addition, let A have the property that for all words x, y, and z, if xyz is in A then y is in A. Then $\alpha' - A$ is sequentially definable for each sequentially definable set α'.*

PROOF. Let α' be any sequentially definable set. Let $f(\xi) = (f_1, \cdots, f_n)$ be a n-tuple sequentially standard function, with α its minimal fixed point and $\alpha^{(n)} = \alpha'$. For $1 \leqq i \leqq n$ let $F_i = A \cap \alpha^{(i)}$, $G_i = f_i(F_1, \cdots, F_i)$, and $D_i = G_i - F_i$. By hypothesis all of the sets F_i and D_i are sequentially definable. For each i, write $f_i(\xi^{(1)}+F_1, \cdots, \xi^{(i)}+F_i)$ in the form $g_i(\xi^{(1)}, \cdots, \xi^{(i)}) + G_i$, where g_i is a polynomial in $\xi^{(1)}, \cdots, \xi^{(i)}$ with the constant term missing. For each i the polynomial $h_i = g_i + D_i$ has only $F_1, \cdots, F_i, D_i$, and the finite sets as coefficients. Let $\beta = (\beta^{(1)}, \cdots, \beta^{(n)})$ be the minimal fixed point of $h(\xi^{(1)}, \cdots, \xi^{(n)}) =$

[8] See footnote 7.

$(h_1, \cdots, h_n)$. Then $\beta = \bigcup_k \beta_k$, where $\beta_0 = (\beta_0^{(1)}, \cdots, \beta_0^{(n)}) = h(\varphi, \cdots, \varphi)$ and each $\beta_{k+1} = (\beta_{k+1}^{(1)}, \cdots, \beta_{k+1}^{(n)}) = h(\beta_k)$. By Theorem C, each $\beta^{(j)}$ is a sequentially definable set. We shall show that $\beta^{(i)} = \alpha^{(i)} - F_i = \alpha^{(i)} - A$ for each i. Since $\beta^{(n)}$ is sequentially definable this will prove the lemma.

Now $\beta_0^{(i)} = D_i \subseteq \alpha^{(i)} - F_i$. Proceeding by induction, suppose that $\beta_m^{(i)} \subseteq \alpha^{(i)} - F_i$, thus, as $F_i \subseteq \alpha^{(i)}$, $\beta_m^{(i)} + F_i \subseteq \alpha^{(i)}$, for each i. Then

$$\begin{aligned} \beta_{m+1}^{(i)} &= g_i(\beta_m^{(1)}, \cdots, \beta_m^{(i)}) + D_i \\ &\subseteq g_i(\beta_m^{(1)}, \cdots, \beta_m^{(i)}) + G_i \\ &= f_i(\beta_m^{(1)} + F_1, \cdots, \beta_m^{(i)} + F_i) \\ &\subseteq f_i(\alpha^{(1)}, \cdots, \alpha^{(i)}) \\ &= \alpha^{(i)}. \end{aligned}$$

Suppose that $\beta_{m+1}^{(i)} \cap A$ is non-empty, that is, contains some word v. As $\beta_{m+1}^{(i)} \subseteq \alpha^{(i)}$, v is in $\alpha^{(i)} \cap A$. Then v is not in D_i since $D_i = G_i - (\alpha^{(i)} \cap A)$. Since $\beta_{m+1}^{(i)} = g_i(\beta_m^{(i)}, \cdots, \beta_m^{(i)}) + D_i$, v is in $g_i(\beta_m^{(1)}, \cdots, \beta_m^{(i)})$. From the definition of g_i, $g_i(\beta_m^{(1)}, \cdots, \beta_m^{(i)})$ is the sum of terms each of which is a product with at least one set $\beta_m^{(k)}$ as a factor. Therefore $v = xyz$, where y is a word in one of the sets $\beta_m^{(1)}, \cdots, \beta_m^{(i)}$, say $\beta_m^{(j)}$. From the hypothesis, y is in A since v is in A. Hence y is in $\beta_m^{(j)} \cap A$. But

$$\beta_m^{(j)} \cap A \subseteq (\alpha^{(j)} - F_j) \cap A = \varphi.$$

From this contradiction it follows that $\beta_{m+1}^{(i)} \cap A$ is empty. Thus $\beta_{m+1}^{(i)} \subseteq \alpha^{(i)} - A = \alpha^{(i)} - F_i$. By mathematical induction, therefore, $\beta_m^{(i)} \subseteq \alpha^{(i)} - F_i$ for each i and each m. Then $\beta^{(i)} = \bigcup_m \beta_m^{(i)} \subseteq \alpha^{(i)} - F_i$ for each i.

As to the reverse inequality, for each i

$$\alpha_0^{(i)} - F_i \subseteq G_i - F_i = D_i \subseteq \beta^{(i)}.$$

Again by induction, for each i suppose that $\alpha_m^{(i)} - F_i \subseteq \beta^{(i)}$. Then $\alpha_m^{(i)} \subseteq \beta^{(i)} + F_i$. Then

$$\begin{aligned} \alpha_{m+1}^{(i)} - F_i &= f_i(\alpha_m^{(1)}, \cdots, \alpha_m^{(i)}) - F_i \\ &\subseteq f_i(\beta^{(1)} + F_1, \cdots, \beta^{(i)} + F_i) - F_i \\ &= [g_i(\beta^{(1)}, \cdots, \beta^{(i)}) + G_i] - F_i \\ &\subseteq g_i(\beta^{(1)}, \cdots, \beta^{(i)}) + (G_i - F_i) \\ &= \beta^{(i)}. \end{aligned}$$

Thus $\alpha_m^{(i)} - F_i \subseteq \beta^{(i)}$ for all m, whence $\alpha^{(i)} - F_i \subseteq \beta^{(i)}$. Therefore $\alpha^{(i)} - F_i = \beta^{(i)}$. Q.E.D.

LEMMA 2. *Let α' be a sequentially definable set and Σ_2 a subset of the basic alphabet Σ_1. Then $\alpha' \cap \theta(\Sigma_2)$ is sequentially definable.*

PROOF. For each element x in Σ_2 let $f(x) = \{x\}$. For each element x in $\Sigma_1 -$

Σ_2 let $f(x) = \phi$. Let $\beta' = \bigcup_{(w \text{ in } \alpha')(w=x_1 \cdots x_r)} f(x_1) \cdots f(x_r)$. Clearly $\alpha' \cap \theta(\Sigma_2) = \beta'$. By Theorem B, β' is sequentially definable.

THEOREM 3. *Let B be either a finite set, or a sequentially definable set of words all generated by the same one letter. Then $\alpha' - B$ is sequentially definable for every sequentially definable set α'.*

PROOF. First suppose that B is a finite set consisting of words $w_1, \cdots, w_r$. For each word x in θ let $L(x)$ denote the length of x. (In particular $L(\epsilon) = 0$.) Let $h = \max\{L(w_i) \mid 1 \leqq i \leqq r\}$ and let $A = \{x \mid x \text{ in } \theta, L(x) \leqq h\}$. Clearly A satisfies the hypotheses of Lemma 1. Therefore $\alpha' - A$ is sequentially definable. Since $\alpha' \cap A$ and B are both finite, the set $(\alpha' \cap A) - B$ is sequentially definable. Then $\alpha' - B = (\alpha' - A) + [(\alpha' \cap A) - B]$ is sequentially definable.

Now suppose that B is a sequentially definable set of words, all generated by the same one letter, say a. Then $B = \{a^s \mid s \text{ in } \lambda\}$ where, as is shown in Corollary 2 of Theorem 4, λ is an ultimately periodic set of non-negative integers.[9] Denote by A the set $A = \{a^t \mid t \geqq 0\}$, where $a^0 = \epsilon$. By Lemma 2, if γ is a sequentially definable set then so is $\gamma \cap A$. By Corollary 2 of Theorem 4, the family of all sequentially definable sets of words of the same one letter coincides with the family of regular sets. Let $g(\xi^{(1)}, \cdots, \xi^{(n)})$ be any standard function. Denote by $\nu \cap A$ the n-tuple $(\nu^{(1)} \cap A, \cdots, \nu^{(n)} \cap A)$. Each $\nu^{(i)} \cap A$ is regular. Then $g(\nu \cap A)$ is the sum and product of regular sets. Since the family of regular sets is closed under multiplication, addition, and subtraction, $g(\nu \cap A) - (\nu^{(i)} \cap A)$ is regular, thus sequentially definable. Therefore A satisfies the hypotheses of Lemma 1. Therefore $\alpha' - A$ is sequentially definable. Furthermore, $(\alpha' \cap A) - B$ is sequentially definable since $\alpha' \cap A$ and B are both regular. Then $\alpha' - B = (\alpha' - A) + [(\alpha' \cap A) - B]$ is sequentially definable. Q.E.D.

4. *An Example*

The question arises: Is the family of definable sets identical with the family of sequentially definable sets? The answer is in the negative. For we shall exhibit a definable set η' which is not sequentially definable.

We first prove some preliminary lemmas.

LEMMA 3. *Let $f(\xi) = (f_1, \cdots, f_n)$ be an n-tuple sequentially standard function and α its minimal fixed point. Let $u(1), \cdots, u(s)$ be a subsequence of $1, \cdots, n$. For each i let $g_i(\xi^{(u(1))}, \cdots, \xi^{(u(s))})$ be the function f_i with each variable $\xi^{(j)}$, j not one of the $u(k)$, replaced by $\alpha^{(j)}$. Then the minimal fixed point of the function $g(\xi^{(u(1))}, \cdots, \xi^{(u(s))}) = (g_{u(1)}, \cdots, g_{u(s)})$ is $\gamma = (\alpha^{(u(1))}, \cdots, \alpha^{(u(s))})$.*

PROOF. Since α is a fixed point of f, $\alpha^{(j)} = f_j(\alpha)$ for each j. Thus $\alpha^{(u(i))} = f_{u(i)}(\alpha) = g_{u(i)}(\gamma)$. Therefore γ is a fixed point of g. It remains to show that γ is the *minimal* fixed point of g.

[9] Let λ be a set of non-negative integers and let $\{x_n\}_{n \geqq 1}$ be the sequence of its elements, ordered by magnitude. Then λ is said to be *ultimately periodic* if the auxiliary sequence $\{y_n\}_{n \geqq 1}$, where $y_1 = x_1$ and $y_{i+1} = x_{i+1} - x_i$, is ultimately periodic, i.e., there exist integers n_0 and p so that $y_{m+p} = y_m$ for all $m \geqq n_0$.

Let $\beta = (\beta^{(u(1))}, \cdots, \beta^{(u(s))})$ be the minimal fixed point of g. By Theorem 1, $\beta = \bigcup_{k=0}^{\infty}\beta_k$, where $\beta_0 = (\beta_0^{u(1)}, \cdots, \beta_0^{(u(s))}) = g(\varphi, \cdots, \varphi)$ and $\beta_{k+1} = (\beta_{k+1}^{(u(1))}, \cdots, \beta_{k+1}^{(u(s))}) = g(\beta_k)$ for each k. Then for each $u(j)$,

$$\alpha_0^{(u(j))} = f_{u(j)}(\varphi) \subseteq f_{u(j)}(\nu_1, \cdots, \nu_n) = \beta_0^{(j)},$$

where $\nu_i = \varphi$ if $i = u(x)$ for some x and $\nu_i = \alpha^{(i)}$ otherwise. For each $i \leqq n$ and each k, let $\nu_k^{(i)} = \beta_k^{(i)}$ if $i = u(x)$ for some x, $\nu_k^{(i)} = \alpha^{(i)}$ otherwise, and $\nu_k = (\nu_k^{(1)}, \cdots, \nu_k^{(n)})$. Continuing by induction, suppose that $(\alpha_k^{(u(1))}, \cdots, \alpha_k^{(u(s))}) \subseteq \beta_k \subseteq \beta$. Then for each $u(j)$:

$$\begin{aligned} \alpha_{k+1}^{(u(j))} = f_{u(j)}(\alpha_k) &\subseteq f_{u(j)}(\nu_k) \\ &= \beta_{k+1}^{(u(j))} \\ &\subseteq \beta^{(u(j))}. \end{aligned}$$

Thus $\bigcup_{k=0}^{\infty}\alpha_k^{(u(j))} = \alpha^{(u(j))} \subseteq \beta^{(u(j))}$, whence $\gamma \subseteq \beta$. From the minimality property of β, it follows that $\gamma = \beta$. Thus γ is the minimal fixed point of g.

As a corollary we obtain

LEMMA 4. *Let $f(\xi) = (f_1, \cdots, f_n)$ be an n-tuple sequentially standard function and α its minimal fixed point. Suppose that $\alpha^{(i)}$ is finite for some integer i. Let $g = (g_1, \cdots, g_{i-1}, g_{i+1}, \cdots, g_n)$ where for each integer $j \neq i$, g_j is the function*

$$g_j(\xi^{(1)}, \cdots, \xi^{(i-1)}, \xi^{(i+1)}, \cdots, \xi^{(n)}) = f_j(\xi^{(1)}, \cdots, \xi^{(i-1)}, \alpha^{(i)}, \xi^{(i+1)}, \cdots, \xi^{(n)}).$$

Then g is a $(n-1)$-tuple sequentially standard function and $\gamma = (\alpha^{(1)}, \cdots, \alpha^{(i-1)}, \alpha^{(i+1)}, \cdots, \alpha^{(n)})$ is its minimal fixed point.

LEMMA 5. *Let $f(\xi) = (f_1, \cdots, f_n)$ be an n-tuple sequentially standard function and α its minimal fixed point. For some integer i let $f_i(\xi) = \xi^{(i)} + h(\xi)$. Then the function $g(\xi) = (g_1, \cdots, g_n)$, where $g_i(\xi) = h(\xi)$ and for $j \neq i$, $g_j(\xi) = f_j(\xi)$, is an n-tuple sequentially standard function and α is its minimal fixed point.*

The proof of Lemma 5 follows from Theorem 2 and the fact that a production of the form $\xi^{(i)} \to \xi^{(i)}$ is not needed.

Another consequence of Lemma 3 is the next result.

LEMMA 6. *Let $(\alpha^{(1)}, \cdots, \alpha^{(n)})$ be the minimal fixed point of the n-tuple sequentially standard function $(f_1, \cdots, f_n)$. Then $\alpha^{(i)}$ is the minimal fixed point of $f_i(\alpha^{(1)}, \cdots, \alpha^{(i-1)}, \xi^{(i)})$.*

Let $\Sigma = \{a, b, c, d\}$ and $g(\nu, \eta) = (g_1, g_2)$, where $g_1 = b\nu b + bd\eta db$ and $g_2 = ad\nu da + a\eta a + aca$. Denote by (ν', η') the minimal fixed point of (g_1, g_2). It is readily verified that η' is the set of those words having the symmetric form

$$[n_1, \cdots, n_{2k-1}] = a^{n_{2k-1}}d \cdots db^{n_2}da^{n_1}ca^{n_1}db^{n_2}d \cdots b^{n_{2k-2}}\, da^{n_{2k-1}},$$

where k, n_1, $\cdots$, and n_{2k-1} are positive integers. Therefore η' is a definable set.

Suppose that η' is sequentially definable. Let $f(\xi^{(1)}, \cdots, \xi^{(n)}) = (f_1, \cdots, f_n)$ be an n-tuple sequentially standard function whose minimal fixed point $(\alpha^{(1)}, \cdots, \alpha^{(n)})$ is such that $\alpha^{(n)} = \eta'$. By Lemma 4 we may assume that each

$\alpha^{(i)}$ is infinite. Let m be the smallest integer satisfying the following two conditions.

(a) There exists a pair (u, v) of words such that $u\alpha^{(m)}v$ is a subset of $\alpha^{(n)}$.

(b) The set $u\alpha^{(m)}v$ contains an infinite number of words $[n_1, \cdots, n_s]$, with different r, $r < s$, such that $n_1 < \cdots < n_r$.

Since n is an integer satisfying (a) and (b), the integer m exists. We shall show that η' is not sequentially definable by proving that the existence of the integer m leads to a contradiction.

We first prove that

for any $\alpha^{(j)}$, if W and Y are two sets such that $W\alpha^{(j)}Y \subseteq \eta'$; then

$$W \text{ and } Y \text{ each contain just one word, say } w \text{ and } y \text{ respectively.} \tag{6}$$

To see this let w and y be any words in W and Y, respectively. Let x_1 and x_2 be two words in the infinite set $\alpha^{(j)}$. Suppose that w contains the letter c. Then wx_1y and wx_2y are two words in η' with c in w. Clearly one of these two words cannot be symmetric. However, this contradicts the symmetry of each word in η'. Thus w does not contain c. Similarly y does not contain c. Hence each word in $\alpha^{(j)}$ contains the letter c. Suppose that W contains a second word w_1. Then w_1x_1y and wx_1y are two words in η' with c in x_1. One of these two words cannot be symmetric, again contradicting the symmetry of each word in η'. Thus W contains just one word. Similarly Y contains just one word. This proves (6).

Consider the set $\alpha^{(m)}$. By assumption, $(\alpha^{(1)}, \cdots, \alpha^{(n)})$ is a fixed point of f. Thus $\alpha^{(m)} = f_m(\alpha^{(1)}, \cdots, \alpha^{(m)})$. As f_m is a polynomial in $\xi^{(1)}, \cdots, \xi^{(m)}$ we may write f_m in the form

$$f_m(\xi^{(1)}, \cdots, \xi^{(m)}) = A_1\xi^{(m)}B_1 + \cdots + A_t\xi^{(m)}B_t + K,$$

where $K = f_m(\xi^{(1)}, \cdots, \xi^{(m-1)}, \phi)$ and, for each i, $A_i = A_i(\xi^{(1)}, \cdots, \xi^{(m)})$ and $B_i = B_i(\xi^{(1)}, \cdots, \xi^{(m)})$. By Lemma 5 we may assume that $\xi^{(m)}$ is not identical with any one of the summands $A_i\xi^{(m)}B_i$. Let $K' = f_m(\alpha^{(1)}, \cdots, \alpha^{(m-1)}, \phi)$ and, for each i, let $A_i' = A_i(\alpha^{(1)}, \cdots, \alpha^{(m)})$ and $B_i' = B_i(\alpha^{(1)}, \cdots, \alpha^{(m)})$. Thus

$$\alpha^{(m)} = f_m(\alpha^{(1)}, \cdots, \alpha^{(m)}) = A_1'\alpha^{(m)}B_1' + \cdots + A_t'\alpha^{(m)}B_t' + K'.$$

Since $uA_i'\alpha^{(m)}B_i'v \subseteq u\alpha^{(m)}v \subseteq \eta'$, by (6), uA_i' and $B_i'v$ each contain one word.

We are now in a position to show that the existence of the integer m leads to a contradiction. We do this by examining the set K'. In particular, we shall see that K' must satisfy one of two alternatives; and that each alternative effects a contradiction.

Case 1. Suppose that the set $uK'v$ contains an infinite number of words $[n_1, \cdots, n_s]$, with different r, $r < s$, such that $n_1 < \cdots < n_r$. Write the function $K = f_m(\xi^{(1)}, \cdots, \xi^{(m-1)}, \phi)$ in the form

$$K = \Pi_1(\xi^{(1)}, \cdots, \xi^{(m-1)}) + \cdots + \Pi_h(\xi^{(1)}, \cdots, \xi^{(m-1)}),$$

where each Π_i is a product of finite sets and variables $\xi^{(1)}, \cdots, \xi^{(m-1)}$. Replacing each $\xi^{(j)}$ by $\alpha^{(j)}$ we have

$$K' = P_1 + \cdots + P_h,$$

where each $P_i = \Pi_i(\alpha^{(1)}, \cdots, \alpha^{(m-1)})$. The set K' is the sum of a finite number of P_i. Hence for one of the terms, call it P_1, uP_1v contains an infinite number of words $[n_1, \cdots, n_s]$, with different r, $r < s$, such that $n_1 < \cdots < n_r$. The set P_1 is the product of finite sets and $\alpha^{(1)}, \cdots, \alpha^{(m-1)}$. One of the factors of P_1 must be one of the sets $\alpha^{(1)}, \cdots, \alpha^{(m-1)}$, say $\alpha^{(p)}$. For if not, then P_1 would be the product of finite sets, and thus finite, contradicting the fact that uP_1v is infinite. Thus P_1 may be written as $Q_1\alpha^{(p)}Q_2$. As $uQ_1\alpha^{(p)}Q_2v \subseteq \eta'$, by (6), uQ_1 and Q_2v each contain just one word, say w and y respectively. Then $w\alpha^{(p)}y = uP_1v$. Then p is an integer smaller than m which satisfies (a) and (b). This contradicts the minimality property of m. Therefore Case 1 does not arise.

Case 2. Suppose that the set $uK'v$ contains just a finite number of words $[n_1, \cdots, n_s]$, with different r, $r < s$, such that $n_1 < \cdots < n_r$. By Lemma 6, $\alpha^{(m)}$ is the minimal fixed point of $f_m(\alpha^{(1)}, \cdots, \alpha^{(m-1)}, \xi^{(m)}) = \sum_i A_i'\xi^{(m)}B_i' + K'$. It has already been shown that for each i there exist words w_i and y_i so that $A_i' = \{w_i\}$ and $B_i' = \{y_i\}$. Thus $\alpha^{(m)}$ is the minimal fixed point of $\Sigma w_i\xi^{(m)}y_i + K'$. Therefore $\alpha^{(m)}$ is the set union of all sets of the form $z_q \cdots z_1K'z_1'z_2' \cdots z_q'$, and $u\alpha^{(m)}v$ is the set union of all sets of the form $uz_q \cdots z_1K'z_1'z_2' \cdots z_q'v$, where q varies, each z_i is one of the words w_j, and z_i' the corresponding word y_j. Now $uK'v$ contains just a finite number of words $[n_1, \cdots, n_s]$, with different r, $r < s$, such that $n_1 < \cdots < n_r$; and $u\alpha^{(m)}v$ contains an infinite number of such words. Thus words of the form $uz_q \cdots z_1$ must contain arbitrarily long strings of a's respectively b's as subwords. Thus one of the words w_i, say w_1 is of the form a^e for some $e > 0$. Similarly one of the words w_i, say w_2 is of the form $b^{e'}$, $e' > 0$. Let x_0 be a word in K'. Then the word $uw_2w_1x_0y_1y_2v$ is in η' and contains a subword ba. This is impossible. Hence Case 2 cannot occur.

Since the set K' is either in Case 1 or Case 2, and both alternatives yield contradictions; it follows that the set K' cannot exist. Therefore the integer m does not exist. Thus the definable set η' is not sequentially definable.

It would be interesting to select one of the definable sets occurring in ALGOL and show that it is not sequentially definable. We do not know if such a set exists because of the massive structure of ALGOL. We strongly suspect that the set of arithmetic expressions in ALGOL is not sequentially definable. Since it requires more than 15 equations to define the set of arithmetic expressions, we have been unable to explicitly determine this set preliminary to showing it is not sequentially definable.

5. *Functions Which Produce Sequentially Definable Sets*

We have just seen that there exist definable sets which are not sequentially definable. In Theorem 4 below we shall show that if all the coefficients in a n-tuple standard function f commute with each other, then each of the coordinates in the minimal fixed point of f is sequentially definable. We shall accomplish this by showing that we may (i) restrict the variables to a family of sets in which commutativity holds; (ii) replace a function by one in which one of the variables is linear; (iii) "solve" the function obtained in (ii), i.e., express one of the variables in terms of the remaining; (iv) replace one of the variables in each of the remaining functions by the "solution" obtained in (iii), thereby obtaining one less function and one less variable; (v) repeat (i)–(iv) until there is just one function and one variable left; (vi) solve this last function, thereby obtaining the last coordinate in the minimal fixed point of the original n-tuple function; and (vii) evaluate each of the remaining coordinates of the original n-tuple function by repeated substitution.

In order to carry out this seven step program we shall need some auxiliary concepts and results.

Notation. As in footnote 5, for each set A let $A^* = \bigcup_{i=0}^{\infty} A^i$.

Definition. A *star polynomial* $f(\xi^{(1)}, \cdots, \xi^{(n)})$ is a function constructed from the finite number of set variables $\xi^{(1)}, \cdots, \xi^{(n)}$, each $\xi^{(i)}$ ranging over all subsets of θ, and having the following form: f is the sum of a finite number of terms, $f = \sum_{i=1}^{s} \Pi_i$, each term Π_i having the form $\Pi_i = \delta_1 \cdots \delta_r$ (r varying). Here each δ_i is either a sequentially definable set; or δ_i is one of the variables $\xi^{(1)}, \cdots, \xi^{(n)}$; or $\delta_i = \tau^*$, where τ has the form $\tau = \gamma_1 \cdots \gamma_t$ (t varying), each γ_j being either a sequentially definable set or one of the variables $\xi^{(1)}, \cdots, \xi^{(n)}$. Each of the δ_i or γ_j which is sequentially definable is called a *coefficient* of f.

For example, $f = A(\xi^{(1)}B\xi^{(2)})^* + A$ is a star polynomial, but $f = A[(\xi^{(1)}B)^*\xi^{(2)}]^* + B$ is not, A and B being sequentially definable sets.

Definition. A family Δ of subsets of θ is called a *base* if

(i) ϵ is in Δ;

(ii) for each pair of sets A and B in Δ, AB is in Δ and $AB = BA$; and

(iii) for each sequence $\{A_i\}$ of sets in Δ, $\bigcup_i A_i$ is in Δ.

From (iii) and (ii) it follows that for each pair A and B of sets in Δ, $A + B$ is in Δ as well as A^*.

In the sequel Δ will be a fixed base.

Definition. A function $f(\xi^{(1)}, \cdots, \xi^{(n)})$ is said to be a *base polynomial*, acting on Δ, if (i) f is a star polynomial, with each variable $\xi^{(i)}$ restricted to Δ, and (ii) each coefficient of f is in Δ.

LEMMA 7. *Let $f(\xi^{(1)}, \cdots, \xi^{(n)})$ be a function built up from the set variables $\xi^{(1)}, \cdots, \xi^{(n)}$, each restricted to Δ, and constants in Δ, each sequentially definable; by using the operations of $+$, $\cdot$, and * a finite number of times. Then*

$$f(\xi^{(1)}, \cdots, \xi^{(n)}) = g_1(\xi^{(1)}, \cdots, \xi^{(n)})\xi^{(1)} + g_2(\xi^{(2)}, \cdots, \xi^{(n)}),$$

where g_1 and g_2 are base polynomials acting on Δ. Furthermore, the coefficients of g_1 and g_2 are formed from the coefficients of f together with ϵ by using the operations of $+$, $\cdot$, and * a finite number of times.

PROOF. We first reduce f to a base polynomial $h(\xi^{(1)}, \cdots, \xi^{(n)})$, acting on Δ, by means of the identities:

$$(1)\ \ (A^*)^m = (A^*)^* = A^*,$$

$$(2)\ (A + B)^* = A^*B^*,$$

$$(3)\ (A^*B)^* = \epsilon + A^*BB^*,$$

where A and B are sets in Δ. These identities are easily derived from the definition $A^* = \bigcup_0^\infty A^i$ and the commutativity of A and B.

The first step in reducing f to h is to apply identity (2) until no $+$ appears within the range of a *. The result is a sum, each term of which involves the operations of $\cdot$ and * only.

We now introduce the idea of a "nest" with its "depth" and "width." The * of a product of variables and constants is defined to be a nest of depth 0 and width 0. An expression $(\mu_1 \cdots \mu_i\nu)^*$, where the μ's are nests and ν is a product (perhaps empty) of constants and variables, is defined to be a nest of width i and depth $1 + \max\{\text{depth of } \mu_j/j\}$. Clearly the width is 0 if and only if the depth is.

Now if a term of the sum has as a factor a nest of width greater than 1, or of width 1 and a nonempty ν, we apply identity (3) to replace the term $(\mu_1 \cdots \mu_i\nu)^*\gamma$ by $\gamma + \mu_1 \cdots \mu_i\nu(\mu_2 \cdots \mu_i\nu)^*\gamma$, i.e., by two terms, each a product of nests, each nest (except γ) of which has either smaller depth, or the same depth and smaller width, than the original factor.

When no term contains as a factor a nest with width greater than or equal to 1 and nonempty ν, then we apply identity (1) to all nests with width 1 and empty ν, reducing their depth by 1. When no such nests remain, we repeat the process of the preceding paragraph. The alternation of these processes must end with no nests remaining of width or depth greater than 0. This is exactly the form of a base polynomial $h(\xi^{(1)}, \cdots, \xi^{(n)})$ over Δ.

We now form from h the two base polynomials g_1 and g_2. Due to the commutativity of the sets in Δ, we can write each term of h which has $\xi^{(1)}$ as a factor as $\gamma\xi^{(1)}$. For each such term, place the corresponding γ in g_1. Place each term of h in which $\xi^{(1)}$ does not occur in g_2. This leaves only those terms of h in which $\xi^{(1)}$ occurs only inside the nests (the nests being factors). Using the identity

$$(AB \cdots)^* = \epsilon + (AB \cdots)(AB \cdots)^*,$$

we generate from each such term in the obvious way two new terms. One of these is of the form $\gamma\xi^{(1)}$ and we place γ in g_1; the other (obtained from ϵ) has one fewer nests (as factors) containing $\xi^{(1)}$ than the original term. This process continues until no nests remain, and this residue we place in g_2. From the construction it is clear that $h = g_1\xi^{(1)} + g_2(\xi^{(2)}, \cdots, \xi^{(n)})$, with g_1 and g_2 being base polynomials acting on Δ. From the method of construction, it is also clear that the coefficients of g_1 and g_2 have the asserted property. Q.E.D.

To illustrate the procedure given in Lemma 7, let $f = [(A\xi^*)^* + (B\eta)^*\xi^* + \eta^*]^*$, where ξ and η represent $\xi^{(1)}$ and $\xi^{(2)}$ respectively. Applying identity (2) twice we get

$$f = [(A\xi^*)^* + (B\eta)^*\xi^*]^*(\eta^*)^* = [(A\xi^*)^*]^*[(B\eta)^*\xi^*]^*(\eta^*)^*.$$

Now $[(B\eta)^*\xi^*]^*$ is the only nest of width >1; and there is no nest of width 1 and nonempty ν. Applying identity (3) we get

$$[(B\eta)^*\xi^*]^* = \epsilon + (B\eta)^*\xi^*(\xi^*)^*,$$

whence

$$f = [(A\xi^*)^*]^*(\eta^*)^* + [(A\xi^*)^*]^*(\eta^*)^*(B\eta)^*\xi^*(\xi^*)^*.$$

There is no term which has as a factor a nest of either width greater than or equal to 1 and nonempty ν. Applying identity (1) to all nests with width 1 and empty ν we get

$$f = (A\xi^*)^*\eta^* + (A\xi^*)^*\eta^*(B\eta)^*\xi^*\xi^*.$$

There is only one nest of width 1 and nonempty ν, and that is $(A\xi^*)^*$ (occurring twice). Write $(A\xi^*)^* = (\xi^*A)^*$. By identity (3),

$$(\xi^*A)^* = \epsilon + \xi^*AA^*.$$

Then

$$f = \eta^* + \xi^*AA^*\eta^* + \eta^*(B\eta)^*\xi^*\xi^* + \eta^*(B\eta)^*\xi^*\xi^*\xi^*AA^*.$$

This is the base polynomial h.

We now determine g_1 and g_2. Place η^* in g_2. Using the identity $\xi^* = \epsilon + \xi\xi^*$, replace $\xi^*AA^*\eta^*$ by $AA^*\eta^* + \xi\xi^*AA^*\eta^*$. Place $AA^*\eta^*$ in g_2 and $\xi^*AA^*\eta^*$ in g_1. Replace $\eta^*(B\eta)^*\xi^*\xi^*$ by $\eta^*(B\eta)^*\xi^* + \eta^*(B\eta)^*\xi^*\xi\xi^*$. Place $\eta^*(B\eta)^*\xi^*\xi^*$ in g_1. Replace $\eta^*(B\eta)^*\xi^*$ by $\eta^*(B\eta)^* + \eta^*(B\eta)^*\xi\xi^*$. Place $\eta^*(B\eta)^*$ in g_2 and $\eta^*(B\eta)^*\xi^*$ in g_1. Similarly the last term, $\eta^*(B\eta)^*\xi^*\xi^*\xi^*AA^*$, in h is ultimately replaced by $\eta^*(B\eta)^*\xi^*\xi^*\xi\xi^*AA^* + \eta^*(B\eta)^*\xi^*\xi\xi^*AA^* + \eta^*(B\eta)^*\xi\xi^*AA^* + \eta^*(B\eta)^*AA^*$. This leads to the following forms for g_1 and g_2:

$$g_1 = \xi^*AA^*\eta^* + \eta^*(B\eta)^*\xi^*\xi^* + \eta^*(B\eta)^*\xi^* + \eta^*(B\eta)^*\xi^*AA^* + \eta^*(B\eta)^*\xi^*\xi^*AA^* + \eta^*(B\eta)^*\xi^*\xi^*\xi^*AA^*,$$

and

$$g_2 = \eta^* + AA^*\eta^* + \eta^*(B\eta)^* + \eta^*(B\eta)^*AA^*.$$

These are not the "simplest" forms for g_1 and g_2. For example, observing that $\xi^*\xi^* = \xi^*$ we get

$$g_1 = AA^*\xi^*\eta^* + (B\eta)^*\xi^*\eta^* + AA^*(B\eta)^*\xi^*\eta^*$$

and

$$g_2 = \eta^* + AA^*\eta^* + (B\eta)^*\eta^* + AA^*(B\eta)^*\eta^*.$$

If we had made this observation at an earlier stage we would have reduced the computation considerably.

Let $f_1, \cdots, f_n$ be n functions, each constructed from sets in Δ and the variables $\xi^{(1)}, \cdots, \xi^{(n)}$, each $\xi^{(i)}$ ranging over all subsets of θ; by using the operations of $+$, $\cdot$, and * a finite number of times. It is readily seen that Theorem 1 is still valid. Furthermore, each $\alpha_k^{(i)}$ is in Δ, whence so is each $\alpha^{(i)} = \bigcup \alpha_k^{(i)}$. Thus as far as finding minimal fixed points is concerned, each set variable may be restricted to sets in Δ. This is hereafter done. This enables us to use Lemma 7 and also to assume the commutativity of all sets under discussion.

LEMMA 8. *Let α be the minimal fixed point of $f(\xi^{(1)}, \cdots, \xi^{(n)}) = (f_1, \cdots, f_n)$, where each f_i is a base polynomial acting on Δ. Suppose that $f_1 = g_1(\xi^{(1)}, \cdots, \xi^{(n)})\xi^{(1)} + g_2(\xi^{(2)}, \cdots, \xi^{(n)})$, where g_1 and g_2 are base polynomials acting on Δ. Let $h_1(\xi^{(1)}, \cdots, \xi^{(n)}) = g_1[g_2(\xi^{(2)}, \cdots, \xi^{(n)}), \xi^{(2)}, \cdots, \xi^{(n)}]\xi^{(1)} + g_2(\xi^{(2)}, \cdots, \xi^{(n)})$, and for $i \geqq 2$ let $h_i = f_i$. Then α is the minimal fixed point of $h = (h_1, \cdots, h_n)$.*

PROOF. Let $\beta = (\beta^{(1)}, \cdots, \beta^{(n)})$ be the minimal fixed point of h. Let $\alpha = (\alpha^{(1)}, \cdots, \alpha^{(n)})$. Let $\alpha_k = (\alpha_k^{(1)}, \cdots, \alpha_k^{(n)})$ and $\beta_k = (\beta_k^{(1)}, \cdots, \beta_k^{(n)})$ have their usual significance. To prove the lemma it suffices to show that $\alpha^{(i)} \subseteq \beta^{(i)}$ and $\beta^{(i)} \subseteq \alpha^{(i)}$ for each i. The proof of the latter is straightforward and is left to the reader. The proof of the former is delicate and is now given.

Clearly $\alpha_0^{(i)} = \beta_0^{(i)} \subseteq \beta^{(i)}$ for each i. Let $D = g_2(\beta^{(2)}, \cdots, \beta^{(n)})$. Then $\alpha_0^{(1)} \subseteq D$. As β is a fixed point of h, $\beta^{(1)} = g_1[D, \beta^{(2)}, \cdots, \beta^{(n)}]\beta^{(1)} + D$, whence $D \subseteq \beta^{(1)}$. Suppose that $\alpha_k^{(i)} \subseteq \beta^{(i)}$ for each i, and that $\alpha_k^{(1)}\beta^{(1)} \subseteq D\beta^{(1)}$. For $i \geqq 2$, $\alpha_{k+1}^{(i)} = f_i(\alpha_k) \subseteq f_i(\beta) = \beta^{(i)}$. It remains to show that $\alpha_{k+1}^{(1)} \subseteq \beta^{(1)}$ [for then $\alpha^{(1)} = \bigcup_k \alpha_k^{(1)} \subseteq \beta^{(1)}$] and that $\alpha_{k+1}^{(1)}\beta^{(1)} \subseteq D\beta^{(1)}$.

Let $\delta_1(\xi^{(1)}, \cdots, \xi^{(n)}), \cdots, \delta_t(\xi^{(1)}, \cdots, \xi^{(n)})$ be any finite sequence, where each $\delta_i(\xi^{(1)}, \cdots, \xi^{(n)})$ is either a constant in Δ or one of the variables $\xi^{(1)}, \cdots, \xi^{(n)}$ restricted to Δ. Define $\tau(\xi^{(1)}, \cdots, \xi^{(n)})$ as the product $\tau(\xi^{(1)}, \cdots, \xi^{(n)}) = \delta_1 \cdots \delta_t$. Then

$$\tau(\alpha_k)^*\beta^{(1)} = \bigcup_{j=0}^{\infty} \tau(\alpha_k)^j\beta^{(1)} = \bigcup_{j=0}^{\infty} \delta_1(\alpha_k)^j \cdots \delta_t(\alpha_k)^j\beta^{(1)}.$$

For each i define γ_i to be the set $\delta_i(\alpha_k)$ if $\delta_i(\xi) \not\equiv \xi^{(1)}$ and D if $\delta_i(\xi) \equiv \xi^{(1)}$. By commutativity and repeated application of the induction hypothesis $\alpha_k^{(1)}\beta^{(1)} \subseteq D\beta^{(1)}$, it follows that

$$\bigcup_{j=0}^{\infty} \delta_1(\alpha_k)^j \cdots \delta_t(\alpha_k)^j\beta^{(1)} \subseteq \bigcup_{j}^{\infty} \gamma_1^j \cdots \gamma_t^j\beta^{(1)} = \tau(D, \alpha_k^{(2)}, \cdots, \alpha_k^{(n)})^*\beta^{(1)}.$$

Thus

$$\tau(\alpha_k^{(1)}, \cdots, \alpha_k^{(n)})^*\beta^{(1)} \subseteq \tau(D, \alpha_k^{(2)}, \cdots, \alpha_k^{(n)})^*\beta^{(1)}. \tag{7}$$

Consider the function $g_1(\xi^{(1)}, \cdots, \xi^{(n)})$. Since g_1 is a base polynomial acting on Δ, $g_1(\xi^{(1)}, \cdots, \xi^{(n)}) = \sum_{i=1}^{s} \Pi_i(\xi^{(1)}, \cdots, \xi^{(n)})$, where $\Pi_i(\xi^{(1)}, \cdots, \xi^{(n)}) = \nu_1 \cdots \nu_r$ (r varying). Here each ν_j is either a constant in Δ; or one of the variables $\xi^{(1)}, \cdots, \xi^{(n)}$ restricted to Δ; or a function $\tau(\xi^{(1)}, \cdots, \xi^{(n)})^*$, $\tau(\xi^{(1)}, \cdots, \xi^{(n)})$ being defined as in the preceding paragraph. Then

$$\begin{aligned} g_1(\alpha_k^{(1)}, \cdots, \alpha_k^{(n)})\beta^{(1)} &= \sum^{s} \Pi_i(\alpha_k^{(1)}, \cdots, \alpha_k^{(n)})\beta^{(1)} \\ &\subseteq \sum^{s} \Pi_i(D, \alpha_k^{(2)}, \cdots, \alpha_k^{(n)})\beta^{(1)} \end{aligned}$$

by repeated application of the induction hypothesis $\alpha_k^{(1)}\beta^{(1)} \subseteq D\beta^{(1)}$, the commutativity of the sets, and (7). Thus

$$\begin{aligned} g_1(\alpha_k^{(1)}, \cdots, \alpha_k^{(n)})\beta^{(1)} &\subseteq g_1(D, \alpha_k^{(2)}, \cdots, \alpha_k^{(n)})\beta^{(1)} \subseteq g_1(D, \beta^{(2)}, \cdots, \beta^{(n)})\beta^{(1)} \\ &\subseteq \beta^{(1)}. \end{aligned}$$

Consider $\alpha_{k+1}^{(1)}$. It follows that

$$\begin{aligned}\alpha_{k+1}^{(1)} &= g_1(\alpha_k^{(1)}, \cdots, \alpha_k^{(n)})\alpha_k^{(1)} + g_2(\alpha_k^{(2)}, \cdots, \alpha_k^{(n)})\\ &\subseteq g_1(\alpha_k)\beta^{(1)} + g_2(\beta^{(2)}, \cdots, \beta_k^{(n)})\\ &\subseteq \beta^{(1)} + \beta^{(1)} = \beta^{(1)},\end{aligned}$$

one of the desired relations.

Finally,

$$\begin{aligned}\alpha_{k+1}^{(1)}\beta^{(1)} &= g_1(\alpha_k)\alpha_k^{(1)}\beta^{(1)} + g_2(\alpha_k^{(2)}, \cdots, \alpha_k^{(n)})\beta^{(1)}\\ &\subseteq Dg_1(\alpha_k)\beta^{(1)} + g_2(\beta^{(2)}, \cdots, \beta^{(n)})\beta^{(1)} \subseteq D\beta^{(1)} + D\beta^{(1)}\\ &= D\beta^{(1)}.\end{aligned}$$

Q.E.D.

LEMMA 9. *Let α be the minimal fixed point of $f(\xi^{(1)}, \cdots, \xi^{(n)}) = (f_1, \cdots, f_n)$ where each f_i is a base polynomial acting on Δ. Suppose that $f_1 = g_1(\xi^{(2)}, \cdots, \xi^{(n)})\xi^{(1)} + g_2(\xi^{(2)}, \cdots, \xi^{(n)})$, where g_1 and g_2 are base polynomials acting on Δ. Let $h_1(\xi^{(2)}, \cdots, \xi^{(n)}) = g_2 g_1^*$. For $i \geqq 2$, let $h_i = f_i(h_1, \xi^{(2)}, \cdots, \xi^{(n)})$. Then α is the minimal fixed point of $(h_1, \cdots, h_n)$.*

The proof of the above lemma involves the, by now, familiar line of reasoning. Accordingly, it is omitted.

THEOREM 4. *Let $f = (f_1, \cdots, f_n)$, where each $f_i = f_i(\xi^{(1)}, \cdots, \xi^{(n)})$ is a polynomial. Suppose that all coefficients are sequentially definable and commute with each other. Then each coordinate in the minimal fixed point of f is a sequentially definable set, and, moreover, can be constructed from the coefficients in f together with ϵ by using the operations of $+$, $\cdot$, and * a finite number of times.*

PROOF. Let Δ be the smallest family of sets which contains all the coefficients if f together with ϵ, and is closed under the operations of product and denumerable union. Since commutativity is preserved under product and denumerable union, by transfinite induction it is readily seen that each two sets in Δ commute, i.e., Δ is a base.

Denote by $\alpha = (\alpha^{(1)}, \cdots, \alpha^{(n)})$ the minimal fixed point of f. From the discussion prior to Lemma 8 it may be assumed that each f_i is a base polynomial acting on Δ. Consider the following statement:

> Let $f = (f_1, \cdots, f_n)$, where each $f_i = f_i(\xi^{(1)}, \cdots, \xi^{(n)})$ is a base polynomial acting on Δ. Then each coordinate in the minimal fixed point $\alpha = (\alpha^{(1)}, \cdots, \alpha^{(n)})$ of f is a sequentially definable set, and, moreover, can be constructed from the coefficients in f together with ϵ by using the operations of $+$, $\cdot$, and * a finite number of times. (P_n)

To prove the theorem it is sufficient to show that (P_n) is true for all n. A proof of (P_n) will now be given by mathematical induction.

Suppose that $n = 1$, i.e., $f(\xi^{(1)}) = (f_1)$. By Lemma 7, $f_1(\xi^{(1)}) = g_1(\xi^{(1)})\xi^{(1)} + K$, where $g_1(\xi^{(1)})$ is a base polynomial acting on Δ; and K and the coefficients of g_1 are formed from the coefficients in f_1 together with ϵ by using the operations of $+$, $\cdot$, and * a finite number of times. By Lemma 8, $\alpha^{(1)}$ is the minimal fixed point of $h_1(\xi^{(1)})$, where $h_1(\xi^{(1)}) = g_1(K)\xi^{(1)} + K$. Then $\alpha^{(1)} = g_1(K)^*K$. Since the family of sequentially definable sets is closed under $+$, $\cdot$, and *, the set $\alpha^{(1)}$ is sequentially definable. Furthermore, $\alpha^{(1)}$ is constructed from the coefficients of f_1 in the desired form. Therefore (P_1) is true.

Continuing by induction suppose that (P_n) is true for all integers less than r. Consider the case when $n = r$. By Lemma 7 there exist base polynomials $g_1(\xi^{(1)}, \cdots, \xi^{(n)})$ and $g_2(\xi^{(2)}, \cdots, \xi^{(n)})$ such that

$$f_1(\xi^{(1)}, \cdots, \xi^{(n)}) = g_1(\xi^{(1)}, \cdots, \xi^{(n)})\xi^{(1)} + g_2(\xi^{(2)}, \cdots, \xi^{(n)}).$$

Furthermore, the coefficients of g_1 and g_2 are formed from the coefficients in f together with ϵ by using the operations of $+$, $\cdot$, and * a finite number of times. By Lemma 8, α is the minimal fixed point of $(f_{n+1}, f_2, \cdots, f_n)$, where

$$f_{n+1}(\xi^{(1)}, \cdots, \xi^{(n)}) = g_1[g_2(\xi^2, \cdots, \xi^{(n)}), \xi^{(2)}, \cdots, \xi^{(n)}]\xi^{(1)} + g_2(\xi^{(2)}, \cdots, \xi^{(n)}).$$

By Lemma 7 it may be assumed that (i) $g_1(g_2, \xi^{(2)}, \cdots, \xi^{(n)})$ is a base polynomial $g_3(\xi^{(2)}, \cdots, \xi^{(n)})$, and (ii) the coefficients of g_3 are derived from those of g_1 and g_2, thus from those of f_1, in the desired manner. Then

$$f_{n+1}(\xi^{(1)}, \cdots, \xi^{(n)}) = g_3(\xi^{(2)}, \cdots, \xi^{(n)})\xi^{(1)} + g_2(\xi^{(2)}, \cdots, \xi^{(n)}).$$

By Lemma 9, α is the minimal fixed point of $(h_1, \cdots, h_n)$, where $h_1 = g_3{}^*g_2$ and $h_i = f_i(g_3{}^*g_2, \xi^{(2)}, \cdots, \xi^{(n)})$ for $i \geqq 2$. By Lemma 7, it may be assumed that each $h_i(\xi^{(2)}, \cdots, \xi^{(n)})$ is in the form of a base polynomial the coefficients of which are formed from those of f_i, g_2, g_3, thus from the coefficients in f, in the desired manner.

Consider the minimal fixed point $\beta = (\beta^{(2)}, \cdots, \beta^{(n)})$ of $h(\xi^{(2)}, \cdots, \xi^{(n)}) = (h_2, \cdots, h_n)$. Since α is the minimal solution to the equation

$$[h_1(\xi^{(2)}, \cdots, \xi^{(n)}), h_2(\xi^{(2)}, \cdots, \xi^{(n)}), \cdots, h_n(\xi^{(2)}, \cdots, \xi^{(n)})] = (\xi^{(1)}, \cdots, \xi^{(n)}),$$

it follows that (i) $\alpha^{(1)} = h_1(\alpha^{(2)}, \cdots, \alpha^{(n)})$, and (ii) $(\alpha^{(2)}, \cdots, \alpha^{(n)})$ is a fixed point of $h(\xi^{(2)}, \cdots, \xi^{(n)})$. Therefore $(\beta^{(2)}, \cdots, \beta^{(n)}) \subseteq (\alpha^{(2)}, \cdots, \alpha^{(n)})$. Since $h_i(\beta^{(2)}, \cdots, \beta^{(n)}) = \beta^{(i)}$ for $i \geqq 2$, it follows that $(h_1(\beta^{(2)}, \cdots, \beta^{(n)}), \beta^{(2)}, \cdots, \beta^{(n)})$ is a fixed point of $(h_1, \cdots, h_n)$. Thus $\alpha^{(i)} \subseteq \beta^{(i)}$, whence $\alpha^{(i)} = \beta^{(i)}$, for $i \geqq 2$. In other words, $(\alpha^{(2)}, \cdots, \alpha^{(n)})$ is the minimal fixed point of $(h_2, \cdots, h_n)$.

Now $h(\xi^{(2)}, \cdots, \xi^{(n)})$ involves fewer than r functions, each of which is a base polynomial acting on Δ. By induction, (P_n) is true for $n < r$. Hence, for $i \geqq 2$, each $\alpha^{(i)}$ is sequentially definable and derived from the coefficients in h together with ϵ, thus from the coefficients in f together with ϵ, in the desired fashion. As $\alpha^{(1)} = h_1(\alpha^{(2)}, \cdots, \alpha^{(n)})$, the same is true for $\alpha^{(1)}$. Therefore (P_r) is true and the theorem is proved.

If the coefficients in f are powers of the same set then the coefficients commute. Thus

COROLLARY 1. *If $f = (f_1, \cdots, f_n)$ is a n-tuple standard function in which all the coefficients are powers of the same set, say A; then each coordinate in the minimal fixed point of f is sequentially definable and is obtainable from A together with ϵ by using the operations of $+$, $\cdot$, and * a finite number of times. Thus each coordinate is a regular set.*

Corollary 1 is obviously satisfied if the alphabet Σ consists of just one letter.

Thus each definable set is a regular set. The reverse, of course, is also true. Now it is a well-known but unpublished result that a set $A = \{a^n \mid n \text{ in } \lambda\}$ is regular if and only if λ is an ultimately periodic set of non-negative integers.[10]. From this there occurs

COROLLARY 2. *If the alphabet Σ consists of just one letter then the (sequentially) definable sets are equivalent to the regular sets, i.e.,* $\{a^n \mid n \text{ in } \lambda\}$ *is (sequentially) definable if and only if λ is an ultimately periodic set of nonnegative integers.*

Remark. A careful examination of Theorem 4 and the lemmas upon which it depends will reveal that the coordinates of the minimal fixed point are constructed only from the coefficients in f which are not ϵ. In fact, the entire paper could have been developed without introducing the empty word ϵ. This would mean, of course, that no (sequentially) definable set would contain the empty word ϵ; and would necessitate several minor changes, such as defining A^* to be $\bigcup_{i=1}^{\infty} A^i$.

To illustrate Theorem 4 let

$$f_1 \equiv A^2\xi^2 + A^3\xi\nu + A^4 \text{ and } f_2 \equiv A^4\xi + A^3\xi\nu + A^2\nu + A^2,$$

where A is some finite set of words and ξ and ν represent $\xi^{(1)}$ and $\xi^{(2)}$ respectively. We shall determine the minimal fixed point $(\bar{\xi}, \bar{\nu})$ of $f = (f_1, f_2)$.

Now f_1 becomes

$$(A^2\xi + A^3\nu)\xi + A^4. \tag{8}$$

The function in (1) is transformed into

$$[A^2A^4 + A^3\nu]\xi + A^4. \tag{9}$$

Then $(\bar{\xi}, \bar{\nu})$ is the minimal fixed point of (f_3, f_4), where

$$f_3(\xi, \nu) \equiv (A^3\nu + A^6)^*A^4$$

and

$$f_4(\nu) \equiv A^4(A^3\nu + A^6)^*A^4 + A^3\nu(A^3\nu + A^6)^*A^4 + A^2\nu + A^2.$$

Then $\bar{\nu}$ is the minimal fixed point of $f_4(\nu)$. Rewriting $f_4(\nu)$ in the form $g_1(\nu)\nu + K$, we have

$$\begin{aligned} f_4 &= A^8(A^3\nu)^*(A^6)^* + A^7(A^3\nu + A^6)^*\nu + A^2\nu + A^2 \\ &= A^8(A^6)^* + A^8(A^6)^*A^3\nu(A^3\nu)^* + A^7(A^3\nu + A^6)^*\nu + A^2\nu + A^2 \qquad (10) \\ &= [A^{11}(A^6)^*(A^3\nu)^* + A^7(A^3\nu + A^6)^* + A^2]\nu + A^8(A^6)^* + A^2. \end{aligned}$$

The function in (10) is transformed into

$$f_5 = H\nu + K,$$

where $H = A^{11}(A^6)^*(A^3[A^8(A^6)^* + A^2])^* + A^7(A^3[A^8(A^6)^* + A^2] + A^6)^* + A^2$ and $K = A^8(A^6)^* + A^2$. Then $\bar{\nu} = H^*K$ and $\bar{\xi} = (A^3H^*K + A^6)^*A^4$. If so desired, Lemma 7 could be employed to reduce the depth and width of the nests appearing in $\bar{\nu}$ and $\bar{\xi}$ (considering powers of A as variables of course).

[10] A proof of this fact easily follows from the last sentence in footnote 7.

There is another system of functions of interest which yields sequentially definable sets whereas its form suggests definable sets. This is the case when $f(\xi^{(1)}, \cdots, \xi^{(n)}) = (f_1, \cdots, f_n)$, where each f_i is linear on the right, i.e., $f_i = \sum_{j=1}^{n} A_{i,j}\xi^{(j)} + A_i$, and all coefficients are sequentially definable. It can be shown that each coordinate in the minimal fixed point $\alpha = (\alpha^{(1)}, \cdots, \alpha^{(n)})$ of f is sequentially definable and, moreover, is constructed from the coefficients in f by using the operations of $+$, $\cdot$, and * a finite-number of times. A proof can be given which involves the notion of *self-embedding* as defined in [6]. Another proof can be given which depends upon a result in [3]. The outline of an alternative proof, which involves eliminating the variables one at a time, is as follows. From f_1 one gets

$$\xi^{(1)} = A_{1,1}^*\left(\sum_{j=2}^{n} A_{1,j}\xi^{(j)} + A_1\right) = \sum_{j=2}^{n} A_{1,1}^*A_{1,j}\xi^{(j)} + A_{1,1}^*A_1 .$$

Replacing $\xi^{(1)}$ by $\sum_{j=2}^{n} A_{1,1}^*A_{1,j}\xi^{(j)} + A_{1,1}^*A_1$ in $f_2, \cdots, f_n$, one obtains $n - 1$ functions $g_2(\xi^{(2)}, \cdots, \xi^{(n)}), \cdots, g_n(\xi^{(2)}, \cdots, \xi^{(n)})$, each g_i being linear on the right and having all coefficients sequentially definable. The minimal fixed point of $(g_2, \cdots, g_n)$ is $(\alpha^{(2)}, \cdots, \alpha^{(n)})$. This eliminates one of the variables and one of the functions. The procedure is continued until there is just one variable and one function. Then $\alpha^{(n)}$ is determined. By repeated substitution, $\alpha^{(n-1)}, \cdots, \alpha^{(1)}$ are obtained in turn.

The result and the outline of proof described in the preceding paragraph are also valid, with minor changes of course, if in $f = (f_1, \cdots, f_n)$ each f_i is linear on the left, i.e., $f_i = \sum_{j=1}^{n} \xi^{(j)}A_{i,j} + A_i$.

There are no functions in ALGOL which illustrate either Theorem 4 or the situations considered in the two preceding paragraphs.

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