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ORIGINAL ARTICLE

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On sufficient second order optimality conditions in multiobjective optimization

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Abstract A second order sufficient optimality criterion is presented for a multi-objective problem subject to a constraint given just as a set. To this aim, we first refine known necessary conditions in such a way that the sufficient ones differ by the replacement of inequalities by strict inequalities. Furthermore, we show that no relationship holds between this criterion and a sufficient multipliers rule, when the constraint is described by inequalities and equalities. Finally, improvements of this criterion for the unconstrained case are presented, stressing the differences with single-objective optimization.

Keywords Second order contingent sets · Sufficient optimality criteria

Mathematics Subject classification 90C29 · 90C48 · 46N10

1 Introduction

The aim of this paper is to provide sufficient second order optimality conditions for the multiobjective program

$$\min_{\mathbb{R}^\ell_+} f(x) \quad \text{subject to} \quad x \in X \tag{1}$$

where $f : \mathbb{R}^n \to \mathbb{R}^\ell$ and $X \subseteq \mathbb{R}^n$ is any subset, matching the corresponding necessary conditions presented in Bigi and Castellani (2000). The notation $\min_{\mathbb{R}^\ell_+}$ marks vector minimum with respect to the cone $\mathbb{R}^\ell_+$: $\bar{x} \in X$ is said to be a *local vector minimum point* of (1) if there exists a neighbourhood N of $\bar{x}$ such that no $x \in X \cap N$ satisfies $f(x) - f(\bar{x}) \in -\mathbb{R}^\ell_+$ with $f(x) \neq f(\bar{x})$, that is $f_k(x) \leq f_k(\bar{x})$ for all $k \in K := \{1, ..., \ell\}$ with $f_{\bar{k}}(x) < f_{\bar{k}}(\bar{x})$ for at least one index.

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Relying on the idea originally exploited in Ben-Tal (1980) and further developed in Cominetti (1990) and Penot (1994, 1999) for the single-objective case, second order necessary optimality conditions for (1) have been proved in Bigi and Castellani (2000), checking optimality just along those curves that allow to achieve a second order expansion (parabolic curves). In order to know which parabolic curves allow to move away from a given point preserving feasibility, the *second order contingent set* $T^2(X, \bar{x}, d)$ has therefore been considered: we just recall that $w \in T^2(X, \bar{x}, d)$ if there exist sequences $t_n \downarrow 0$ and $w_n \to w$ such that $\bar{x} + t_n d + 2^{-1} t_n^2 w_n \in X$ (see Bonnans et al. 1999 for further details). Exploiting this tool, Theorem 3.1 of Bigi and Castellani (2000) states that given any local vector minimum point $\bar{x} \in X$ of (1), setting $K(\bar{x}, d) := \{k \in K : \nabla f_k(\bar{x}) \cdot d = 0\}$, the condition

$$\max_{k \in K(\bar{x}, d)} [\nabla f_k(\bar{x}) \cdot w + \nabla^2 f_k(\bar{x})(d, d)] \geq 0 \tag{2}$$

holds for any $d \in D_{\leq}(f, \bar{x}) \cap T(X, \bar{x})$ and any $w \in T^2(X, \bar{x}, d)$, where $D_{\leq}(f, \bar{x})$ denotes the set of the *descent directions* for f at $\bar{x}$, i.e. $d \in D_{\leq}(f, \bar{x})$ if $\nabla f_k(\bar{x}) \cdot d \leq 0$ for any $k \in K$, and $T(X, \bar{x}) := T^2(X, \bar{x}, 0)$ denotes the Bouligand contingent cone.

A standard procedure to obtain sufficient optimality conditions from necessary ones is to replace inequalities with strict inequalities (see for instance Ben-Tal 1980; Ben-Tal and Zowe 1982; Wang 1991). This procedure does not work when dealing with the above second order necessary conditions, as shown by the following example.

Example 1 Consider (1) with $n = 3$, $\ell = 2$, and

$$f_1(x_1, x_2, x_3) = x_3^2 - x_1, \quad f_2(x_1, x_2, x_3) = x_2^2 - \sqrt[3]{x_3^7},$$
$$X = \{\, x \in \mathbb{R}^3 \ : \ x_1^2 \leq x_2^3 \,\}.$$

Choosing the point $\bar{x} = (0, 0, 0)$, we have

$$T^2(X, \bar{x}, d) = \begin{cases} \{0\} \times \mathbb{R}_+ \times \mathbb{R} & \text{if } d_2 = 0 \\ \mathbb{R}^3 & \text{if } d_2 \neq 0 \end{cases}$$

for any nonzero $d \in T(X, \bar{x}) \cap D_{\leq}(f, \bar{x}) = \{0\} \times \mathbb{R}_+ \times \mathbb{R}$. Therefore,

$$\max_{k \in K(\bar{x}, d)} \nabla f_k(\bar{x}) \cdot w + \nabla^2 f_k(\bar{x})(d, d)$$

becomes

$$\max\{\, -w_1 + 2d_3^2,\ 2d_2^2 \,\}$$

which is positive for any $d \in T(X, \bar{x}) \cap D_{\leq}(f, \bar{x})$ and $w \in T^2(X, \bar{x}, d)$. However, $\bar{x}$ is not a local vector minimum point since f_1 is zero and f_2 is negative along the curve described by the feasible points $x_t = (t^6, t^4, t^3)$ for $t \in (0, 1)$.

A class of sets X, for which such a replacement yields sufficiency in the single-objective case, has been identified in Bonnans et al. (1999). However, we aim to consider a different kind of approach, which does not require any particular assumption on X and has already been analysed for single-objective optimization in Penot (1999). Another second order approximation of the constraining set is considered in order to achieve an additional necessary condition. Coupling it together with Theorem 3.1 of Bigi and Castellani (2000), we can turn them into a sufficient criterion mainly replacing inequalities with strict inequalities. In the unconstrained case the additional condition is not actually needed, since no constraint has to be taken into account; however, some meaningful differences with the well-known second order optimality conditions for unconstrained single-objective optimization hold and are analysed in the last section.

2 Sufficient optimality criterion

There are two explanations why changing the inequality in (2) into a strict one does not lead to sufficiency. The second order contingent sets may be empty (see the example in Penot 1999) and the corresponding necessary optimality conditions are meaningless in such a case, since they are obviously satisfied by any objective function. Furthermore, there is no convincing reason why it should be enough to test optimality only along parabolic curves, as the above example corroborates. These two drawbacks can be overcome considering in addition to $T^2(X, \bar{x}, d)$ also the following sets that take into account curves different from parabolas, which allow to move away from the given point preserving feasibility.

Definition 1 (*Penot* 1999) *$T_0^2(X, \bar{x}, d)$ denotes the asymptotic second order contingent set of X at $\bar{x} \in \operatorname{cl} X$ in the direction $d \in \mathbb{R}^n$, namely $w \in T_0^2(X, \bar{x}, d)$ if there exist sequences $t_n \downarrow 0$, $\gamma_n \downarrow 0$ and $w_n \to w$ such that*

$$\gamma_n^{-1} t_n^2 \to 0 \ \ \text{and} \ \ \bar{x} + t_n d + \gamma_n w_n \in X.$$

It is worth noting that the above sets do not consider all possible curves but just those that, roughly speaking, approach $\bar{x}$ slower than a parabola.

Proposition 1 *If $d \in T(X, \bar{x})$, then $T^2(X, \bar{x}, d) \cup T_0^2(X, \bar{x}, d) \neq \emptyset$.*

Proof By assumption there exist $t_n \downarrow 0$ and $d_n \to d$ such that $\bar{x} + t_n d_n \in X$. Let $s_n := ||d_n - d||$; if $s_n = 0$ for at least a subsequence, then $0 \in T^2(X, \bar{x}, d)$. Otherwise, let $w_n := s_n^{-1}(d_n - d)$; taking the suitable subsequence, we can suppose $s_n \downarrow 0$ and $w_n \to w$ for some $w \neq 0$. Furthermore, we have that $\bar{x} + t_n d + t_n s_n w_n \in X$; considering the suitable subsequence, we can suppose $a_n := t_n^{-1} s_n \to r$ for some $r \geq 0$ or $a_n \to +\infty$. In the latter case we have $w \in T_0^2(X, \bar{x}, d)$ while in the former we have $2rw \in T^2(X, \bar{x}, d)$ since $\hat{w}_n := 2a_n w_n \to 2rw$ and $\bar{x} + t_n d + 2^{-1} t_n^2 \hat{w}_n \in X$. □

Therefore, the first drawback can be avoided if the necessary conditions of Bigi and Castellani (2000) are refined, taking into account also the asymptotic second order contingent set. It is worth noting that the following result requires that f is twice continuously differentiable even though only the Jacobians of the components of f are involved in the condition.

Theorem 1 *If $\bar{x} \in X$ is a local vector minimum point of (1), then*

$$\max_{k \in K} [\nabla f_k(\bar{x}) \cdot w] \geq 0, \quad \forall\, w \in T_0^2(X, \bar{x}, d) \tag{3}$$

holds for any direction $d \in D_{\leq}(f, \bar{x}) \cap T(X, \bar{x})$.

Proof Ab absurdo, suppose that $\nabla f_k(\bar{x}) \cdot w < 0$ holds for all $k \in K$ and some $d \in D_{\leq}(f, \bar{x}) \cap T(X, \bar{x})$ and $w \in T_0^2(X, \bar{x}, d)$. Furthermore, there exist $t_n \downarrow 0$, $\gamma_n \downarrow 0$ and $w_n \to w$ such that $\gamma_n^{-1} t_n^2 \to 0$ and $x_n := \bar{x} + t_n d + \gamma_n w_n \in X$. Since f_k is twice differentiable, we have

$$f_k(x_n) - f_k(\bar{x}) \leq \nabla f_k(\bar{x}) \cdot \gamma_n w_n + 2^{-1} \nabla^2 f_k(\bar{x})(t_n d + \gamma_n w_n, t_n d + \gamma_n w_n) + t_n^2 \varepsilon_n$$

with $\varepsilon_n \to 0$. Dividing by γ_n and taking the limit as $n \to +\infty$, the right handside goes to $\nabla f_k(\bar{x}) \cdot w < 0$ and therefore $f_k(x_n) < f_k(\bar{x})$ whenever n is large enough. This contradicts the local optimality of $\bar{x}$. □

Therefore, Theorems 3.1 of Bigi and Castellani (2000) and 1 together allow to check optimality along any curve preserving feasibility: also the second drawback is overcome. In fact, replacing inequalities with strict inequalities in these two necessary conditions, we achieve the following sufficient optimality criterion.

Theorem 2 *Let $\bar{x} \in X$. If for each nonzero direction $d \in D_{\leq}(f, \bar{x}) \cap T(X, \bar{x})$ the condition*

$$\max_{k \in K(\bar{x}, d)} [\nabla f_k(\bar{x}) \cdot w + \nabla^2 f_k(\bar{x})(d, d)] > 0, \tag{4}$$

holds for any $w \in T^2(X, \bar{x}, d)$ such that $w \cdot d = 0$ and the condition

$$\max_{k \in K(\bar{x}, d)} [\nabla f_k(\bar{x}) \cdot w] > 0, \tag{5}$$

holds for any nonzero $w \in T_0^2(X, \bar{x}, d)$ such that $w \cdot d = 0$, then $\bar{x}$ is a local vector minimum point of (1).

Proof Ab absurdo, suppose that there exists a sequence $\{x_n\} \subseteq X$ with $x_n \to \bar{x}$ such that $f(x_n) \neq f(\bar{x})$ and $f_k(x_n) \leq f_k(\bar{x})$ for all $k \in K$. Let $t_n = ||x_n - \bar{x}||$ and $d_n = t_n^{-1}(x_n - \bar{x})$. Taking the suitable subsequence, we can suppose that $t_n \downarrow 0$ and $d_n \to d$ for some unit vector d; therefore, $d \in T(X, \bar{x})$ and it is easy to check that $d \in D_{\leq}(f, \bar{x})$. Choosing s_n and w_n as in Proposition 1, we have $x_n = \bar{x} + t_n d + t_n s_n w_n$ with $w_n \to w$ for some suitable $w \neq 0$ and $s_n \downarrow 0$ unless $s_n = 0$ for all indexes (in which case we can set $w_n = w = 0$). Since d_n and d are both unit vectors, then

$$||d_n||^2 = ||d||^2 + ||s_n w_n||^2 + 2 s_n w_n \cdot d$$

implies $w_n \cdot d = -2^{-1} s_n ||w_n||^2$ and therefore $w \cdot d = 0$. Given any $k \in K(\bar{x}, d)$ (notice that the assumptions guarantee that this set is nonempty), we have

$$\begin{aligned} 0 > 2t_n^{-2}[f_k(x_n) - f_k(\bar{x})] = {} & 2t_n^{-1} s_n \nabla f_k(\bar{x}) \cdot w_n + \nabla^2 f_k(\bar{x})(d, d) \\ & + 2 s_n \nabla^2 f_k(\bar{x})(d, w_n) + 2 s_n^2 \nabla^2 f_k(\bar{x})(w_n, w_n) + \varepsilon_n \end{aligned} \tag{6}$$

with $\varepsilon_n \to 0$. If $t_n^{-1} s_n \to r$ for some $r \geq 0$, taking the limit as $n \to +\infty$ we get

$$0 \geq 2r\nabla f_k(\bar{x}) \cdot w + \nabla^2 f_k(\bar{x})(d, d)$$

which contradicts (4) since $2rw \in T^2(X, \bar{x}, d)$ as already shown in Proposition 1. If $t_n^{-1} s_n \to +\infty$, then $w \in T_0^2(X, \bar{x}, d)$; therefore, (5) implies that $\nabla f_k(\bar{x}) \cdot w > 0$ for some $k \in K(\bar{x}, d)$ and therefore $t_n^{-1} s_n \nabla f_k(\bar{x}) \cdot w_n \to +\infty$ in contradiction with (6). □

Beyond the standard replacement of inequalities with strict inequalities there are two other small gaps between the necessary and sufficient criteria: (5) is not exactly the sufficient counterpart of (3), since it involves only some suitable components of the objective function; furthermore, sufficient conditions have to be checked only for directions d such that $w \cdot d = 0$.

3 Comparison with sufficient multipliers rules

When the feasible region is described by inequality and equality constraints, i.e.

$$X = \{ x \in \mathbb{R}^n \; : \; g_i(x) \leq 0, \; h_j(x) = 0, \quad i \in I, \; j \in J \} \tag{7}$$

where $g = (g_1, ..., g_m) : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ and $h = (h_1, ..., h_p) : \mathbb{R}^n \longrightarrow \mathbb{R}^p$ are twice continuously differentiable and $I := \{1, ..., m\}$ and $J := \{1, ..., p\}$ denote the corresponding index sets, second order necessary multipliers rules (see Aghezzaf and Hachimi 1999; Bolintinéanu and Maghri 1998; Wang 1991) can be turned into sufficient ones, replacing the inequality involving the second order derivatives with a strict inequality. Considering a set of descent directions $D(\bar{x})$ suitable for (1) along with (7), namely $d \in D(\bar{x})$ if $d \in D_{\leq}(f, \bar{x})$ and $\nabla g_i(\bar{x}) \cdot d \leq 0$ for any $i \in I(\bar{x}) := \{i \in I : g_i(\bar{x}) = 0\}$ and $\nabla h_j(\bar{x}) \cdot d = 0$ for any $j \in J$, the following criterion has been proved in Wang (1991).

Theorem 3 *Let X be given as in (7) and $\bar{x} \in X$. If for each nonzero descent direction $d \in D(\bar{x})$, there exist $\theta \in \mathbb{R}^{\ell}_+$, $\lambda \in \mathbb{R}^m_+$ and $\mu \in \mathbb{R}^p$ satisfying*

$$\sum_{k \in K} \theta_k \nabla f_k(\bar{x}) + \sum_{i \in I} \lambda_i \nabla g_i(\bar{x}) + \sum_{j \in J} \mu_j \nabla h_j(\bar{x}) = 0, \tag{8}$$

$$\lambda_i g_i(\bar{x}) = 0, \quad i \in I \tag{9}$$

and moreover

$$\left(\sum_{k \in K} \theta_k \nabla^2 f_k(\bar{x}) + \sum_{i \in I} \lambda_i \nabla^2 g_i(\bar{x}) + \sum_{j \in J} \mu_j \nabla^2 h_j(\bar{x}) \right) (d, d) > 0, \tag{10}$$

then $\bar{x}$ is a local vector minimum point for (1).

The multipliers in the above rule depend upon the considered direction and in particular the vector multiplier θ can even be zero for some directions; moreover, it is easy to check that actually conditions (8) and (9) imply that $\theta_k = 0$ for any $k \notin K(\bar{x}, d)$ and $\lambda_i = 0$ for any $i \notin I(\bar{x}, d)$, where this latter set consists of all the indexes $i \in I(\bar{x})$ such that $\nabla g_i(\bar{x}) \cdot d = 0$.

In Lemma 1 the above sufficient multipliers rule is equivalently expressed as the impossibility of a family of nonhomogeneous linear systems, just relying on a theorem of the alternative (such as Gale's theorem 2.4.10 in Mangasarian 1969).

Lemma 1 *Let X be given as in (7) and $\bar{x} \in X$. If for each nonzero descent direction $d \in D(\bar{x})$ the following system*

$$\begin{cases} \nabla f_k(\bar{x}) \cdot w + \nabla^2 f_k(\bar{x})(d,d) \leq 0, & k \in K(\bar{x}, d), \\ \nabla g_i(\bar{x}) \cdot w + \nabla^2 g_i(\bar{x})(d,d) \leq 0, & i \in I(\bar{x}, d), \\ \nabla h_j(\bar{x}) \cdot w + \nabla^2 h_j(\bar{x})(d,d) = 0, & j \in J, \end{cases} \tag{11}$$

has no solution $w \in \mathbb{R}^n$, then $\bar{x}$ is a local vector minimum point of (1).

It is straightforward to show, see e.g. Lemma 4.1 in Bigi and Castellani (2000), that the impossibility of the above linear system implies that (4) holds for any $w \in T^2(X, \bar{x}, d)$; obviously, the converse does not hold, since the second order contingent set alone is not related to any sufficient optimality criterion. Furthermore, the following examples show that actually no relationship holds between the sufficient criteria of Lemma 1 (or equivalently Theorem 3) and Theorem 2.

Example 2 Consider (1) and (7) with $n = 3$, $\ell = m = 2$, $p = 1$ and

$$\begin{aligned} &f_1(x_1, x_2, x_3) = x_1^2 + x_2, \quad f_2(x_1, x_2, x_3) = x_2 + x_3^2 \\ &g_1(x_1, x_2, x_3) = -x_2, \quad g_2(x_1, x_2, x_3) = x_2 - x_1^2, \quad h_1(x_1, x_2, x_3) = x_2 x_3. \end{aligned}$$

It is easy to check that $\bar{x} = (0, 0, 0)$ is a vector minimum point. However, chosen any nonzero $d \in D_{\leq}(f, \bar{x}) \cap T(X, \bar{x}) = \mathbb{R} \times \{0\} \times \mathbb{R}$, condition (5) does not hold for the choice $w = (d_3, 0, -d_1) \in T_0^2(X, \bar{x}, d) = \mathbb{R} \times \{0\} \times \mathbb{R}$ since $\nabla f_k(\bar{x}) \cdot w = 0$ for $k = 1, 2$. Therefore, Theorem 2 does not allow to recognize the optimality of $\bar{x}$; in contrast, Lemma 1 or equivalently Theorem 3 does. In fact, chosen any nonzero $d \in D(\bar{x}) = \mathbb{R} \times \{0\} \times \mathbb{R}$, system (11) becomes

$$w_2 + 2d_1^2 \leq 0, \quad w_2 + 2d_3^2 \leq 0, \quad w_2 \geq 0, \quad w_2 - 2d_1^2 \leq 0, \quad d_2 d_3 = 0$$

which admits no solution $w \in \mathbb{R}^n$.

Example 3 Consider (1) and (7) with $n = \ell = 3$, $m = p = 1$ and

$$\begin{aligned} &f_1(x_1, x_2, x_3) = x_1 + x_2^2, \quad f_2(x_1, x_2, x_3) = -x_1 - x_2^2 + x_3, \\ &f_3(x_1, x_2, x_3) = x_1^2 - x_2^2 + x_3, \quad g_1(x_1, x_2, x_3) = -x_3, \quad h_1(x_1, x_2, x_3) = x_1 x_2. \end{aligned}$$

It is easy to show that $\bar{x} = (0, 0, 0)$ is a vector minimum point. However, given any nonzero $d \in D(\bar{x}) = \{0\} \times \mathbb{R} \times \{0\}$, system (11) becomes

$$w_1 + 2d_2^2 \leq 0, \quad w_3 - w_1 - 2d_2^2 \leq 0, \quad w_3 + 2d_1^2 - 2d_2^2 \leq 0, \quad w_3 \geq 0, \quad d_1 d_2 = 0$$

which admits the solution $w = (-2d_2^2, 0, 0)$. Therefore, Lemma 1 and Theorem 3 do not allow to recognize the optimality of $\bar{x}$; in contrast, Theorem 2 does: given any nonzero direction $d \in D_{\leq}(f, \bar{x}) \cap T(X, \bar{x}) = \{0\} \times \mathbb{R} \times \{0\}$ condition (4) becomes

$$\max \left\{ w_1 + 2d_2^2, \; w_3 - w_1 - 2d_2^2, \; w_3 + 2d_1^2 - 2d_2^2 \right\} > 0$$

and it is satisfied by any $w \in T^2(X, \bar{x}, d) = \{0\} \times \mathbb{R} \times \mathbb{R}_+$ since $d_2 \neq 0$; condition (5) becomes

$$\max \{ w_1, \ -w_1 + w_3, \ w_3 \} > 0$$

and it is satisfied by any nonzero $w \in T_0^2(X, \bar{x}, d)$ such that $w \cdot d = 0$ (which means $w_2 = 0$) since

$$T_0^2(X, \bar{x}, d) = \begin{cases} [\{0\} \times \mathbb{R} \times \mathbb{R}_+] \cup [\mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}_+] & \text{if } d_2 < 0, \\ [\{0\} \times \mathbb{R} \times \mathbb{R}_+] \cup [\mathbb{R} \times \mathbb{R}_- \times \mathbb{R}_+] & \text{if } d_2 > 0. \end{cases}$$

4 The unconstrained case

Obviously, Theorem 2 provides a sufficient optimality criterion also in the unconstrained case, i.e. $X = \mathbb{R}^n$. However, in this case condition (5) is very unlikely to be satisfied by all direction $w \in T_0^2(X, \bar{x}, d) = \mathbb{R}^n$ such that $w \cdot d = 0$ and moreover it can never happen in the single-objective case. Actually, in the unconstrained case there is no real need to consider any condition related to the asymptotic second order set (notice, for instance, that Theorem 1 collapses into the well-known first order necessary optimality condition). In fact, the following sufficient criterion, which involves only condition (4), holds.

Theorem 4 *Let $X = \mathbb{R}^n$ and $\bar{x} \in \mathbb{R}^n$. If (4) holds for each nonzero direction $d \in D_{\leq}(f, \bar{x})$ and any $w \in \mathbb{R}^n$ such that $w \cdot d = 0$, then $\bar{x}$ is a local vector minimum point of (1).*

Proof Ab absurdo, suppose there exist $t_n \downarrow 0$ and $d_n \to d$ with $d \neq 0$ such that $x_n = \bar{x} + t_n d_n$ satisfies $f(x_n) \neq f(\bar{x})$ and $f_k(x_n) \leq f_k(\bar{x})$ for all $k \in K$. Therefore, we have

$$0 \geq t_n^{-1}[f_k(x_n) - f_k(\bar{x})] = \nabla f_k(\bar{x}) \cdot d_n + 2^{-1} t_n \nabla^2 f_k(\bar{x})(d_n, d_n) + 2^{-1} t_n \varepsilon_n^k \tag{12}$$

with $\varepsilon_n^k \to 0$. Taking the limit as $n \to +\infty$, we get that $d \in D_{\leq}(f, \bar{x})$. Hence the assumptions imply that $K(\bar{x}, d) \neq \emptyset$ and that the system

$$\begin{gathered} \nabla f_k(\bar{x}) \cdot w + \nabla^2 f_k(\bar{x})(d, d) \leq 0, \ k \in K(\bar{x}, d) \\ d \cdot w = 0 \end{gathered}$$

has no solution $w \in \mathbb{R}^n$. Therefore, Gale's theorem of the alternative (see 2.4.10 in Mangasarian 1969) implies that there exist numbers $\theta_k \geq 0$ not all zero such that

$$\sum_{k \in K(\bar{x}, d)} \theta_k \nabla f_k(\bar{x}) + \theta_0 d = 0 \ \text{ and } \sum_{k \in K(\bar{x}, d)} \theta_k \nabla^2 f_k(\bar{x})(d, d) > 0. \tag{13}$$

Since

$$0 = 0 \cdot d = \sum_{k \in K(\bar{x}, d)} \theta_k \nabla f_k(\bar{x}) \cdot d + \theta_0 d \cdot d = \theta_0 ||d||^2,$$

we have that $\theta_0 = 0$. Therefore, multiplying both sides of (12) by $2t_n^{-1}\theta_k$ and summing up, we get

$$0 \geq \sum_{k \in K(\bar{x},d)} \theta_k \nabla^2 f_k(\bar{x})(d_n, d_n) + \varepsilon_n^k.$$

Taking the limit as $n \to +\infty$, we get a contradiction with (13). □

In the single-objective case the above result collapses into the classical sufficient conditions, i.e. the gradient is the zero vector and the Hessian matrix is positive definite. Therefore, summing up together the first and second order terms as in (4) does not lead to a better criterion than the classical one, in which they are considered separately. On the contrary, in the multiobjective case it is not so. It could be checked directly that local vector optimality is achieved at $\bar{x}$ if condition

$$\max_{k \in K} \left[\nabla f_k(\bar{x}) \cdot w\right] \geq 0 \tag{14}$$

holds for any $w \in \mathbb{R}^n$ and condition

$$\min_{k \in K(\bar{x},d)} \nabla^2 f_k(\bar{x})(d, d) > 0 \tag{15}$$

holds for any nonzero descent direction $d \in D_{\leq}(f, \bar{x})$. However, the following relationship holds.

Proposition 2 *Let a nonzero direction $\bar{d} \in D_{\leq}(f, \bar{x})$ be given. If (14) holds for any $w \in \mathbb{R}^n$ and (15) holds for $\bar{d}$, then (4) holds for $\bar{d}$ and any $w \in \mathbb{R}^n$.*

Proof Ab absurdo, suppose there exists $\bar{w} \in \mathbb{R}^n$ such that

$$\nabla f_k(\bar{x}) \cdot \bar{w} + \nabla^2 f_k(\bar{x})(\bar{d}, \bar{d}) \leq 0$$

holds for any $k \in K(\bar{x}, \bar{d})$. Therefore, (15) implies that $\nabla f_k(\bar{x}) \cdot \bar{w} < 0$ for any $k \in K(\bar{x}, \bar{d})$. If this set is the whole K, then $\bar{w}$ contradicts (14); otherwise, take

$$\tau := 2^{-1} \min\{\, -(\nabla f_k(\bar{x}) \cdot \bar{d})/(\nabla f_k(\bar{x}) \cdot \bar{w}) \;:\; k \notin K(\bar{x}, \bar{d}),\ \nabla f_k(\bar{x}) \cdot \bar{w} > 0 \,\}.$$

It is obvious that $\nabla f_k(\bar{x}) \cdot (\bar{d} + \tau \bar{w}) < 0$ whenever $k \in K(\bar{x}, \bar{d})$ or alternatively $\nabla f_k(\bar{x}) \cdot \bar{w} \leq 0$; for all other indexes we have

$$\nabla f_k(\bar{x}) \cdot (\bar{d} + \tau \bar{w}) = \nabla f_k(\bar{x}) \cdot \bar{d} + \tau \nabla f_k(\bar{x}) \cdot \bar{w} \leq 2^{-1} \nabla f_k(\bar{x}) \cdot \bar{d} < 0.$$

Therefore, $\bar{d} + \tau \bar{w}$ does not satisfy (14) in contradiction with the assumption. □

The converse does not hold; in fact, the following example shows that relying on condition (4) we achieve a sharper sufficient optimality criterion.

Example 4 Consider (1) with $n = 2$, $\ell = 2$, $X = \mathbb{R}^2$ and

$$f_1(x_1, x_2) = x_2, \quad f_2(x_1, x_2) = x_1^2 - x_2.$$

It is easy to check that $\bar{x} = (0, 0)$ is a vector minimum point. However, given any $d \in D_{\leq}(f, \bar{x}) = \mathbb{R} \times \{0\}$, condition (15) becomes

$$\min\{\, 0,\ 2d_1^2 \,\} > 0$$

which can never hold. On the contrary, (4) becomes

$$\max\{\, w_2,\ -w_2 + 2d_1^2 \,\} > 0$$

which holds for any nonzero descent direction d and any $w \in \mathbb{R}^2$.

Condition (15) may appear a little to strong since it involves the minimum and not the maximum; the following example shows that such a replacement does not allow to preserve sufficiency.

Example 5 Consider (1) with $n = 2$, $\ell = 3$, $X = \mathbb{R}^2$ and

$$f_1(x_1, x_2) = x_1 \sin x_2, \quad f_2(x_1, x_2) = x_1 + x_2^2, \quad f_3(x_1, x_2) = -(\sin x_1 + \sin x_2).$$

Since f_1 and f_2 are negative and f_3 is identically zero along the line described by the points $x_t = (-t, t)$ as $t \in (0, 1)$, then $\bar{x} = (0, 0)$ is not a local vector minimum point of (1). It is easy to check that (14) holds for any $w \in \mathbb{R}^n$. Given any nonzero $\bar{d} \in D_{\leq}(f, \bar{x}) = \{d \in \mathbb{R}^2 : d_1 \leq 0,\ d_1 + d_2 \geq 0,\ d_2 \geq 0\}$, we have $1 \in K(\bar{x}, \bar{d})$ and

$$\nabla^2 f_1(\bar{x})(\bar{d}, \bar{d}) = 2\bar{d}_1\bar{d}_2 > 0$$

unless $\bar{d}_1 = 0$ (notice that $\bar{d}_2 = 0$ implies $\bar{d}_1 = 0$). In this latter case we have also $2 \in K(\bar{x}, \bar{d})$ and

$$\nabla^2 f_2(\bar{x})(\bar{d}, \bar{d}) = 2\bar{d}_2^2 > 0$$

Therefore, we have $\max\limits_{k \in K(\bar{x}, d)} \nabla^2 f_k(\bar{x})(d, d) > 0$ for any nonzero $d \in D_{\leq}(f, \bar{x})$.

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