An Event Structure Semantics for P/T Contextual Nets: Asymmetric Event Structures*

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Abstract. We propose an event based semantics for *contextual nets*, i.e. an extension of Place/Transition Petri nets where transitions can also have *context* conditions, modelling resources that can be read without being consumed. The result is a generalization of Winskel's work on safe nets: the event based semantics is given at categorical level via a chain of coreflections leading from the category **WS-CN** of weakly safe contextual nets to the category **Dom** of finitary prime algebraic domains. A fundamental rôle is played by the notion of *asymmetric event structures* that generalize Winskel's prime event structures, following an idea similar to that of "possible flow" introduced by Pinna and Poignè. Asymmetric event structures have the usual causal relation of traditional prime event structures, but replace the symmetric conflict with a relation modelling *asymmetric conflict* or *weak causality*. Such relation allows one to represent the new kind of dependency between events arising in contextual nets, as well as the usual symmetric conflict. Moreover it is used in a non-trivial way in the definition of the ordering of configurations, which is different from the standard set-inclusion.

1 Introduction

Contextual nets, as introduced in [14], extend classical Petri nets, a formalism for the specification of the behaviour of concurrent systems, with the possibility of handling contexts: in a contextual net transitions can have not only preconditions and postconditions, but also *context* conditions, that, intuitively, specify something which is necessary for the transition to be fired, but is not affected by the firing of the transition. In other words, a context can be thought of as an item which is *read but not consumed* by the transition, in the same way as preconditions can be considered as being read and consumed and postconditions

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being simply written. Consistently with this view, the same token can be used as context by many transitions at the same time and with multiplicity greater than one by the same transition. Context conditions of [14] are also called *test arcs* in [5], *activator arcs* in [10] or *read arcs* in [18, 19].

The possibility of faithfully representing the "reading of resources" allows contextual nets to model a lot of concrete situations more naturally than classical nets. In recent years they have been used to model concurrent access to shared data (e.g. reading in a database) [17, 7], to provide a concurrent semantics to concurrent constraint (CC) programs [13], to model priorities [9], to specify a net semantics for the π-calculus [3]. Moreover they have been studied for their connections with another powerful formalism for the representation of concurrent computations, namely graph grammars [14, 6].

In this paper we consider *marked contextual P/T nets* (shortly *c-nets*), that following the lines suggested in [14] for C/E systems, add contexts to classical P/T nets. The problem of giving a truly concurrent semantics based on (deterministic) processes has been faced by various authors (see, e.g., [9, 14, 4, 19]). Each process of a c-net records the events occurring in a *single* computation of the net and the relations existing between such events.

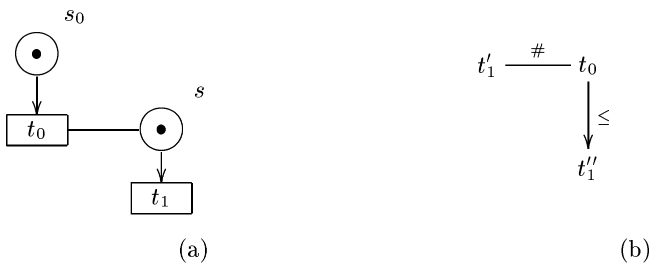

Fig. 1. A simple contextual net and a prime event structure representing its behaviour.

Here we provide (weakly safe) c-nets with a truly concurrent event structure semantics following another classical approach. Generalizing Winskel's construction for safe nets [20], we associate to each c-net an event structure that describes *all* the possible behaviours of the net. Recall that *prime event structures (PES)* are a simple event based model of (concurrent) computations in which events are considered as atomic, indivisible and instantaneous steps, which can appear only once in a computation. An event can occur only after some other events (its causes) have taken place and the execution of an event can inhibit the execution of other events. This is formalized via two binary relations: *causality*, modelled by a partial order relation and *conflict*, modeled by a symmetric and irreflexive relation, hereditary w.r.t. causality. When working with c-nets the main critical point is represented by the fact that the presence of context conditions leads to *asymmetric conflicts* or *weak dependencies* between events. To understand this basic concept, consider two transitions t_0 and t_1 such that the same place s is a context for t_0 and a precondition for t_1. Following [14], such a situation is represented pictorially as in Fig. 1.(a), i.e., non-directed arcs are used to represent context conditions. The possible firing sequences are t_0, t_1 and $t_0; t_1$, while $t_1; t_0$ is not allowed. This situation cannot be modelled in a direct way within a traditional prime event structure: t_0 and t_1 are neither in conflict nor concurrent nor causal dependent. Simply, as for a traditional conflict, the firing of t_1 prevents t_0 to be executed, so that t_0 can never follow t_1 in a computation. But the converse is not true, since t_0 *can* fire before t_1. This situation can be naturally interpreted as an *asymmetric conflict* between the two transitions. Equivalently, since t_0 precedes t_1 in any computation where both are fired, in such computations, t_0 acts as a cause of t_1. However, differently from a true cause, t_0 is not necessary for t_1 to be fired. Therefore we can also think of the relation between the two transitions as a *weak* form of *causal dependency.*

A reasonable way to encode this situation in a PES is to represent the firing of t_1 with two distinct mutually exclusive events (as shown in Fig. 1.(b)): t_1', representing the execution of t_1 that prevents t_0, thus mutually exclusive with t_0, and t_1'', representing the execution of t_1 after t_0 (caused by t_0). This encoding can be unsatisfactory since it leads to a "duplication" of events (e.g., see [1]). The events of the prime event structure associated to a system would not represent the elementary actions of the system, but the possible histories of such actions.

Several authors pointed out the inadequacy of event structures for faithfully modeling general concurrent computations and proposed alternative definitions of event structures (flow event structures [2], bundle event structures [11], prioritized event structures [8]). Asymmetric conflicts have been specifically treated by Pinna and Poigné in [15, 16], where the "operational" notion of event automaton suggests an enrichment of prime event structures and flow event structures with *possible causes.* The basic idea is that if e is a possible cause of e' then e can precede e' or it can be ignored, but the execution of e never follows e'. This is formalized by introducing an explicit subset of possible events in prime event structures or adding a "possible flow relation" in flow event structures. Similar ideas are developed, under a different perspective, in [8], where PES are enriched with a partial order relation modeling priorities between events.

In order to provide a more direct, event based representation of c-nets we introduce a new kind of event structure, called *asymmetric event structure (aES).* Despite of some differences in the definition and in the related notions, aES's can be seen as a generalization of event structures with possible events and of prioritized event structures. Besides of the usual causal relation ($\leq$) of a traditional prime event structure, an aES has a relation $\nearrow$, that allows us to specify the new situation analyzed above simply as $t_0 \nearrow t_1$. As just remarked, the same relation has two natural interpretations: it can be thought of as an asymmetric version of conflict or as a weak form of causality. We decided to call it *asymmetric conflict*, but the reader should keep in mind both views, since in some situations it will be preferable to refer to the *weak causality* interpretation. *Configurations* of an aES are then introduced and the set of configurations of an aES, ordered in a suitable way using the asymmetric conflict relation, turns out to be a finitary prime algebraic domain. We prove that such a construction extends to a functor from the category **aES** of asymmetric event structures to the category **Dom** of finitary prime algebraic domain, that establishes a coreflection between **aES** and **Dom**. Recalling that **Dom** is equivalent to the category **PES** of prime event structures we can recover a semantics in terms of traditional prime event structures.

The seminal work by Winskel presents an adjunction between event structures and a subclass of P/T nets, namely *safe* nets. Such a result is extended in [12] to the wider category of *weakly safe* nets, i.e. P/T nets in which the initial marking is a set and transitions can generate at most one token in each post-condition. Similarly, we restrict here to a (full) subcategory of contextual nets, called *weakly safe* c-nets and we show how, given a weakly-safe c-net N, an *unfolding* construction allows us to obtain an occurrence c-net $\mathcal{U}_a(N)$. i.e. an "acyclic c-net" that describes in a static way the behaviour of N, by expressing possible events and the dependency relations between them. The unfolding operation can be extended to a functor $\mathcal{U}_a$ from **WS-CN** to the category **O-CN** of occurrence c-net, that is right adjoint of the inclusion functor $\mathcal{I}_O$: **O-CN** $\rightarrow$ **WS-CN**.

Transitions of an occurrence c-net are related by causal dependency and asymmetric conflict, while mutual exclusion is a derived relation. Thus, the semantics of weakly safe c-nets given in terms of occurrence c-nets can be naturally abstracted to an aES semantics. Again this construction extends, at categorical level, to a coreflection from **aES** to **O-CN**.

Finally we exploit the coreflection between **aES** and **Dom**, to complete the chain of coreflections from **WS-CN** to **Dom**.

2 Asymmetric event structures

We stressed in the introduction that PES's (and in general Winskel's event structures) are too poor to model in a direct way the behaviour of models of computation allowing context sensitive firing of events, such as string, term and graph rewriting, and contextual nets. The fact that an event to be fired requires the presence of some resources that are not "consumed", but just read, leads to a new kind of dependency between events that can be seen as an asymmetric version of conflict or a weak form of causality. Technically speaking, the problem is essentially the axiom of event structures (see [20]) stating that the enabling relation $\vdash$ is "monotone" w.r.t. set inclusion:

$$A \vdash e \ \wedge \ A \subseteq B \ \wedge \ B \ \textit{consistent} \quad \Rightarrow \quad B \vdash e.$$

As a consequence the computational order between configurations is set inclusion, the idea being that if $A \subseteq B$ are finite configurations then starting from A we can reach B by performing the events in $B - A$. This means that the conflict is symmetric, i.e. it cannot be the case that the execution of an event e_1 prevents e_0 to be executed but e_0 can precede e_1 in a computation.

To faithfully represent the dependencies existing between events in such models, avoiding the unpleasant phenomenon of duplication of events (see Fig. 1), we generalize prime event structures by replacing the usual symmetric conflict relation with a new binary relation $\nearrow$, called *asymmetric conflict*. If $e_0 \nearrow e_1$ then the firing of e_1 inhibits e_0: the execution of e_0 may precede the execution of e_1 or e_0 can be ignored, but e_0 cannot follow e_1. By using the terminology of Pinna and Poigné [16], we can say that e_0 is a "possible" cause of e_1. Nicely, the symmetric binary conflict can be represented easily with cycles of asymmetric conflict. Therefore symmetric conflict will be a derived relation.

We first introduce some basic notations. Let $r \subseteq X \times X$ be a binary relation and let $Y \subseteq X$. Then r_Y denotes the restriction of r to $Y \times Y$, i.e. $r \cap (Y \times Y)$, r^+ denotes the transitive closure of r and r^* denotes the reflexive and transitive closure of r. We say that r is *well-founded* if it has no infinite descending chains, i.e. $\langle e_i \rangle_{i \in I\!N}$ with $e_{i+1}\, r\, e_i$, $e_i \neq e_{i+1}$, for all $i \in I\!N$. The relation r is *acyclic* if it has no "cycles" $e_0\, r\, e_1\, r\, \ldots\, r\, e_n\, r\, e_0$, with $e_i \in X$. In particular, if r is well-founded it has no (non-trivial) cycles. The powerset of X is denoted by $\mathbf{2}^X$, while $\mathbf{2}^X_{\mathit{fin}}$ denotes the set of finite subsets of X.

Definition 1 (asymmetric event structure). An *asymmetric event structure* (*aES*) is a tuple $G = \langle E, \leq, \nearrow \rangle$, where E is a set of *events* and $\leq$, $\nearrow$ are binary relations on E called *causality relation* and *asymmetric conflict* respectively, s.t.:

1. the relation $\leq$ is a partial order and $\lfloor e \rfloor = \{e' \in E : e' \leq e\}$ is finite for all $e \in E$;
2. the relation $\nearrow$ satisfies for all $e, e' \in E$:
   (a) $e < e' \quad \Rightarrow \quad e \nearrow e'$;[1]
   (b) $\nearrow_{\lfloor e \rfloor}$ is acyclic;[2]

If $e \nearrow e'$, accordingly to the double interpretation of $\nearrow$, we say that e *is prevented* by e' or e *weakly causes* e'. Moreover we say that e *is strictly prevented* by e' (or e *strictly weakly causes* e'), written $e \leadsto e'$, if $e \nearrow e'$ and $\neg(e < e')$.

The definition can be easily understood by giving a more formal account of the ideas presented at the beginning of the section. Let *Fired*(e) denote the fact that the event e has been fired in a computation and let *prec*(e, e') denote that e precedes e' in the computation. Then

$$e < e' \quad \stackrel{def}{\equiv} \quad \mathit{Fired}(e') \;\Rightarrow\; \mathit{Fired}(e) \;\wedge\; \mathit{prec}(e, e')$$
$$e \nearrow e' \quad \stackrel{def}{\equiv} \quad \mathit{Fired}(e) \wedge \mathit{Fired}(e') \;\Rightarrow\; \mathit{prec}(e, e')$$

Therefore $<$ represents a global order of execution, while $\nearrow$ determines an order of execution only locally, in each configuration (computation). Thus it is natural to impose $\nearrow$ to be an extension of $<$. Moreover if a set of events forms an asymmetric conflict cycle $e_0 \nearrow e_1 \nearrow \ldots \nearrow e_n \nearrow e_0$, then such events cannot appear

[1] With $e < e'$ we mean $e \leq e$ and $e \neq e'$.
[2] Equivalently, we can require $(\nearrow_{\lfloor e \rfloor})^+$ irreflexive. This implies that, in particular, $\nearrow$ is irreflexive.

in the same computation, otherwise the execution of each event should precede the execution of the event itself. This explains why we require the acyclicity of $\nearrow$, restricted to the causes $\lfloor e \rfloor$ of an event e. Otherwise not all causes of e can be executed in the same computation and thus e itself cannot be executed. The informal interpretation makes also clear that $\nearrow$ is *not* in general transitive. If $e \nearrow e' \nearrow e''$ it is not true that e must precede e'' when both fire. This holds only in a computation where also e' fires.

The fact that a set of n events in a weak-causality cycle can never occur in the same computation can be naturally interpreted as a form of n-ary conflict. More formally, it is useful to associate to each aES an explicit conflict relation (on sets of events) defined in the following way:

Definition 2 (induced conflict relation). Let $G = \langle E, \leq, \nearrow \rangle$ be an aES. The *conflict* relation $\#^a \subseteq \mathbf{2}^E_{\mathit{fin}}$ associated to G is defined as:

$$\frac{e_0 \nearrow e_1 \nearrow \ldots \nearrow e_n \nearrow e_0}{\#^a\{e_0, e_1, \ldots, e_n\}} \qquad\qquad \frac{\#^a(A \cup \{e\}) \quad e \leq e'}{\#^a(A \cup \{e'\})}$$

where A denotes a generic finite subset of E. The superscript a in $\#^a$ reminds that this relation is induced by asymmetric conflict. Sometimes we use the infix notation for the “binary version” of the conflict, i.e. we write $e\#^a e'$ for $\#^a\{e, e'\}$.

It is worth noticing that the binary version of the conflict relation $\#^a$, satisfies all the properties of the conflict relation of traditional PES's, i.e. it is irreflexive, symmetric and hereditary w.r.t. the causal dependency relation.

The notion of aES morphism is a quite natural extension of that of PES morphism. Intuitively, it is a (possibly partial) mapping of events that “preserves computations”.

Definition 3 (category aES). Let $G_0 = \langle E_0, \leq_0, \nearrow_0 \rangle$ and $G_1 = \langle E_1, \leq_1, \nearrow_1 \rangle$ be two aES. An *aES-morphism* $f : G_0 \to G_1$ is a partial function $f : E_0 \to E_1$ such that:

1. for all $e_0 \in E_0$, if $f(e_0)$ is defined then $\lfloor f(e_0) \rfloor \subseteq f(\lfloor e_0 \rfloor)$;
2. for all $e_0, e_0' \in E_0$
   (a) $(f(e_0) = f(e_0')) \wedge (e_0 \neq e_0') \quad \Rightarrow \quad e_0 \#^a_0 e_0'$;
   (b) $f(e_0) \nearrow_1 f(e_0') \quad \Rightarrow \quad (e_0 \nearrow_0 e_0') \vee (e_0 \#^a_0 e_0')$.

We denote with **aES** the category of asymmetric event structures and aES morphisms.

It can be shown that aES morphisms are closed under composition and thus category **aES** is well-defined. Moreover, analogously to what happens for PES's, one can prove that aES morphisms reflect the (n-ary derived) conflict relation.

Lemma 4 (prime and asymmetric event structures). *Let $ES = \langle E, \leq, \# \rangle$ be a prime event structure. Then $G = \langle E, \leq, < \cup \# \rangle$ is and aES, where the* *asymmetric conflict relation is defined as the union of the "strict" causality and conflict relations.*

Moreover, if $f : ES_0 \to ES_1$ is an event structure morphism then f is an aES-morphism between the corresponding aES's G_0 and G_1, and if $g : G_0 \to G_1$ is an aES morphism then it is also a PES morphism between the original PES's.

By the lemma, there is a full embedding functor $\mathcal{J} : \mathbf{PES} \to \mathbf{aES}$ defined on objects as $\mathcal{J}(\langle E, \leq, \# \rangle) = \langle E, \leq, < \cup \# \rangle$ and on arrows as $\mathcal{J}(f : ES_0 \to ES_1) = f$.

A configuration of an event structure is a set of events representing a possible computation of the system modelled by the event structure. The presence of the asymmetric conflict relation makes such definition slightly more involved w.r.t. the traditional one.

Definition 5 (configuration). Let $G = \langle E, \leq, \nearrow \rangle$ be an aES. A *configuration* of G is a set of events $C \subseteq E$ such that

1. $\nearrow_C$ is well-founded;
2. $\{e' \in C : e' \nearrow e\}$ is finite for all $e \in C$;
3. C is left-closed w.r.t. $\leq$, i.e. for all $e \in C$, $e' \in E$, $e' \leq e$ implies $e' \in C$.

The set of all configurations of G is denoted by $Conf(G)$.

Condition 1 first ensures that in C there are no $\nearrow$ cycles, and thus excludes the possibility of having in C a subset of events in conflict (formally, for any $A \subseteq_{fin} C$, we have $\neg(\#^a A)$). Moreover it guarantees that $\nearrow$ has no infinite descending chain in C, that, together with Condition 2, implies that the set $\{e' \in C : e'(\nearrow_C)^+ e\}$ is finite for each event e in C; thus each event has to be preceded only by finitely many other events of the configuration. Finally Condition 3 requires that all the causes of each event are present.

If a set of events A satisfies only the first two properties of Definition 5 it is called *consistent* and we write $co(S)$. Notice that, unlike for traditional event structures, consistency is not a finitary property.[3] For instance, let $A = \{e_i : i \in \mathbb{N}\} \subseteq E$ be a set of events such that all e_i's are distinct and $e_{i+1} \nearrow e_i$ for all $i \in \mathbb{N}$. Then A is not consistent, but each finite subset of A is.

A remarkable difference w.r.t. to the classical approach is that the order on configurations is not simply set-inclusion, since a configuration C cannot be extended with an event inhibited by some of the events already present in C.

Definition 6 (extension). Let $G = \langle E, \leq, \nearrow \rangle$ be an aES and let $A, A' \subseteq E$ be sets of events. We say that A' *extends* A and we write $A \sqsubseteq A'$, if

1. $A \subseteq A'$;
2. $\neg(e' \nearrow e)$ for all $e \in A, e' \in A' - A$.

[3] A property Q on the subsets of a set X is *finitary* if given any $Y \subseteq X$, from $Q(Z)$ for all finite subsets $Z \subseteq Y$ it follows $Q(Y)$.

An important result is the fact that the set $Conf(G)$ of configurations of an aES endowed with the extension relation is a finitary prime algebraic domain, i.e. a coherent, prime algebraic, finitary partial order, in the following simply referred to as *domain*. Therefore asymmetric event structures, as well as prime [20] and flow [1] event structures, provide a concrete presentation of prime algebraic domains.

The proof of such result is technically involved and will appear in the full paper: only a sketch is presented here. The fact that $\langle Conf(G), \sqsubseteq \rangle$ is a partial order immediately follows from the definition. Moreover for pairwise compatible sets of configurations the least upper bound and the greatest lower bound are given by union and intersection.

Interestingly, the primes of the domain of configurations turn out to be the possible histories of the various events. We call *history* of an event e in a configuration C the set of events of C that *must* be executed before e (together with e itself). Recall that in a prime event structure an event e uniquely determines its history, that is the set $\lfloor e \rfloor$ of its causes, independently from the configuration at hand. In the case of asymmetric event structures, instead, an event e may have different histories. In fact, given a configuration C, the set of events that must precede e is $C[\![e]\!] = \{e' \in C : e' (\nearrow_C)^* e\}$, and clearly, such a set depends on the configuration C. The set of all possible histories of an event e, namely $\{C[\![e]\!] : C \in Conf(G)\}$ is denoted by $Hist(e)$.

Theorem 7. *Let G be an aES. Then $\langle Conf(G), \sqsubseteq \rangle$ is a (finitary prime algebraic) domain. The primes of $Conf(G)$ are the possible histories of events in G, i.e. the configurations in $\bigcup_{e \in E} Hist(e)$.*

Winskel in his seminal work [20] proved the equivalence between the category **PES** of prime event structures and the category **Dom** of domains and additive, stable, immediate precedence-preserving functions.

$$\mathbf{PES} \underset{\mathcal{L}}{\overset{\mathcal{P}}{\underset{\sim}{\rightleftarrows}}} \mathbf{Dom}$$

The functor $\mathcal{L}$ associates to each PES the domain of its configurations, while the functor $\mathcal{P}$ associates to each domain a PES having its prime elements as events.

We want now to generalize this result to our framework by showing the existence of a coreflection between **aES** and **Dom**. One can prove that aES morphisms preserve configurations and that the natural function between the domains of configurations induced by an aES morphism is a domain morphism. These results, together with Theorem 7, ensure that the functor $\mathcal{L}_a$ leading from the category **aES** of asymmetric event structures to the category **Dom** of finitary prime algebraic domains is well-defined. The functor $\mathcal{P}_a$ performing the backward step is obtained simply by embedding in **aES** the Winskel's construction.

Definition 8. Let $\mathcal{L}_a : \mathbf{aES} \rightarrow \mathbf{Dom}$ be the functor defined as:
- $\mathcal{L}_a(G) = \langle Conf(G), \sqsubseteq \rangle$, for any **aES**-object G;

- $\mathcal{L}_a(f) = f^* : \mathcal{L}_a(G_0) \to \mathcal{L}_a(G_1)$, for any **aES**-morphism $f : G_0 \to G_1$.[4]
The functor $\mathcal{P}_a : \mathbf{Dom} \to \mathbf{aES}$ is defined as $\mathcal{J} \circ \mathcal{P}$.

The proof of the following main result will appear in the full paper.

Theorem 9. *The functor $\mathcal{P}_a$ is left adjoint of $\mathcal{L}_a$. The counit of the adjunction $\epsilon : \mathcal{P}_a \circ \mathcal{L}_a \dot{\to} 1$ is defined by $\epsilon_G(C) = e$, if $C \in \mathit{Hist}(e)$.*

3 Contextual nets

We introduce here *marked contextual P/T nets (c-nets)*, that, following the lines suggested in [14] for C/E systems, add contexts to classical P/T nets. We first recall some notation for multisets. Let A be a set; a *multiset* of A is a function $m : A \to \mathbb{N}$. Such a multiset will be denoted sometimes as a formal sum $m = \sum_{a \in A} n_a \cdot a$, where $n_a = m(a)$. The set of multisets of A is denoted as μA. The usual operations and relations on multisets of A are used. As an example, multiset union is denoted by $+$ and defined as $(m + m')(a) = m(a) + m'(a)$; multiset difference $(m - m')$ is defined as $(m - m')(a) = m(a) - m'(a)$ if $m(a) \geq m'(a)$ and $(m - m')(a) = 0$ otherwise. We write $m \leq m'$ if $m(a) \leq m'(a)$ for all $a \in A$. If m is a multiset of A, we denote by $[\![m]\!]$ the multiset $\sum_{\{a \in A | m(a) > 0\}} 1 \cdot a$, obtained by changing all non-zero coefficients of m to 1. Sometimes we will confuse the multisets $[\![m]\!]$ with the corresponding subsets $\{a \in A : m(a) > 0\}$ of A, and use on them the usual set operations and relations. A *multirelation* $f : A \to B$ is a multiset of $A \times B$. It induces in an obvious way a function $\mu f : \mu A \to \mu B$, defined as $\mu f(\sum_{a \in A} n_a \cdot a) = \sum_{b \in B} \sum_{a \in A} (n_a \cdot f(a, b)) \cdot b$. If the multirelation f satisfies $f(a, b) \leq 1$ for all $a \in A$ and $b \in B$ then we sometimes confuse it with the corresponding set-relation and write $f(a, b)$ for $f(a, b) = 1$.

Definition 10 (c-net). A *(marked) contextual Petri net* (*c-net*) is a tuple $N = \langle S, T, F, C, m \rangle$, where

- S is a set of *places*;
- T is a set of *transitions*;
- $F = \langle F_{pre}, F_{post} \rangle$ is a pair of multirelations from T to S;
- C is a multirelation from T to S, called the *context relation*;
- m is a multiset of S, called the *initial marking*.

We assume, without loss of generality, that $S \cap T = \emptyset$. Moreover, we require that for each transition $t \in T$, there exists a place $s \in S$ such that $F_{pre}(t, s) > 0$.[5]

Let N be a c-net. As usual, the functions from μT to μS induced by the multirelations F_{pre} and F_{post} are denoted by ${}^{\bullet}(\ )$ and $(\ )^{\bullet}$, respectively. If $A \in \mu T$ is a multiset of transitions, ${}^{\bullet}A$ is called its *pre-set*, while $A^{\bullet}$ is called its *post-set*. Moreover, by $\underline{A}$ we denote the *context* of A, defined as $\underline{A} = \mu C(A)$.

[4] With f^* we denote the natural extension of the function f to the powerset of E_0 (i.e., $f^*(A) = \{f(a) : a \in A\}$, for $A \subseteq E_0$).

[5] This is a weak version of the condition of *T-restrictness* that requires also $F_{post}(t, s) > 0$, for some $s \in S$.

The same notation is used to denote the functions from S to $\mathbf{2}^T$ defined as, for $s \in S$, ${}^{\bullet}s = \{t \in T : F_{post}(t,s) > 0\}$, $s^{\bullet} = \{t \in T : F_{pre}(t,s) > 0\}$, $\underline{s} = \{t \in T : C(t,s) > 0\}$.

In the following when considering a c-net N, we implicitly assume that $N = \langle S, T, F, C, m \rangle$. Moreover superscripts and subscripts on the nets names carry over the names of the involved sets, functions and relations. For instance $N_i = \langle S_i, T_i, F_i, C_i, m_i \rangle$.

For a finite multiset of transitions A to be enabled by a marking M, it is sufficient that M contains the pre-set of A and at least one *additional* token in each place of the context of A. This corresponds to the intuition that a token in a place can be used as context by many transitions at the same time and with multiplicity greater than one by the same transition.

Definition 11 (token game). Let N be a c-net and let M be a *marking* of N, that is a multiset $M \in \mu S$. Given a finite multiset $A \in \mu T$, we say that A is *enabled* by M if ${}^{\bullet}A + [\![\underline{A}]\!] \leq M$.[6] The *transition relation* between markings is defined as

$$M\,[A\rangle\,M' \qquad \text{iff} \qquad A \text{ is enabled by } M \text{ and } M' = M - {}^{\bullet}A + A^{\bullet}.$$

We call $M\,[A\rangle\,M'$ a *step*. A *simple step* or *firing* is a step involving just one transition. A marking M is called *reachable* if there exists a finite *step sequence* $m\,[A_0\rangle\,M_1\,[A_1\rangle\,M_2 \ldots [A_n\rangle\,M$, starting from the initial marking and leading to M.

A c-net morphism is a partial mapping between transitions that "preserves" pre- and post-sets, and also contexts in a weak sense.

Definition 12 (c-net morphism). Let N_0 and N_1 be c-nets. A *c-net morphism* $h : N_0 \to N_1$ is a pair $h = \langle h_T, h_S \rangle$, where $h_T : T_0 \to T_1$ is a partial function and $h_S : S_0 \to S_1$ is a multirelation such that (1) $\mu h_S(m_0) = m_1$ and, for each $A \in \mu T$, (2) $\mu h_S({}^{\bullet}A) = {}^{\bullet}\mu h_T(A)$, (3) $\mu h_S(A^{\bullet}) = \mu h_T(A)^{\bullet}$ and (4) $[\![\underline{\mu h_T(A)}]\!] \leq \mu h_S(\underline{A}) \leq \underline{\mu h_T(A)}$.
We denote by $\mathbf{CN}$ the category having c-nets as objects and c-net morphisms as arrows.

Conditions (1)-(3) are standard, but condition (4), regarding contexts, deserves some comments. It can be explained by recalling that, since in our model a single token can be used as context with multiplicity greater than one, the firing of a transition t can use as context any multiset X satisfying $[\![\underline{t}]\!] \leq X \leq \underline{t}$. Given any multiset of tokens that can be used as context in a firing of a transition,

[6] Other approaches (e.g. [9, 18]) allow for the concurrent firing of transitions that use the same token as context and precondition. For instance, in [9] the formal condition for a multiset A of transitions to be enabled by a marking M is ${}^{\bullet}A \leq M$ and $\underline{A} \leq M$. We do not admit such steps, the idea being that two concurrent transitions should be allowed to fire also in any order.

its image should be a set of tokens that can be used as context by the image of the transition. This can be formalized by requiring that $[\![\underline{\mu h_T(A)}]\!] \leq \mu h_S(X) \leq \underline{\mu h_T(A)}$ for any $X \in \mu S_0$ such that $[\![\underline{A}]\!] \leq X \leq \underline{A}$, which is equivalent to the above condition (4).

The basic result to prove (to check that the definition of morphism is "meaningful") is that the token game is preserved by c-net morphisms.

Theorem 13 (morphisms preserve the token game). *Let N_0 and N_1 be c-nets, and let $h = \langle h_T, h_S \rangle : N_0 \to N_1$ be a c-net morphism. Then for each $M, M' \in \mu S$ and $A \in \mu T$*

$$M\,[A\rangle\,M' \quad \Rightarrow \quad \mu h_S(M)\,[\mu h_T(A)\rangle\,\mu h_S(M').$$

Therefore c-net morphisms preserve reachable markings, i.e. if M_0 is a reachable marking in N_0 then $\mu h_S(M_0)$ is reachable in N_1.

The seminal work by Winskel [20] presents a coreflection between event structures and a subclass of P/T nets, namely *safe* nets. In [12] it is shown that essentially the same constructions work for the larger category of "weakly safe nets" as well (while the generalization to the whole category of P/T nets requires some original technical machinery and allows one to obtain a proper adjunction rather than a coreflection). In the next sections we will relate by a coreflection event structures and "weakly safe c-nets".

Definition 14 (weakly safe c-nets). A *weakly safe* c-net is a c-net N such that the initial marking m is a set and F_{post} is a relation (i.e. $t^\bullet$ is a set for all $t \in T$). We denote by **WS-CN** the full subcategory of **CN** having weakly safe c-nets as objects.

A weakly safe c-net is called *safe* if also F_{pre} and C are relations (i.e., ${}^\bullet t$ and $\underline{t}$ are sets for all $t \in T$) and each reachable marking is a set.

4 Occurrence c-nets and the unfolding construction

Occurrence c-nets are intended to represent, via an unfolding construction, the behaviour of general c-nets in a static way, by expressing the events (firing of transitions) which can appear in a computation and the dependency relations between them. Occurrence c-nets will be defined as safe c-nets such that the dependency relations between transitions satisfy suitable acyclicity and well-foundedness requirements. While for traditional occurrence nets one has to take into account the causal dependency and the conflict relations, by the presence of contexts, we have to consider an asymmetric conflict (or weak dependency) relation as well. Interestingly, the conflict relation turns out to be a derived (from asymmetric conflict) relation.

Causal dependency is defined as for traditional nets, with an additional clause stating that transition t causes t' if it generates a token in a context place of t'.

Definition 15 (causal dependency). Let N be a safe c-net. The *causal dependency relation* $<_N$ is the transitive closure of the relation $\prec$ defined by:

1. if $s \in {}^\bullet t$ then $s \prec t$;
2. if $s \in t^\bullet$ then $t \prec s$;
3. if $t^\bullet \cap \underline{t'} \neq \emptyset$ then $t \prec t'$.

Given a place or transition $x \in S \cup T$, we denote with $\lfloor x \rfloor$ the set of *causes* of x, defined as $\lfloor x \rfloor = \{t \in T : t \leq_N x\} \subseteq T$, where $\leq_N$ is the reflexive closure of $<_N$.

Definition 16 (asymmetric conflict). Let N be a safe c-net. The *strict asymmetric conflict relation* $\rightsquigarrow_N$ is defined as

$$t \rightsquigarrow_N t' \qquad \text{iff} \qquad \underline{t} \cap {}^\bullet t' \neq \emptyset \ \text{ or } \ (t \neq t' \ \wedge \ {}^\bullet t \cap {}^\bullet t' \neq \emptyset).$$

The *asymmetric conflict relation* $\nearrow_N$ is the union of the strict asymmetric conflict and causal dependency relations:

$$t \nearrow_N t' \qquad \text{iff} \qquad t <_N t' \text{ or } t \rightsquigarrow_N t'.$$

In our informal interpretation, $t \nearrow_N t'$ if t must precede t' in each computation in which both fire or, equivalently, t' prevents t to be fired:

$$t \nearrow t' \quad \stackrel{def}{\equiv} \quad \mathit{Fired}(t) \wedge \mathit{Fired}(t') \ \Rightarrow \ \mathit{prec}(t, t') \tag{$\dagger$}$$

As noticed in the introduction, this is surely the case when the same place s appears as context for t and as precondition for t'. But ($\dagger$) is trivially true (with t and t' in interchangeable roles) when t and t' have a common precondition, since they never fire in the same computation. This is apparently a little tricky but corresponds to the clear intuition that a (usual) symmetric (direct) conflict leads to asymmetric conflict in both directions. Finally, since, as noticed for the general model of aES, ($\dagger$) is weaker than the condition that expresses causality, the condition ($\dagger$) is satisfied when t causes (in the usual sense) t'.[7] For technical reasons it is convenient to distinguish the first two cases from the last one.

The c-net in Fig. 2 shows that, as expected, the relation $\nearrow_N$ is not transitive. In fact we have $t_1 \nearrow_N t_3 \nearrow_N t_2 \nearrow_N t_1$, but, for instance, it is not true that $t_1 \nearrow_N t_2$.

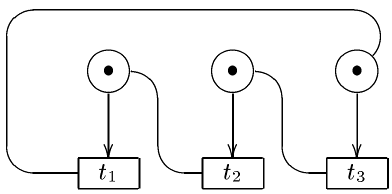

Fig. 2. An occurrence c-net with a cycle of asymmetric conflict.

An occurrence c-net is a safe c-net that exhibits an acyclic behaviour and such that each transition in it can be fired.

[7] This is the origin of the weak causality interpretation of $\nearrow$.

Definition 17 (occurrence c-nets). An *occurrence c-net* is a safe c-net N such that

- each place $s \in S$ is in the post-set of at most one transition, i.e. $|{}^\bullet s| \leq 1$;
- the *causal relation* $<_N$ is irreflexive and its reflexive closure $\leq_N$ is a partial order, such that $\lfloor t \rfloor$ is finite for any $t \in T$;
- the *initial marking* m is the set of minimal places w.r.t. $\leq_N$, i.e., $m = \{s \in S : {}^\bullet s = \emptyset\}$;
- $(\nearrow_N)_{\lfloor t \rfloor}$ is acyclic[8] for all transitions $t \in T$.

The full subcategory of **WS-CN** having occurrence c-nets as objects is denoted by **O-CN**.

The last condition corresponds to the requirement of irreflexivity for the conflict relation in ordinary occurrence nets. In fact, if a transition t has a $\nearrow_N$ cycle in its causes then it can never fire, since in an occurrence c-net N, the order in which transitions appear in a firing sequence must be compatible with the transitive closure of the (restriction to the transitions in the sequence of the) asymmetric conflict relation.

As anticipated, the asymmetric conflict relation induces a symmetric conflict relation (on sets of transitions) defined in the following way:

Definition 18 (conflict). Let N be a c-net. The *conflict* relation $\# \subseteq \mathbf{2}^T_{fin}$ associated to N is defined as:

$$\frac{t_0 \nearrow t_1 \nearrow \ldots \nearrow t_n \nearrow t_0}{\#\{t_0, t_1, \ldots, t_n\}} \qquad\qquad \frac{\#(A \cup \{t\}) \quad t \leq t'}{\#(A \cup \{t'\})}$$

where A denotes a generic finite subset of T. As for aES's, we use the infix notation $t \# t'$ for $\#\{t, t'\}$.

For instance, referring to Fig. 2, we have $\#\{t_1, t_2, t_3\}$, but not $\#\{t_i, t_j\}$ for any $i, j \in \{1, 2, 3\}$.

As for traditional occurrence nets, a set of places M is concurrent if there is some reachable marking in which all the places in M contain a token. However for the presence of contexts some places that a transition needs to be fired (contexts) can be concurrent with the places it produces.

Definition 19 (concurrency relation). Let N be an occurrence c-net. A set of places $M \subseteq S$ is called *concurrent*, written *conc*(M), if

1. $\forall s, s' \in M.\ \neg(s < s')$;
2. $\lfloor M \rfloor$ is finite, where $\lfloor M \rfloor = \bigcup\{\lfloor s \rfloor : s \in M\}$;
3. $\nearrow_{\lfloor M \rfloor}$ is acyclic (and thus well-founded, since $\lfloor M \rfloor$ is finite).

[8] We can equivalently require $((\nearrow_N)_{\lfloor t \rfloor})^+$ to be irreflexive. In particular this implies $\nearrow_N$ irreflexive.

It can be shown that, indeed, the concurrent sets of places of an occurrence c-net coincide with the (subsets of) reachable markings. In particular, for each transition t in an occurrence c-net, since $conc({}^{\bullet}t + \underline{t})$, there is a reachable marking $M \supseteq {}^{\bullet}t + \underline{t}$, in which t is enabled.

It is possible to prove that c-net morphisms preserve the concurrency relation. Moreover, they preserve the "amount of concurrency" also on transitions. More precisely, they reflect causal dependency and conflicts, while asymmetric conflict is reflected or becomes conflict. These results are fundamental for establishing a connection between occurrence c-nets and aES's.

Theorem 20. *Let N_0 and N_1 be c-nets and let $h : N_0 \to N_1$ be a morphism. Then, for all $t_0, t_0' \in T_0$*

1. $\lfloor h_T(t_0) \rfloor \subseteq h_T(\lfloor t_0 \rfloor)$;
2. $(h_T(t_0) = h_T(t_0')) \ \wedge \ (t_0 \neq t_0') \quad \Rightarrow \quad t_0 \#_0 t_0'$;
3. $h_T(t_0) \nearrow_1 h_T(t_0') \quad \Rightarrow \quad (t_0 \nearrow_0 t_0') \ \vee \ (t_0 \#_0 t_0')$;
4. $\# h_T(A) \quad \Rightarrow \quad \# A$.

Given a weakly-safe c-net N, an *unfolding* construction allows us to obtain an occurrence c-net $\mathcal{U}_a(N)$ that describes the behaviour of N. As for traditional nets, each transition in $\mathcal{U}_a(N)$ represents an instance of a precise firing of a transition in N, and places in $\mathcal{U}_a(N)$ represent occurrences of tokens in the places of N. The unfolding operation can be extended to a functor $\mathcal{U}_a$: **WS-CN** $\to$ **O-CN** that is right adjoint of the inclusion functor $\mathcal{I}_O$: **O-CN** $\to$ **WS-CN** and thus establishes a coreflection between **WS-CN** and **O-CN**.

Definition 21 (unfolding). Let $N = \langle S, T, F, C, m \rangle$ be a weakly safe c-net. The unfolding $\mathcal{U}_a(N) = \langle S', T', F', C', m' \rangle$ of the net N and the *folding morphism* $f_N : \mathcal{U}_a(N) \to N$ are the unique occurrence c-net and c-net morphism satisfying the following equations.

$$
\begin{aligned}
m' &= \{\langle \emptyset, s \rangle : s \in m\} \\
S' &= m' \cup \{\langle t', s \rangle : t' = \langle M_p, M_c, t \rangle \in T' \ \wedge \ s \in t^{\bullet}\} \\
T' &= \{\langle M_p, M_c, t \rangle : M_p, M_c \subseteq S' \ \wedge \ M_p \cap M_c = \emptyset \ \wedge \ conc(M_p \cup M_c) \ \wedge \\
&\qquad t \in T \ \wedge \ \mu f_S(M_p) = {}^{\bullet}t \ \wedge \ [\![\underline{t}]\!] \leq \mu f_S(M_c) \leq \underline{t}\}
\end{aligned}
$$

$$
\begin{array}{lll}
F'_{pre}(t', s') & \text{iff} & t' = \langle M_p, M_c, t \rangle \ \wedge \ s' \in M_p \quad (t \in T) \\
C'(t', s') & \text{iff} & t' = \langle M_p, M_c, t \rangle \ \wedge \ s' \in M_c \quad (t \in T) \\
F'_{post}(t', s') & \text{iff} & s' = \langle t', s \rangle \quad (s \in S) \\[1ex]
f_T(t') = t & \text{iff} & t' = \langle M_p, M_c, t \rangle \\
f_S(s', s) & \text{iff} & s' = \langle x, s \rangle \quad (x \in T' \cup \{\emptyset\})
\end{array}
$$

The unfolding can be effectively constructed by giving an inductive definition. Uniqueness follows from the fact that to each item in a occurrence c-net we can associate a finite depth.

Places and transitions in the unfolding of a c-net represent respectively tokens and firing of transitions in the original net. Each place in the unfolding is a pair recording the "history" of the token and the corresponding place in the original net. Each transition is a triple recording the precondition and context used in the firing, and the corresponding transition in the original net. A new place with empty history $\langle \emptyset, s \rangle$ is generated for each place s in the initial marking. Moreover a new transition $t' = \langle M_p, M_c, t \rangle$ is inserted in the unfolding whenever we can find a concurrent set of places (precondition M_p and context M_c) that corresponds, in the original net, to a marking that enables t. For each place s in the post-set of such t, a new place $\langle t', s \rangle$ is generated, belonging to the post-set of t'. The folding morphism f maps each place (transition) of the unfolding to the corresponding place (transition) in the original net.

We can now state the main result of this section, establishing a coreflection between weakly safe c-nets and occurrence c-nets.

Theorem 22. *The unfolding construction extends to a functor* $\mathcal{U}_a : \mathbf{WS\text{-}CN} \to \mathbf{O\text{-}CN}$ *which is right adjoint to the obvious inclusion functor* $\mathcal{I}_O : \mathbf{O\text{-}CN} \to \mathbf{WS\text{-}CN}$ *and thus establishes a coreflection between* **WS-CN** *and* **O-CN**.

The component at an object N *in* **WS-CN** *of the counit of the adjunction,* $f : \mathcal{I}_O \circ \mathcal{U}_a \dot{\to} 1$*, is the folding morphism* $f_N : \mathcal{U}_a(N) \to N$.

5 Occurrence c-nets and asymmetric event structures

We now show that the semantics of weakly safe c-nets given in terms of occurrence c-nets can be related with event structures and prime algebraic domains semantics. First we show that there exists a coreflection from **aES** to **O-CN** and thus aES's represent a suitable model for giving event based semantics to c-nets. Given an occurrence c-net we obtain an aES simply forgetting the places, but remembering the dependency relations that they induce between transitions, namely causality and asymmetric conflict. In the same way a morphism between occurrence c-nets naturally restricts to a morphism between the corresponding aES's.

Definition 23. Let $\mathcal{E}_a : \mathbf{O\text{-}CN} \to \mathbf{aES}$ be the functor defined as:

- $\mathcal{E}_a(N) = \langle T, \leq_N, \nearrow_N \rangle$, for each occurrence c-net N;
- $\mathcal{E}_a(h : N_0 \to N_1) = h_T$, for each morphism $h : N_0 \to N_1$.

Notice that the induced conflict relation $\#^a$ in the aES $\mathcal{E}_a(N)$, given by Definition 2, is the restriction to transitions of the induced conflict relation in the net N, given by Definition 18. Therefore in the following we will confuse the two relations and simply write $\#$.

An aES can be identified with a canonical occurrence c-net, via a free construction that mimics Winskel's: for each set of events related in a certain way by causal dependency or asymmetric conflict relations we generate a unique place that induces such kind of relation on the events.

Definition 24. Let $G = \langle E, \leq, \nearrow \rangle$ be an aES. Then $\mathcal{N}_a(G)$ is the net $N = \langle S, T, F, C, m \rangle$ defined as follows:

- $m = \left\{ \langle \emptyset, A, B \rangle : \begin{array}{l} A, B \subseteq E,\ \forall a \in A.\ \forall b \in B.\ a \nearrow b\ \vee\ a\#b, \\ \forall b, b' \in B.\ b \neq b' \Rightarrow b\#b' \end{array} \right\}$;
- $S = m \cup \left\{ \langle e, A, B \rangle : \begin{array}{l} A, B \subseteq E,\ e \in E,\ \forall x \in A \cup B.\ e < x, \\ \forall a \in A.\ \forall b \in B.\ a \nearrow b\ \vee\ a\#b, \\ \forall b, b' \in B.\ b \neq b' \Rightarrow b\#b' \end{array} \right\}$;
- $T = E$;
- $F = \langle F_{pre}, F_{post} \rangle$, with
  $F_{pre} = \{(e, s) : s = \langle x, A, B \rangle \in S,\ e \in B\}$,
  $F_{post} = \{(e, s) : s = \langle e, A, B \rangle \in S\}$;
- $C = \{(e, s) : s = \langle x, A, B \rangle \in S,\ e \in A\}$.

The generation process extends to a functor $\mathcal{N}_a : \mathbf{aES} \to \mathbf{O\text{-}CN}$

The only unexpected thing for the reader could be the fact that we insert a place that gives rise to asymmetric conflicts between the transitions of B and A, but we require only that all the transition of B are in asymmetric conflict *or in conflict* with all the transitions in A. Therefore we add asymmetric conflicts between events that are in conflict. Abstracting from the formal details, this becomes very natural since, being $\#$ the symmetric conflict relation, we can think that conceptually $t\#t'$ implies $t \nearrow t'$.

The next proposition relates the causal dependency and asymmetric conflict relations of an aES with the corresponding relations of the c-net $\mathcal{N}_a(G)$. In particular it is useful in proving that $\mathcal{N}_a(G)$ is indeed an occurrence c-net.

Proposition 25. *Let $G = \langle E, \leq, \nearrow \rangle$ be an aES and let $\mathcal{N}_a(G)$ be the net $N = \langle S, T, F, C, m \rangle$. Then for all $e, e' \in E$:*

1. *$e <_N e'$ iff $e < e'$;*
2. *$e \nearrow_N e'$ iff $e \nearrow e'$ or $e\#e'$.*

Let $G = \langle E, \leq, \nearrow \rangle$ be an aES. By Proposition 25, $\mathcal{E}_a(\mathcal{N}_a(G)) = \langle E, \leq, \nearrow \cup \# \rangle$. Therefore the identity on events $\eta_G : G \to \mathcal{E}_a(\mathcal{N}_a(G))$, defined by $\eta_G(e) = e$, for all $e \in E$, is an aES morphism. Moreover $\eta_G^{-1} : \mathcal{E}_a(\mathcal{N}_a(G)) \to G$, again defined as identity on events is clearly a morphism, and η_G and η_G^{-1} are one the inverse of the other. Therefore η_G is an isomorphism. We are now able to state the main result of this section.

Theorem 26. *The functor $\mathcal{N}_a : \mathbf{aES} \to \mathbf{O\text{-}CN}$ is left adjoint to $\mathcal{E}_a : \mathbf{O\text{-}CN} \to \mathbf{aES}$ and it establishes a coreflection from $\mathbf{aES}$ to $\mathbf{O\text{-}CN}$. The unit of the the coreflection is $\eta : 1 \dot{\to} \mathcal{N}_a \circ \mathcal{E}_a$.*

Such a result completes the chain of coreflections leading from **WS-CN** to **Dom**. Therefore, as claimed at the beginning, we provide weakly safe c-nets with a truly concurrent semantics, by associating to each weakly safe c-net a finitary prime algebraic domain. The construction works at categorical level and establishes a coreflection between the corresponding categories.

Finally, notice that, as an easy extension, Winskel's coreflection between **PES** and **Dom** can be used to provide weakly safe c-nets with a traditional event structure semantics. The PES semantics is obtained from the aES semantics by introducing an event for each possible different history of events in the aES. This reflects the idea of duplication of events discussed in the introduction.

6 Conclusions and future work

We presented a truly concurrent event-based semantics for (weakly safe) P/T contextual nets. The semantics is given at categorical level via a coreflection between the categories **WS-CN** of weakly safe c-nets and **Dom** of finitary prime algebraic domains (or equivalently **PES** of prime event structures). Such a coreflection factorizes through the following chain of coreflections:

$$\begin{tikzcd} \mathbf{WS\text{-}CN} \arrow[r, shift right, "\mathcal{U}_a"'] \arrow[r, phantom, "\perp"] & \mathbf{O\text{-}CN} \arrow[l, hook', shift right, "\mathcal{I}_O"'] \arrow[r, shift right, "\mathcal{E}_a"'] \arrow[r, phantom, "\perp"] & \mathbf{aES} \arrow[l, shift right, "\mathcal{N}_a"'] \arrow[r, shift right, "\mathcal{L}_a"'] \arrow[r, phantom, "\perp"] & \mathbf{Dom} \arrow[l, shift right, "\mathcal{P}_a"'] \end{tikzcd}$$

It is worth noticing that such a construction associates to a safe c-net without context places (thus essentially a traditional safe net), the same domain produced by Winskel's construction and therefore can be considered as a consistent extension of Winskel's result. The use of finitary prime algebraic domains, widely accepted as standard semantics models for concurrency, makes our result satisfactory. Moreover the existence of a coreflection provides an abstract semantics (the domain associated to each c-net) and a standard choice in each class of equivalent c-nets (the c-net obtained by embedding the semantics into the category of nets), defined by a universal property. This is one of the more pleasant semantic frameworks one can desire.

An immediate future work should be the generalization of these results to general P/T c-nets, based on a suitable extension of the notions of decorated occurrence net and family morphism introduced in [12] to give unfolding semantics to traditional P/T nets. Moreover, notions and results on c-nets can be seen as a first step towards the definition of an unfolding semantics for graph grammars. We think that the work on c-nets could be a guide for the introduction of the notions of non-deterministic occurrence graph grammar and graph grammar unfolding that are still lacking or not consolidated.

Apart from the application to c-nets analyzed in this paper, asymmetric event structures seem to be rather promising in the semantic treatment of models of computation, such as string, term and graph rewriting, allowing context sensitive firing of events. Therefore, as suggested in [16], it would be interesting to investigate the possibility of developing a general theory of event structures with asymmetric conflict (or weak causality) similar to that in [20].

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