Lecture20
Foundation of Functional Languages: Higher Order Functional Programming

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Foundation of Functional Languages: Higher Order Functional Programming

- Functional Languages: The main Features
- Syntax Essentials
- Programming Methodologies in Functional Languages
- Higher Order Programming, Iterative and Combinatory Programming
- Foundations: Term Reduction, Reduction Strategies, Combinators and Graph Reduction

Functional Languages: The Main Features

- Referential Transparency in Pure Functional (see Lecture9-10)
- List Types and Operators (see Lecture11)
- Structured Values are Fully Expressible Values (see Lecture11)
- Garbage Collection for Heap re-allocation (see Lecture18-19)
- **First-Class Function Values and Higher Order Functions**
- **Extensive Polymorphism** (see Lecture11 and Lecture18-19)
- **Functions may return Structured Values**

Functional Languages: Syntax Essentials

Table20 – Functional Languages : Syntax	
Program Structures	
$\mathbf{D} ::= \mathtt{F} \mid \mathtt{T} \mid \ldots$	
$\mathbf{F} ::= \mathtt{I}= \mathtt{A}$	
$\mathbf{A} ::= \mathtt{fun\ I} \to \mathtt{E}$	
$\mathbf{E} ::= \mathtt{A} \mid \mathtt{E\ E} \mid \mathtt{I} \mid \mathtt{D\ E} \mid \ldots$	
$\mathbf{T} ::= \ldots$	(*scalar*, *tuple*, **map**, ...)
$\mid \ldots$	(*list* : *fund*. *meth*.)
$\mid \ldots$	(*Concrete* : *fund*. *meth*.)
$\mid \ldots$	(*Abstract* : *fund*. *meth*.)

- It is impressive how compact is the syntax of functional language (only Functions and Types definitions and Expressions), and
- How few are the mechanisms that are needed in functional programming, and
- The amount of different programming methodologies that are well supported by functional programming

All these facts are a trivial consequence of only one fact:

Functions are First-Class Values

Programming Methodologies in Functional Programming

- Decomposition Based Programming (see Lecture15-17)

Example

Decomposition Programming is used in getting the program of the memoized factorial below. The solution consists of 4 components: A shared memory (hashTab), 2 distinct, autonomous, independent, functions (hash and set), and a code that combines all them and realizes the memoized factorial

- Inductive Programming (see Lecture15-17)
- Tail Recursion Programming
- Memoization Based Programming

Example

```
let rec fact = fun n → if (n=0) then 1 else n*(fact (n-1))
                     -- Recursive definition of an inductive algorithm for fact
let fact = fun n → let rec factT = fun n r → if (n=0) then r else (fact (n-1) (n*r)) in
                      factT n 1
                     -- Tail Recursive definition of inductive factorial
let fact = let hashTab = ref [] in
           let hash = fun n → if (mem_assoc n (!hashTab)) then (true,(assoc n(!hashTab)))
                                                            else (false,raise IntUndef) in
           let set = fun n v → hashTab:=(n,v)::(!hashTab) in
                     -- Memoized Part, to be completed with the code of lecture 15-17, of inducti-
                        ve factorial
```

- Divide and Conquer (see Lecture15-17)

Example

We apply the methodology to define QuickS on lists of a generic type

Programming Methodologies in Functional Programming/2

- Divide and Conquer (see Lecture15-17)

Example

We apply the methodology to define QuickS on lists of a generic type

```
let rec quickS =
       fun c o → if (length c) < 2 then c
                 else let sel = hd in
                      let u = (sel c) in
                      let gt = filter (fun x → (o u x)) c in   – use of Higher Order
                      let lt = filter (fun x → (o x u)) c in   – use of H.O.: Lambda Abstraction
                      (quickS lt o)@(u::(quickS gt o));
```

– It uses the module "List.ml" but it is not enough to guarantee the full generalization of the algorithm. The module has only "hd" that behaves as a selection function
– Apply to the computation of: quickS [3;5;1;5;0;8;9] (>)

- Polymorphism. quickS has Ocaml type:
  `'a list -> ('a -> 'a -> bool) -> 'a list`

Example

– Apply quickS to sorting [("aba",78),("a",13),("ab",0)] according to two different pair orderings at your choice.

- Higher Order Programming
- Iterative and Combinators based Programming

Programming Methodologies in Functional Programming/3

- Divide and Conquer (see Lecture15-17)

Example

We apply the methodology to define QuickS on lists of a generic type

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– Apply to the computation of: quickS [3;5;1;5;0;8;9] (>)

- Higher Order Programming, HOP
  - It is pervasive of Programming in Functional Languages
  - Hence, it appear also, in combination with all other programming methodologies used in functional Programming
  - For instance, in quickS, HOP furnishes the values to make quickS parametric w.r.t. the ordering relation.
- Iterative and Combinators Programming

Methodologies: Higher Order Programming - Values

- Higher Order Programming, HOP
  - It is pervasive of Programming in Functional Languages
  - Hence, it appear also, in combination with all other programming methodologies used in functional Programming
- But what are the ingredients of the methodology HOP
  - **Data Extensions through Functional Abstractions**
    - New Domains of values are introduced through New Sets of functions
    - The new functions behave according to the way in which the new values have to be used in the program to be developed
    - Implementation details of the new values have not to be provided from programmer
    - This is much more than of abstract data types of programming languages, since ADT require the definition of an implementation module in order to be used in computation

Example

We apply HOP to the definition of ENV in Lecture 11

```
# let bindP = fun i d e -> fun j -> if (j=i) then d else e j;;
val bindP : 'a -> 'b -> ('a -> 'b) -> 'a -> 'b = <fun>
# let findP = fun i e -> e i and emptyP = fun i -> i;;
# let anEnv = bindP 3 5 emptyP;;
val anEnv :  int -> int = <fun>
```

  - **Control Extensions through Functional Abstractions**

Methodologies: Higher Order Programming - Values/2

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val bindP : 'a -> 'b -> ('a -> 'b) -> 'a -> 'b = <fun>
# let findP = fun i e -> e i and emptyP = fun i -> i;;
# let anEnv = bindP 3 5 emptyP;;
val anEnv :  int -> int = <fun>
# findP 3 anEnv;;
- :  int = 5
```

    - Ocaml can define it better, by using types that highlight the different set of values and forbid illegal uses of the values

Example

```
We apply HOP to the definition of ENV in Lecture 11
# type ide = I of string;;
# type den = L of int | C of int;;
# type env = ide -> den;;
# let bind = fun (i:ide) (d:den) (e:env) -> fun (j:ide) -> if (j=i) then d else e j;;
val bind :  ide -> den -> env -> ide -> den = <fun>
...  -- complete and run
```

  - Control Extensions through Functional Abstractions

Methodologies: Higher Order Programming - Control

- But what are the ingredients of the methodology HOP
  - Data Extensions through Functional Abstractions
  - **Control Extensions through Functional Abstractions**
    - New Control Structures are introduced through New Sets of functions
    - The new functions combines in new ways the data and the functions to be used in the program to be developed
    - Implementation details of the new control structures (namely, the structure of the Activation Records, etc) have not to be provided from programmer
    - This is the why, in giving denotational semantics, we can use a functional language as the defining metalanguage: All the control mechanisms of all languages are easily formalized without adding useless, implementation details

Example

We apply HOP to a combinatory, recursive, definition of factorial

```
let cITE = fun p f g -> fun n -> if (p n) then (f n) else (g n);;
let cmp = fun f g n -> g(f n) and cC = fun f g n -> g n (f n) and cK = fun f g -> f ;;
let im = (+)(-1) and ip = fun n m -> n*m;;
let rec fact = fun n -> (cITE ((=)0) (cK 1) (cC (cmp im fact) ip) n);;[a]
```

Comment the use of the argument n, (apply denotational semantic) and run: fact 3

---

[a] by η-conversion it is equivalent to combinatory cITE ((=)0) (cK 1) (cC (cmp im fact) ip)

Methodologies: HOP - Control/2 - NO

- But what are the ingredients of the methodology HOP
    - Data Extensions through Functional Abstractions
    - **Control Extensions through Functional Abstractions**
        - We revisit step by step, the program development

Example

```
(1) We introduce "cITE" that applies to 3 functions and a value "n" and returns the application
of "if-then-else" to the 3 expressions resulting from distributing "n" to each function:
let cITE = fun p f g n -> if (p n) then (f n) else (g n);;
```

Example

```
(2) We introduce "cmp" that computes ordinary function composition:
(3) cC that compute a different way of doing function composition
(4) cK that ignores second argument and returns the first one
let cmp = fun f g n -> g(f n) and cC = fun f g n -> g n (f n) and cK = fun f g -> f ;;
```

Example

```
(5) We introduce im and ip to use subtraction and multiplication as values in Ocaml (that has
various syntactic idiosyncracies)
let im = (+)(-1) and ip = fun n m -> n*m;;
let rec fact = fun n -> cITE ((=)0) (cK 1) (cC (cmp im fact) ip) n;;
```

Example

```
let rec fact = fun n -> cITE ((=)0) (cK 1) (cC (cmp im fact) ip) n;;
```

Methodologies: HOP - Control/3 - NO

But which are the ingredients of the methodology HOP

- Data Extensions through Functional Abstractions
- **Control Extensions through Functional Abstractions**
  - The previous slide showed the way to use HOP in Control Extensions
  - Programmer defines the HOP control functions of which the program needs:
    - In order to implement a specific algorithm or way of computing,
    - Or to satisfy any other requirement of the program development or of its use
  - However, 40 years of Functional Programming produced a great quantity of HOP control functions
  - Functional Languages are equipped with several Libraries of **Functionals** with this purpose
  - These Libraries of Functionals differ one another for the kind of applications in which such Functionals are of general use

Methodologies: HOP - Control/4

But which are the ingredients of the methodology HOP

- Data Extensions through Functional Abstractions
- **Control Extensions through Functional Abstractions**
  - The previous slide showed the way to use HOP in Control Extensions
  - Programmer defines the HOP control functions ...
  - However, 40 years of FP produced a great quantity of HOP control ...
  - Functional map is one of them: It applies one function to each element of one list

Example

```
map behaves in this way:  map g [e1;...;en] returns [(g e1);...;(g en)]

Some applications:
map (fun x -> x+1);;   computes...
map fact;;   computes...
map (fun x -> x > 10);;   computes...

An Ocaml, sequential definition of Map follows (but in parallel computing it has an obvious,
different definition):
let rec map = fun f l -> match l with
                  |[] -> []
                  |x::lR -> (f x)::(map f lR)
                  ;;
```

Methodologies: HOP - Iterative Programming

But which are the ingredients of the methodology HOP

- Data Extensions through Functional Abstractions
- **Control Extensions through Functional Abstractions**
  - However, 40 years of FP produced a great quantity of HOP control ...
  - Iteration in FP: Functionals may be used to iterate functions on index ranges that are collected into lists
  - Iteration in FP: Fold's are collectively called the functionals having this use

Example

```
fold_left behaves in this way:  fold_left g a [e1;...;en] returns g(...(g (g a e1) e2)...)en
fold_right behaves in this way:  fold_right g [e1;...;en] b returns g e1 (g e2 (...(g en b)...))

Some applications:
   fold_left (-) 100;;    computes...
   fold_right (+);;    computes...

and again, by introducing intervals:
   let rec nTom = fun n m -> if n>m then [] else (if n=m then [m] else n::(nTom (n+1) m));;
We give an iterative, combinatory, factorial:
   let fact = cmp (ntom 1) (fold_left (ip) 1);;
where cmp and ip are the function composition and the integer product of the previous slides
Comment and run fact for computing the 3!
```

Example

```
Use iterative HOP in getting a program for defining the size of lists
```

Foundations: Term Reduction Semantics - NO

Table20 – Functional Languages : Syntax

Program Structures

$\mathbf{D} ::= \mathtt{F} \mid \mathtt{T} \mid \ldots$

$\mathbf{F} ::= \mathtt{I} = \mathtt{A}$

$\mathbf{A} ::= \mathtt{fun\ I} \rightarrow \mathtt{E}$

$\mathbf{E} ::= \mathtt{A} \mid \mathtt{E\ E} \mid \mathtt{I} \mid \mathtt{D\ E} \mid \ldots$

$\mathbf{T} ::= \ldots$ $(\mathit{scalar}, \mathit{tuple}, \mathbf{map}, \ \ldots)$

$\mid \ldots$ $(\mathit{list} : \ \mathit{fund.\ meth.})$

$\mid \ldots$ $(\mathit{Concrete} : \ \mathit{fund.\ meth.})$

$\mid \ldots$ $(\mathit{Abstract} : \ \mathit{fund.\ meth.})$

- α – *reduction*
  $\mathtt{fun\ I_1} \rightarrow \mathtt{E} \Longrightarrow \mathtt{fun\ I_2} \rightarrow [\mathtt{I_2}/\mathtt{I_1}]\mathtt{E}$ $\quad$ (when $\mathtt{I_2} \notin \mathtt{Free(E)}$)
- β – *reduction*
  $(\mathtt{fun\ \ I} \rightarrow \mathtt{E})\mathtt{E_0} \Longrightarrow [\mathtt{E_0}/\mathtt{I}]\mathtt{E}$ $\quad$ (when $\mathtt{Free(E_0)} \cap \mathtt{Bound(E)} = \{\}$)

- The above rules are for function application.
- Additional reduction rules are needed for the constructs introducing special class of data and associated operations, and concrete and abstract data.

Foundations: Term Reduction Semantics/2 - NO

- α − *reduction*

  $\texttt{fun I}_1 \rightarrow \texttt{E} \Longrightarrow \texttt{fun I}_2 \rightarrow [\texttt{I}_2/\texttt{I}_1]\texttt{E} \quad (\text{when } \texttt{I}_2 \notin \texttt{Free(E)})$
- β − *reduction*

  $(\texttt{fun I} \rightarrow \texttt{E})\texttt{E}_0 \Longrightarrow [\texttt{E}_0/\texttt{I}]\texttt{E} \quad (\text{when } \texttt{Free(E}_0\texttt{)} \cap \texttt{Bound(E)} = \{\})$

Example

$\texttt{let f = fun n m -> if n=0 then 1 else m and g = fun n -> g(n+1);;}$

$\texttt{f 0 }\underline{\texttt{(g 3)}} \Longrightarrow \texttt{f 0 }\underline{\texttt{(g 4)}} \Longrightarrow \texttt{...}$

$\underline{\texttt{f 0 (g 3)}} \Longrightarrow \texttt{if }\underline{\texttt{[0/n,(g 3)/m]n=0}}\texttt{ then [0/n,(g 3)/m]1 else [0/n,(g 3)/m]m)} \Longrightarrow \texttt{1}$

- Different Reduction Strategies are used in Functional Languages, in determining the sub-term to be reduced:
- External or Normal evaluation results in defining Most Defined functions (Haskell, Miranda)
- Internal or Eager Evaluation results in defining Less Defined functions (ML, Ocaml)

Example

$\texttt{let add = fun x y -> x+y;;}$

$\underline{\texttt{add}}\texttt{ y} \Longrightarrow \texttt{[z/y](fun x y -> x+y)y} \Longrightarrow \texttt{...}$

$\texttt{it requires that add be } \alpha\texttt{-reduced before ...}$

- Combinatory programs do not need α-reduction and limit β-reduction only to the replacement of function names with their definition

Example

$\texttt{let add = (+);; -- this is the combinatory definition of add}$

$\underline{\texttt{add y}} \Longrightarrow \texttt{(+)y}$

$\underline{\texttt{add y 3}} \Longrightarrow \texttt{y+3}$

Combinators and Reduction - NO

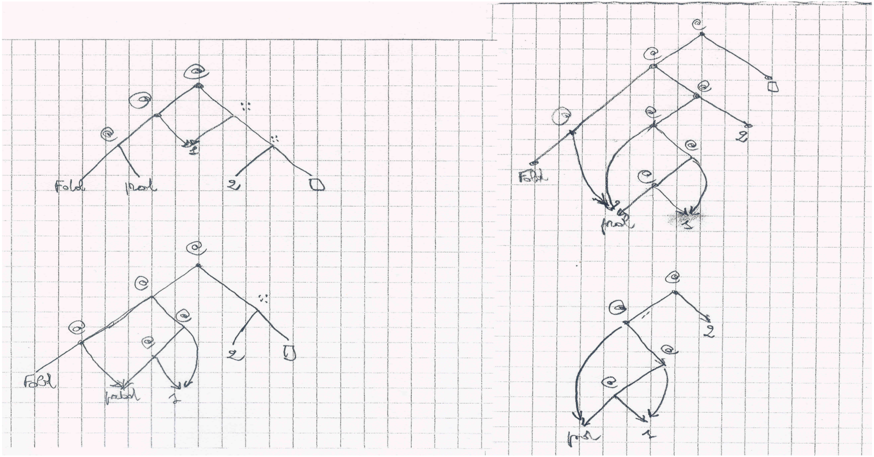
- The use of combinators avoid the use of bindings for parameters
- Thus ruling out the need of α-reduction and limiting the use of β-reduction
- Thus simplifying the reduction semantics and allowing graph reduction instead of term reduction
- We apply it to the iterative, combinatory, factorial:
  `fact = cmp (nTom 1) (fold (prod) 1)` [1]
- in the computation of:
  `fact 2`
- The images in next slide show the reduction graph produced by the reduction of:
  `fold (prod) 1 (nTom 1 2)`

---

[1] where fold stands for fold_left and prod for integer product

Combinators and Reduction: A reduction graph - NO

- Reduction of the expression `fold(prod)1(nTom 1 2)` that results from the invocation `(fact 2)`[2]



---

[2] symbol `@` stands for the application symbol, i.e. `(E1 E2)` is `(E1@E2)` and is drawn as a tree rooted at `@` and having `E1` and `E2` as left and right sub-graphs.

Excercises

Exercise1.

a Apply HOP methodology to the definition, in Ocaml, of the Homogeneous Heap in Lecture 3-4-5-6: In particular, define operation for allocation, deallocation, and for computing the number of blocks that are free. You cannot use the imperative features of Ocaml;

b Show the behaviour of the new data by running the given definitions in the case of an heap of 4 blocks, all initially free. Then, run for the allocation of 4 blocks, followed from the deallocation of the first and then, of the third of them. Finally, run for asking the number of free blocks

c Comment the use of types to get the definitions of point (a). Rewrite the definitions of (a) by using types, if such definitions do not use types in a way to forbid illegal uses of data and of operations.

Exercise2.

a Give, in Ocaml, a memoized definition of the binomial coefficient C^n_k [3];

b Discuss adequately, the choice of the hashTab;

c Run it repeatedly for the computation of the coefficients of $(1+x)^3$ and of $(1+x)^2$ and discuss the execution statistics

Exercise3.

a Give a combinatory, iterative, definition of map in Ocaml;

b Discuss adequately, the choice of the combinators that You have used in the development;

c Run map (prod 5) [2;3] in an interactive session of Ocaml and check execution for the expected answer;

d Show the computation of map (prod 5) [2;3] when map is the combinatory definition, given in (a);

Exercise4.

- Give points (a,b,c,d) of exercise3, in the case of a combinatory, but inductive, definition of fold_left: Use fold_left (prod) 1 [2;3] for the expression to be run in (c) and (d)

---

[3] It is the coefficient of the term x^k in the polynomial development of $(1+x)^n$. Hence, it is such that: $C^n_0 = 1$ (for n$\geq$0), $C^0_k = 0$ (for k>0), $C^n_k = C^{n-1}_{k-1} + C^{n-1}_k$ (for n,k>0)