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K-epiderivatives for set-valued functions and optimization

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Abstract. Exploiting different tangent cones, many derivatives for set-valued functions have been introduced and considered to study optimality. The main goal of the paper is to address a general concept of K-epiderivative and to employ it to develop a quite general scheme for necesary optimality conditions in set-valued problems.

Key words: Tangent cones, K-derivatives and K-epiderivatives, set-valued optimization, optimality conditions

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1 Introduction

In the last years set-valued optimization problems have been considered by many researchers. Besides intrinsic interest, this type of programs arise quite naturally in the context of duality for vector optimization (see, for instance, [14, 17]). Moreover, when the data of a single-valued optimization problem are not exactly known, it is reasonable to replace the values of the involved functions with sets representing their fuzzy outcomes.

In order to study set-valued problems, some notion of derivative for set-valued functions is required. An useful concept of derivative had been introduced by Aubin [1], relying on the Bouligand contingent cone: given two real normed linear spaces E_1 and E_2, the contingent derivative of a set-valued function $H : E_1 \rightrightarrows E_2$ at the point (x, y) is the map $D_T^g H(x, y)$ whose graph equals the Bouligand contingent cone T of the graph of H, i.e.

$$\operatorname{graph} D_T^g H(x, y) = T(\operatorname{graph} H, (x, y)) \tag{1}$$

where

$$\text{graph}\, H := \{(x, y) \in E_1 \times E_2 : y \in H(x)\}.$$

Even if it was originally employed within the context of differential inclusions, since then many applications also to the study of optimality conditions for vector and set-valued optimization problems have been provided (see for instance [6, 15, 20]). Recently, Jahn and Rauh [13] introduced the contingent epiderivative of a set-valued function, extending the concept of "upper contingent derivative" of real-valued ones [1]. The main difference between the definitions of contingent derivative and epiderivative is that the graph is replaced by the epigraph and the epiderivative is single-valued. When E_2 is partially ordered by a pointed convex cone C_{E_2}, a single-valued map $DH(x, y) : E_1 \to E_2$ such that

$$\text{epi}\, DH(x, y) = T(\text{epi}\, H, (x, y)) \tag{2}$$

is called contingent epiderivative of H, where

$$\text{epi}\, H := \{(x, y) \in E_1 \times E_2 : y \in H(x) + C_{E_2}\}.$$

Though single-valuedness seems useful to develop calculus rules [12], we believe that replacing the graph with the epigraph is even more important: approximating just the graph with the contingent cone may not preserve enough information about the function. In fact, as pointed out in [13], necessary and sufficient optimality conditions based on the contingent derivative do not coincide under convexity assumptions. Moreover, sometimes the domain of the contingent derivative is reduced to only one point (see Example 2).

We stress that both these derivatives rely on the well-known concept of Bouligand contingent cone to a set. Actually, several kinds of derivatives have been developed exploiting different types of concrete tangent cones [2, 6, 18, 20]. Moreover, relying on standard properties, general definitions of tangent cone have been proposed and employed to define generalized derivatives of real-valued functions [8, 11, 23].

Following these ideas, we propose a definition of generalized epiderivative for set-valued functions and we employ it to achieve a general scheme for necessary optimality conditions of set-valued optimization problems. Finally, we show how already known conditions can be recovered within this scheme quite easily.

2 Tangent cones and *K*-epiderivatives

The main concrete tangent cones, which are useful in optimization theory, fall within the following general definition (see, for instance, [22]).

Definition 1. *Let E be a real normed linear space. A set-valued mapping $K : 2^E \times E \rightrightarrows E$ is a* tangent cone *on E if for all $A \subseteq E$ and $x \in E$ the set $K(A, x)$ is a cone such that $0^+A \subseteq 0^+K(A, x)$ where*

$$0^+A := \{d \in E : x + td \in A, \forall x \in A, \forall t \geq 0\}$$

is the recession cone of A.

Actually, other concepts of tangent cones can be considered: an axiomatic definition given through six reasonable properties has been proposed in [8] and the mappings satisfying them have been referred to as local cone approximations; another way to introduce general tangent cones is based on the use of quantificational expressions [11, 23].

The disjunction of two sets is a recurrent situation in optimization theory; in order to analyze it, the knowledge of approximations preserving disjunction can be very helpful. We call a pair (K_1, K_2) of tangent cones *admissible* when $A \cap B = \emptyset$ implies $K_1(A, x) \cap K_2(B, x) = \emptyset$.

Given an isotone tangent cone K, i.e. $K(A, x) \subseteq K(B, x)$ whenever $A \subseteq B$, it is well-known [21] that the pair (K, K_c) is admissible, where

$$K_c(A, x) := (K(A^c, x))^c.$$

The family of tangent cones is very wide; we recall the Bouligand tangent cone

$$T(A, x) := \{w \in E : \exists \{t_n\} \downarrow 0, \exists \{w_n\} \to w \text{ s.t. } x + t_n w_n \in A\}$$

and the Dubovitskij-Miljutin tangent cone

$$DM(A, x) := T_c(A, x) = \{w \in E : \forall \{t_n\} \downarrow 0, \forall \{w_n\} \to w, x + t_n w_n \in A\}.$$

Since the approach developed in this paper is mainly based on tangent cones of epigraphs, it is interesting to consider also tangent cones on product spaces. Relying on the structure of product spaces, some particular ones can be introduced. However, they seem not to be so popular; to the best of our knowledge, only one of them has been considered in a few papers [18, 19]. Slightly modifying classical ones, fairly new tangent cones on product spaces can be introduced. For instance, given $A \subseteq E_1 \times E_2$, a modified Bouligand tangent cone is

$$\begin{aligned} T_m(A, (x, y)) := \{(u, v) \in E_1 \times E_2 : \exists \{t_n\} \downarrow 0, \exists \{u_n\} \to u \text{ s.t.} \\ \forall \{v_n\} \to v, \ (x + t_n u_n, y + t_n v_n) \in A\} \end{aligned}$$

while a modified Dubovitskij-Miljutin tangent cone is

$$\begin{aligned} DM_m(A, (x, y)) &:= (T_m)_c(A, (x, y)) \\ &= \{(u, v) \in E_1 \times E_2 : \forall \{t_n\} \downarrow 0, \\ &\qquad \forall \{u_n\} \to u, \exists \{v_n\} \to v \text{ s.t. } (x + t_n u_n, y + t_n v_n) \in A\}. \end{aligned}$$

The four above tangent cones are all isotone and therefore the pairs (T, DM) and (T_m, DM_m) are admissible.

Concrete epiderivatives for set-valued map have introduced in [6] relying on Bouligand and Clarke tangent cones. Inspired by the axiomatic approach for the real-valued case [8], we propose the following general definition.

Definition 2. *The K-epiderivative of $H : E_1 \rightrightarrows E_2$ at $(x, y) \in$* graph H *is the set-valued map $D_K H(x, y) : E_1 \rightrightarrows E_2$ such that*

$$\operatorname{graph} D_K H(x, y) = K(\operatorname{epi} H, (x, y)),$$

that is

$$v \in D_K H(x, y)(u) \Leftrightarrow (u, v) \in K(\operatorname{epi} H, (x, y)).$$

It worth noting that this K-epiderivative does not collapse into the one of [8] when H is a real single-valued map; in fact, the former is always set-valued while the latter involves the infimum of the "vertical" lines of $K(\operatorname{epi} H, (x, y))$ to achieve single-valuedness, i.e.

$$D_K H(x, y)(u) := \inf\{\beta \in \mathbb{R} : (u, \beta) \in K(\operatorname{epi} H, (x, y))\}.$$

This definition is suitable only for real-valued maps. Since it is equivalent to require

$$K(\operatorname{epi} H, (x, y)) \subseteq \operatorname{epi} D_K H(x, y) \subseteq \operatorname{cl} K(\operatorname{epi} H, (x, y)), \tag{3}$$

it could be reasonable to consider just the above chain of inclusions as a definition of the K-epiderivative also in the set-valued case; such a definition would not guarantee uniqueness. On the contrary, Definition 2 allows to achieve uniqueness, satisfying (3); in fact, we have

$$\begin{aligned} \operatorname{epi} D_K H(x, y) &= \operatorname{graph} D_K H(x, y) + (\{0\} \times C_{E_2}) \\ &= K(\operatorname{epi} H, (x, y)) + (\{0\} \times C_{E_2}) \\ &= K(\operatorname{epi} H, (x, y)) \end{aligned}$$

where the last equality is due to the inclusions $\{0\} \times C_{E_2} \subseteq 0^+ \operatorname{epi} H \subseteq 0^+ K(\operatorname{epi} H, (x, y))$. Hence, the reason why it does not collapse in the scalar case is the lack of single-valuedness. To recover it, the contingent epiderivative of a set-valued map has been introduced in [13] but unfortunately it may not exist (see, for instance, the example in [3]); in fact, to achieve single-valuedness, some particular assumptions are needed [5, 12]. Obviously, the contingent epiderivative can not coincide with the T-epiderivative; however, their epigraphs always do.

Relying on (1), also the concept of contingent derivative can be generalized in the same way.

Definition 3. *The K-derivative of $H : E_1 \rightrightarrows E_2$ at $(x, y) \in \operatorname{graph} H$ is the set-valued map $D_K^g H(x, y) : E_1 \rightrightarrows E_2$ such that*

$$\operatorname{graph} D_K^g H(x, y) = K(\operatorname{graph} H, (x, y)).$$

It is clear that K-epiderivatives and K-derivatives are strictly connected since

$$D_K H(x, y) = D_K^g (H + C_{E_2})(x, y),$$

where $(H + C_{E_2})(x) := H(x) + C_{E_2}$. However, some differences do exist. For instance, necessary optimality conditions based on K-derivatives do not turn out to be sufficient in the convex case while the ones based on K-epiderivatives do (see [6] when K is Bouligand or Clarke tangent cone). Moreover, it may happen that K-derivatives do not exist or have very small domains, as shown by the following examples.

Example 1. Let $H : \mathbb{R} \rightrightarrows \mathbb{R}$ be the function $H(x) = [x^2, 2x^2]$. Considering the point $(0,0)$, we have

$$DM(\operatorname{graph} H, (0,0)) = \varnothing \quad \text{and} \quad DM(\operatorname{epi} H, (0,0)) = \mathbb{R} \times \operatorname{int} \mathbb{R}_+.$$

Example 2. Let $H : \mathbb{R} \rightrightarrows \mathbb{R}$ be the function $H(x) = [-2\sqrt{|x|}, -\sqrt{|x|}]$. Considering the point $(0,0)$, we have

$$T(\operatorname{graph} H, (0,0)) = \{0\} \times \mathbb{R}_- \quad \text{and} \quad T(\operatorname{epi} H, (0,0)) = \mathbb{R}^2$$

and hence

$$D_T^g H(0,0)(u) = \begin{cases} \mathbb{R}_- & \text{if } u = 0 \\ \varnothing & \text{if } u \neq 0 \end{cases} \quad \text{and} \quad D_T H(0,0)(u) = \mathbb{R}.$$

3 A general approach to optimality conditions

Starting from the seminal results presented in [6], in recent years many authors studied optimality conditions for set-valued problems, using different concepts of derivative. Though the most used one has been the contingent derivative (see [6, 15, 19, 20]), also other ones have been employed [6, 19, 20]. More recently concepts of epiderivatives [3, 5, 9, 10, 13] have been introduced and applied in the same framework.

In this section we aim to present a general scheme to obtain necessary optimality conditions based on K-epiderivatives. Such an approach recalls the one presented in [4] for the real single-valued case.

Let X, Y, Z be real normed linear spaces, $C_Y \subseteq Y$, $C_Z \subseteq Z$ be pointed convex cones with nonempty interior and $F : X \rightrightarrows Y$, $G : X \rightrightarrows Z$ be set-valued functions. We will consider the following constrained set-valued optimization program

$$\begin{cases} \min F(x) \\ \text{subject to} \\ G(x) \cap -C_Z \neq \varnothing \\ x \in S, \end{cases} \tag{4}$$

supposing that the feasible region is nonempty. We recall that the pair $(\bar{x}, \bar{y}) \in \operatorname{graph} F$ is called a weak minimizer of (4) if $\bar{x} \in X$ satisfies the constraints and there is no feasible $x \in X$ such that

$$F(x) \cap (\bar{y} - \operatorname{int} C_Y) \neq \varnothing.$$

Relying on the cartesian product of the objective and constraining functions, i.e. $(F, G)(x) := F(x) \times G(x)$, optimality can be obviously written as

$$\text{graph}(F, G) \cap [S \times (\bar{y} - \text{int}\, C_Y) \times -C_Z] = \varnothing, \tag{5}$$

or equivalently

$$\text{epi}(F, G) \cap [S \times (\bar{y} - \text{int}\, C_Y) \times -C_Z] = \varnothing. \tag{6}$$

The characterization given by disjunction (6) is the key to obtain the following optimality condition based on general K-epiderivatives.

Theorem 1. *Let (K_1, K_2) be an admissible pair. Suppose there exists K_3 tangent cone on X such that the inclusion*

$$K_3(S, x) \times -\text{int}\, C_Y \times -\text{int}\,\text{cone}(C_Z + z)$$
$$\subseteq K_2(S \times (y - \text{int}\, C_Y) \times -C_Z, (x, y, z)) \tag{7}$$

holds for all $x \in S$, $y \in Y$, $z \in -C_Z$. If $(\bar{x}, \bar{y}) \in \text{graph}\, F$ is a weak minimizer of (4), then for any given $\bar{z} \in G(\bar{x}) \cap -C_Z$ the following condition

$$D_{K_1}(F, G)(\bar{x}, (\bar{y}, \bar{z}))(u) \cap -[\text{int}\, C_Y \times \text{int}\,\text{cone}(C_Z + \bar{z})] = \varnothing \tag{8}$$

holds for all $u \in K_3(S, \bar{x})$.

Proof. The definition of admissible pair implies that given any $\bar{z} \in G(\bar{x}) \cap -C_Z$, we have

$$K_1(\text{epi}(F, G), (\bar{x}, \bar{y}, \bar{z})) \cap K_2(S \times (\bar{y} - \text{int}\, C_Y) \times -C_Z, (\bar{x}, \bar{y}, \bar{z})) = \varnothing.$$

Thus, assumption (7) implies

$$K_1(\text{epi}(F, G), (\bar{x}, \bar{y}, \bar{z})) \cap (K_3(S, \bar{x}) \times -\text{int}\, C_Y \times -\text{int}\,\text{cone}(C_Z + \bar{z})) = \varnothing. \tag{9}$$

Therefore, given any $u \in K_3(S, \bar{x})$ and $v \in D_{K_1}(F, G)(\bar{x}, \bar{y})(u)$, we have

$$(u, v) \in K_1(\text{epi}(F, G), (\bar{x}, \bar{y}, \bar{z}))$$

and thus (9) implies that

$$v \notin -[\text{int}\, C_Y \times -\text{int}\,\text{cone}(C_Z + \bar{z})].$$

Hence, (8) follows. □

Actually, assumption (7) is not so restrictive as it may seem. On the contrary, it is quite natural, requiring (roughly speaking) that the tangent cone of a "cylinder" with base S contains the "cylinder" whose base is a suitable tangent cone of S; in fact, this assumption is satisfied by a wide range of tangent cones (see Section 4 for some important cases). For instance, when $S = X$, a class of tangent cones such that (7) always holds is given by the local cone approximations.

Remark 1. A set-valued optimization problem with no explicit constraint can be viewed as a particular case of problem (4) setting $G \equiv \{0\}$. Thus

$$D_{K_1}F(\bar{x},\bar{y})(u) \cap -\text{int}\, C_Y = \emptyset, \tag{10}$$

is a necessary optimality condition for all $u \in K_3(S,\bar{x})$ whenever (K_1, K_2) is an admissible pair and K_3 satisfies

$$K_3(S,x) \times -\text{int}\, C_Y \subseteq K_2(S \times (y - \text{int}\, C_Y), (x,y)), \quad \forall x \in S,\ y \in Y.$$

Remark 2. An unconstrained set-valued optimization problem can be viewed as a particular case of problem (4) setting $G \equiv \{0\}$ and $S = X$; thus, at a weak minimizer of (4), condition (10) holds for all $u \in X$ whenever K_2 is a local cone approximation such that the pair (K_1, K_2) is admissible.

The optimality condition of Theorem 1 is expressed as the impossibility of a suitable system and it recalls the well-known Abadie Linearization Lemma of nonlinear programming; thus, no multipliers are involved. Minumum principle type optimality conditions for problem (4) can be obtained from Theorem 1 by applying standard separation arguments. To achieve a set of multipliers, we need to recall that the positive dual cone of a given cone $C \subseteq E$ is

$$C^* := \{x^* \in E^* : \langle x^*, x\rangle \geq 0, \forall x \in C\},$$

where $\langle \cdot , \cdot \rangle$ denotes the duality pairing between E and its topological dual E^*.

Theorem 2. *Let K_1, K_2, K_3 satisfy the assumptions of the previous theorem. Suppose that S and* epi(F,G) *are convex, K_1 and K_3 are convex preserving. If $(\bar{x},\bar{y})$ is a weak minimizer of (4), then for all $\bar{z} \in G(\bar{x}) \cap -C_Z$ there exist $t \in C_Y^*$, $s \in C_Z^*$ not both zero such that $\langle s, \bar{z}\rangle = 0$ and*

$$\langle t, v\rangle + \langle s, w\rangle \geq 0 \tag{11}$$

holds for all $(v,w) \in D_{K_1}(F,G)(\bar{x},(\bar{y},\bar{z}))(u)$ and $u \in K_3(S,\bar{x})$.

Proof. Given any $\bar{z} \in G(\bar{x}) \cap -C_Z$, by the previous theorem we have

$$D_{K_1}(F,G)(\bar{x},\bar{y})(K_3(S,\bar{x})) \cap -[\text{int}\, C_Y \times (\text{int}\,\text{cone}(C_Z + \bar{z})] = \emptyset \tag{12}$$

where

$$D_{K_1}(F,G)(\bar{x},\bar{y})(K_3(S,\bar{x})) := \bigcup_{u \in K_3(S,\bar{x})} D_{K_1}(F,G)(\bar{x},\bar{y})(u) \tag{13}$$

To prove that the above set is a convex cone, let us consider any $\lambda_i \geq 0$, $v_i \in D_{K_1}(F,G)(\bar{x},\bar{y})(u_i)$ and $u_i \in K_3(S,\bar{x})$, $i = 1,\ldots,n$. Since

$$(u_i, v_i) \in K_1(\text{epi}(F,G), (\bar{x},(\bar{y},\bar{z}))), \quad i = 1,\ldots,n$$

and the assumptions imply that $K_1(\text{epi}(F,G),(\bar{x},(\bar{y},\bar{z})))$ is a convex cone, we have

$$\sum_{i=1}^{n} \lambda_i(u_i, v_i) \in K_1(\mathrm{epi}(F, G), (\bar{x}, (\bar{y}, \bar{z}))).$$

Moreover, also $K_3(S, \bar{x})$ is a convex cone; therefore, we have

$$\sum_{i=1}^{n} \lambda_i v_i \in D_{K_1}(F, G)(\bar{x}, \bar{y})(K_3(S, \bar{x})).$$

Hence, (13) is a convex cone and a well-known separation theorem implies that there exist $t \in Y^*$ and $s \in Z^*$ not both zero such that (11) holds for all $(v, w) \in D_{K_1}(F, G)(\bar{x}, (\bar{y}, \bar{z}))(u)$ and $u \in K_3(S, \bar{x})$ and moreover

$$\langle t, c_Y \rangle + \langle s, c_Z + \bar{z} \rangle \geq 0$$

holds for all $c_Y \in C_Y$ and $c_Z \in C_Z$. It is clear that the above inequality implies $t \in C_Y^*$, $s \in C_Z^*$ and $\langle s, \bar{z} \rangle \geq 0$. Since $\bar{z} \in -C_Z$, we have also the complementarity slackness condition $\langle s, \bar{z} \rangle = 0$. □

Since separation theorems require convexity, what is really needed in the above theorem is that the sets in (12) are convex; hence, the assumptions on (F, G) and S can be obviously replaced by the convexity of the tangent cones K_1, K_3.

Remark 3. The proof of Theorem 1 (and consequently also Theorem 2) involves characterization (6) which relies on epigraphs; considering instead characterization (5), we can achieve the analogous results for K-derivatives. However, approximating the epigraph with a tangent cone generally gives more information: in the following example necessary optimality conditions based on the T-epiderivative allow to drop a pair, which is not a weak minimizer, while the ones based on the T-derivative do not. Let $F : \mathbb{R} \rightrightarrows \mathbb{R}$ be the function

$$F(x) = \begin{cases} [2, 3] & \text{if } x \in \mathbb{Q} \\ [0, 1] & \text{if } x \notin \mathbb{Q}. \end{cases}$$

Obviously, the pair $(0, 2)$ is not a weak minimizer of F over $\mathbb{R}$ with respect to the ordering cone $\mathbb{R}_+$. It is easy to check that

$$D_T^g F(0, 2)(u) = \mathbb{R}_+ \quad \text{and} \quad D_T F(0, 2)(u) = \mathbb{R}.$$

Thus, (10) is not satisfied by any $u \in \mathbb{R}$ while the following necessary condition holds

$$D_T^g F(0, 2)(u) \cap -\mathrm{int}\, \mathbb{R}_+ = \varnothing, \quad \forall u \in \mathbb{R}.$$

4 Some applications

The aim of this section is to show that many known optimality conditions can be obtained applying Theorem 1 with some particular choices of the tangent cones.

Since the most popular derivatives are based on Bouligand tangent cone [3, 5, 6, 10, 13], we start considering the choice $K_1 = T$. Therefore, we can take $K_2 = T_c = DM$ and moreover it is easy to check that (7) holds with $K_3 = DM$.

In [13] problem (4) with no explicit constraint is studied. Since the authors suppose that F is defined only on the set S, their problem can be equivalently expressed as an unconstrained one with objective function F_S defined as

$$F_S(x) := \begin{cases} F(x) & \text{if } x \in S \\ \emptyset & \text{if } x \notin S. \end{cases}$$

In their Theorem 7 the following necessary optimality condition has been proved

$$DF_S(\bar{x}, \bar{y})(u) \notin -\text{int}\, C_Y, \quad \forall u \in (S - \bar{x}), \tag{14}$$

where DF_S denotes the contingent epiderivative of F_S as defined in (2). This condition can also be deduced from Theorem 1 with our choice, taking into account Remark 2. In fact, (10) can be equivalently written as

$$\text{graph}\, D_T F_S(\bar{x}, \bar{y}) \cap [X \times -\text{int}\, C_Y] = \emptyset.$$

Since $\text{graph}\, D_T F_S(\bar{x}, \bar{y}) = \text{epi}\, DF_S(\bar{x}, \bar{y})$, then (14) follows. It worth noting that condition (14) does not hold if $F : X \rightrightarrows Y$ and $S \neq X$ even in the real single-valued case, as the following example shows.

Example 3. Let $S = \mathbb{R}^2_+$ and $F : \mathbb{R}^2 \to \mathbb{R}$ defined as follows

$$F(x) = \begin{cases} \|x\| & \text{if } x_2 \geq 0 \\ -\|x\| & \text{if } x_2 < 0. \end{cases}$$

The point $\bar{x} = (0, 0)$ is a global minimum of F on S but, choosing $\bar{u} = (1, 0)$, we have $DF(\bar{x}, 0)(\bar{u}) = -1$.

The same contingent epiderivative has been exploited in [10] to obtain a minimum principle optimality condition for problem (4). Since T and DM are convex preserving, this result can be deduced from Theorem 2 with the same reasoning used to obtain (14).

We notice that if (K_1, K_2) is an admissible pair then also (K_1', K_2) is admissible, whenever $K_1' \subseteq K_1$. Hence, the above choice can be considered with the Clarke tangent cone in place of the Bouligand one. In particular, the seminal results presented in [6] follow immediately.

Another concept of contingent epiderivative $D_\uparrow F(\bar{x}, \bar{y})(u)$ has been proposed in [5] and independently in [3], considering only the minimal points of $D_T F(\bar{x}, \bar{y})(u)$. Since $\text{graph}\, D_\uparrow F(\bar{x}, \bar{y}) \subseteq \text{graph}\, D_T F(\bar{x}, \bar{y})$, Theorem 5 of [5] follows from Theorem 1 in a way analogous to Theorem 7 of [13].

Also necessary conditions for Benson proper minimum points can be recovered within our scheme. In [3] such a condition has been presented for problem (4) with no explicit constrains under the assumption that C_Y has a compact base. It is well-known (see [3]) that under this assumption any Benson proper minimum point is a weak one with respect to the ordering given by some convex cone $\hat{C}_Y$ such that $C_Y \setminus \{0\} \subseteq \operatorname{int} \hat{C}_Y$. Thus, by Theorem 1 we get that

$$D_T F(\bar{x}, \bar{y})(u) \cap -\operatorname{int} \hat{C}_Y = \emptyset, \quad \forall u \in X,$$

and Theorem 1 of [3] follows.

Also other types of derivative have been employed in set-valued optimization; for instance, optimality conditions for problem (4) have been presented in [20], relying on the lower Dini derivative introduced in [16]:

$$\underline{D}F(\bar{x}, \bar{y})(u) = \liminf_{(t,u') \to (0^+, u)} t^{-1}[F(\bar{x} + tu') - \bar{y}]$$

where lim inf is intended in the sense of set-valued analysis [2]. It is easy to check that

$$\underline{D}F(\bar{x}, \bar{y})(u) + C_Y \subseteq D_{DM_m} F(\bar{x}, \bar{y})(u) = \underline{D}(F + C_Y)(\bar{x}, \bar{y})(u) \tag{15}$$

holds for all $u \in X$. Thus, let us consider the choice $K_1 = DM_m$: since $K_2 = T_m$ and $K_3 = T$ satisfy (7), we can apply Theorem 1 to problem (4) with no explicit constraints (see Remark 1) to get

$$D_{DM_m} F(\bar{x}, \bar{y})(u) \cap -\operatorname{int} C_Y = \emptyset, \quad \forall u \in T(S, \bar{x}).$$

Hence, (15) implies Proposition 3.1 of [20]. Moreover, in the same paper problem (4) has been considered also with explicit constraints. Thus, Theorem 4.1 of [20] follows arguing as above and noting that

$$D_{DM_m}(F, G)(\bar{x}, (\bar{y}, \bar{z}))(u) = (D_{DM_m} F(\bar{x}, \bar{y})(u), D_{DM_m} G(\bar{x}, \bar{y})(u)).$$

Though introduced with a different aim [18], the Shi tangent cone

$$\begin{aligned} T_{\text{Shi}}(A, (x, y)) := \{(u, v) \in E_1 \times E_2 : \exists \{t_n\} \subseteq (0, +\infty), \exists \{u_n\} \to u, \\ \exists \{v_n\} \to v \text{ s.t. } t_n u_n \to 0, (x + t_n u_n, y + t_n v_n) \in A\} \end{aligned}$$

has been used to obtain optimality conditions in [19] for unconstrained programs. Since it is isotone, we can choose $K_1 = T_{\text{Shi}}$ and $K_2 = (T_{\text{Shi}})_c$; the further choice $K_3 = DM$ allows to satisfy (7) and therefore we get the new necessary optimality condition

$$\begin{aligned} & D_{T_{\text{Shi}}}(F, G)(\bar{x}, (\bar{y}, \bar{z}))(u) \\ & \quad \cap -[\operatorname{int} C_Y \times \operatorname{int} \operatorname{cone}(C_Z + \bar{z})] = \emptyset, \quad \forall u \in DM(S, \bar{x}) \end{aligned}$$

for the constrained problem (4).

5 Conclusion

Relying on the concept of *K*-epiderivative of a set-valued map, we have proved necessary optimality conditions in a quite general form. In the last section we have shown how some concrete ones can be easily obtained with suitable choices. Actually, many others could be developed in an analogous way. For instance, the closed radial tangent cone could be used following the ideas in [9] or the pair composed by the feasible and weak feasible tangent cones could be considered as already pointed out in [4] for the real single-valued case.

For the sake of simplicity, we showed optimality conditions only for (global) weak minimizers; however, we stress that all the results presented in this paper still hold for local weak minimizers, just supposing that at least one of the involved tangent cones is local. In fact, the characterizations (5) and (6) hold for local optima if disjunction is achieved considering the two sets and a suitable cylinder neighbourhood $I(\bar{x}) \times Y \times Z$. Moreover, it worth noting that all the concrete tangent cones, we considered, are local except the Shi one: however, it satisfies the local property for the above type of cylinder neighbourhoods.

We stress that not all the known optimality conditions fall within this approach. In fact, it is based only on derivatives and tangent cones while some results rely on suitable subdifferentials; for instance, a quite different approach has been proposed in [7], considering the approximate subdifferential of a suitable distance function.

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