# Composing complete and partial knowledge

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## Abstract

The representation of knowledge in the logic OLP-FOL [8], [7], is split in two parts: writing definitions for known concepts, and writing constraints, expressing partial knowledge on other concepts. This is reflected in an OLP-FOL theory $\mathcal{T}$, which is a pair: $\mathcal{T} = (\mathcal{T}_d, \mathcal{T}_c)$. The definition part $\mathcal{T}_d$ contains the definitions for known predicates in the form of a normal open logic program (OLP), whereas the first order logic (FOL) part $\mathcal{T}_c$ is a set of FOL axioms, expressing partial knowledge on other predicates. The semantics of OLP-FOL is a generalisation of the well-founded semantics [16]. An OLP-FOL theory $\mathcal{T} = (\mathcal{T}_d, \mathcal{T}_c)$, divides the set of predicate symbols in two disjoint subsets: the defined predicates, which occur in the head of a clause of $\mathcal{T}_d$, and the open predicates, which occur at the most in the body of the clauses of $\mathcal{T}_d$.
In previous work [19], the composition of two OLP-FOL theories, with non-intersecting sets of defined predicate symbols, was studied. It was argued that their composition is given by the set of common models. Here, we investigate the possibility of composing two OLP-FOL theories, which define the same predicate. Therefore, we introduce two operators on theories: the $p$-opening operator, which opens the definition of the predicate $p$ in a theory completely, and the conditional $p$-opening operator, which maintains the definition of $p$ in a theory if a certain condition holds, and opens $p$ in the other cases. We show that we can compose two theories, which both have an open definition for the same predicate, or which both have a conditional open definition for the same predicate, with non-overlapping conditions.

## 1 Introduction

The *representation of knowledge* in the logic OLP-FOL is split in two parts: writing definitions for known concepts and writing constraints, expressing partial knowledge on other concepts. To clarify this, consider a situation in which an expert wants to represent his knowledge on a subdomain of the world in an OLP-FOL theory $\mathcal{T}$. Some concepts are completely known by the expert, whereas on other concepts, he has only partial knowledge or no knowledge at all. This is reflected in the theory $\mathcal{T}$, which consists of two parts: $\mathcal{T} = (\mathcal{T}_d, \mathcal{T}_c)$. The *definition part* $\mathcal{T}_d$, is a normal open logic program (OLP), representing the definitions for the known predicates. The *FOL part* $\mathcal{T}_c$, is a set of first order logic (FOL) axioms, representing partial knowledge on the other predicates. So, the theory $\mathcal{T}$ divides the set of predicates in two disjoint subsets: the *defined* predicates, which occur in the head of a clause of $\mathcal{T}_d$, and the *open* predicates, which occur at the most in the body of the clauses of $\mathcal{T}_d$. The semantics of OLP-FOL, which is a generalisation of the well-founded semantics [16], is a *possible state semantics*. This means that a model of a theory represents a state in which the problem domain might occur according to the (incomplete) expert knowledge. A theory with several models, expresses the fact that the expert who wrote the theory, has no complete knowledge about the problem area. A theory in which all the predicates are defined can at most have one model (namely the well-founded model [16]).

In previous work [19], the *composition* of two OLP-FOL theories $\mathcal{T}_1$ and $\mathcal{T}_2$, with non-intersecting sets of defined predicate symbols, was investigated. It was argued that the composition of these two theories should contain exactly the sum of the knowledges of the theories. Formally, this means that the set of models of the composition $\mathcal{T}$ of the theories $\mathcal{T}_1$ and $\mathcal{T}_2$, is exactly equal to the intersection of the sets of models of $\mathcal{T}_1$ and $\mathcal{T}_2$: $\mathcal{M}od(\mathcal{T}) = \mathcal{M}od(\mathcal{T}_1) \cap \mathcal{M}od(\mathcal{T}_2)$. In [19], several conditions on $\mathcal{T}_1$ and $\mathcal{T}_2$ are given, such that their composition is simply the union, that is, such that $\mathcal{M}od(\mathcal{T}_1) \cap \mathcal{M}od(\mathcal{T}_2) = \mathcal{M}od(\mathcal{T}_1 \cup \mathcal{T}_2)$. Note that in [19], the composition of $\mathcal{T}_1$ and $\mathcal{T}_2$ is taken, only if $\mathcal{T}_1$ and $\mathcal{T}_2$ define disjoint sets of predicate symbols. The reason for this is that a definition for a predicate expresses the fact that the expert has *complete* knowledge about this predicate, and hence, how can we compose two definitions for the same predicate?

In this paper, we investigate if there is a way to compose two theories, which both give a definition for the same predicate symbol. For this, it is important to note that the definition of a predicate can depend on one or more open predicates, and hence the expert has only complete knowledge of that predicate w.r.t. the interpretation of the open predicates. We will consider two special forms of definitions for a predicate $p$, namely *open* and *conditional open* definitions, which leave the predicate $p$ open in a sense. These two definitions are introduced via two operators on theories defining the predicate $p$: the $p$-opening operator and the conditional $p$-opening operator. More precisely, the application of the $p$-opening operator $\Theta_p$ on a theory which has a definition for the predicate $p$, makes the definition of $p$ open, by letting $p$ depend on an open predicate. We prove that applying the $p$-opening operator on a theory is actually the same as reading the definition of $p$ as a set of FOL axioms, i.e. putting the definition of $p$ in the FOL part of the theory. On the other hand, the application of the conditional $p$-opening operator $\Theta_p^{c,j}$ on a theory which has a definition for the predicate $p$, maintains the definition of $p$ if a certain condition holds and opens $p$ in the other cases. We will show that we can compose two theories which both have an open definition for the same predicate $p$, or which both have a conditional open definition for $p$, if the conditions don't overlap.

In the following section 2 of preliminaries, we first discuss the logic OLP-FOL and knowledge representation in OLP-FOL (subsection 2.1). In subsection 2.2, we introduce some more notations and assumptions, and in subsection 2.3, we recall the compositionality criterion of [19]. In subsection 3.1, open definitions are introduced via the opening operator. We give some properties and as a result, we show in subsection 3.2, that we can compose two theories, which both have an open definition for the same predicate. Conditional open definitions are studied in section 4. In subsection 4.1, we define the conditional opening operator and we show in subsection 4.2, that we can compose two theories, which both have a conditional open definition for the same predicate, with non-overlapping conditions. We end with a conclusion in section 5, where we give some related works. For more details and the proofs of the propositions and theorems in this paper, we refer to [18].

# 2 Preliminaries

## 2.1 The logic OLP-FOL

We assume familiarity with basic concepts of logic and logic programming such as logical alphabets $\mathcal{A}$, logical languages $\mathcal{L}$, atoms, literals, (normal) program clauses or rules based on $\mathcal{L}$, ground instances of rules w.r.t. a language $\mathcal{L}$, 2-valued and 3-valued Herbrand interpretations of $\mathcal{L}$. For the definitions, we refer to [13]. We assume some familiarity with the well-founded semantics [16] as well.

We introduce some auxiliary concepts. Each language $\mathcal{L}$ is assumed to contain propositional predicates $\top$ and $\bot$; in each interpretation $I$ of $\mathcal{L}$, $\top$ is true and $\bot$ is false. The truth function of a Herbrand interpretation $I$ (i.e. the function which maps ground atoms to $\{\mathbf{f}, \mathbf{u}, \mathbf{t}\}$) is denoted by $\mathcal{H}_I$. The truth function of a Herbrand interpretation is described as a set of tuples of ground atoms with truth value (e.g. $\{p^{\mathbf{f}}, q^{\mathbf{u}}, r^{\mathbf{t}}\}$, meaning that $\mathcal{H}_I(p) = \mathbf{f}, \mathcal{H}_I(q) = \mathbf{u}$ and $\mathcal{H}_I(r) = \mathbf{t}$). The truth function of a 2-valued Herbrand interpretation will also be described as a subset of the Herbrand base (denoting the true atoms).

We will work in the logic OLP-FOL [8], [7], which is an expressive logic to represent uncertainty on definitions of concepts and on the problem domain. A *theory* $\mathcal{T}$ in this logic is a pair $(\mathcal{T}_d, \mathcal{T}_c)$ of a First Order Logic (FOL) theory $\mathcal{T}_c$, the *FOL part* of $\mathcal{T}$, and a normal open logic program (OLP) $\mathcal{T}_d$, the *definition* part of the theory. A normal open logic program is a set of normal program clauses $p(t) \leftarrow q_1(t_1), .., q_n(t_n), \neg r_1(s_1), .., \neg r_m(s_m)$ ($t, t_i, s_j$ are tuples of terms). Atomic rules are denoted $p(t) \leftarrow \top$. Predicates occuring in the head of a clause of $\mathcal{T}_d$ are called *defined*. With $\mathcal{D}ef(\mathcal{T})$, we denote the set of defined predicate symbols of the theory $\mathcal{T}$. If $\mathcal{D}ef(\mathcal{T}) = \{q_1, \ldots, q_k\}$, then we also write the definition part of $\mathcal{T}$ as the following disjoint union: $\mathcal{T}_d = \mathcal{T}_d^{q_1} \cup \ldots \cup \mathcal{T}_d^{q_k}$, where $\mathcal{T}_d^{q_i}$ is the set of clauses of $\mathcal{T}_d$ defining $q_i$, i.e. the *definition* of $q_i$. The predicates which are not defined, and hence occur at the most in the body of program clauses, are called *open*. Intuitively, they represent concepts for which no definitions are given. Partial knowledge about these predicates can be expressed in the set of FOL axioms $\mathcal{T}_c$. A definition for a predicate in a theory expresses the fact that this predicate is completely known by (the expert who wrote) the theory. But, it is of course possible that the definition of a predicate depends on one or more open predicates, and hence, the expert has only complete knowledge about that predicate w.r.t. the interpretation of the open predicates. This will become more clear below, where we give the definition of the model semantics of OLP-FOL. For simplicity, we restrict ourselves here to Herbrand interpretations (HI) only. In the general definition of the model semantics of OLP-FOL [8], also non-Herbrand interpretations are considered, hence taking into account uncertainty on the domain of discourse. We want to remark that everything in this paper carries over in a trivial way to the case of general interpretations.

**Definition 2.1** *A* model *of a normal logic program $\mathcal{T}_d$ is a HI $M$, such that $M$ is the well-founded model [16] of the grounding of $\mathcal{T}_d$ augmented with some ground unit clauses of open predicates representing the interpretation of the open predicates in $M$ (i.e. $M$ is the extended well-founded model of $\mathcal{T}_d$ as defined in [14]).*
*A* model *of a theory $(\mathcal{T}_d, \mathcal{T}_c)$ is a HI $M$, such that $M$ is a model of $\mathcal{T}_d$, as defined above, and $M$ satisfies the set of FOL axioms $\mathcal{T}_c$ (in the classical FOL sense).*
*A formula is a* consequence *of a theory $\mathcal{T}$ iff it is true in every model of $\mathcal{T}$.*

Note that a model is necessarily 2-valued on the open predicates. With $\mathcal{M}od(\mathcal{T})$, we will denote the set of models of a theory $\mathcal{T}$.

With a *partial HI* we mean a HI $I$ of which the truth function is defined only on some part of the predicate symbols. We say that a HI $M$ *extends* $I$ if $\mathcal{H}_M$ coincides with $\mathcal{H}_I$ on the predicates to which $I$ gives truth values. As a consequence of definition 2.1, we have that, for each 2-valued partial HI $I$ on the open predicates of a normal logic program $\mathcal{T}_d$, there exists a unique model $M$ of $\mathcal{T}_d$ extending $I$. Note that, if all the predicate symbols of $\mathcal{L}$, except $=$, are defined, $M$ is the unique well-founded model of $\mathcal{T}_d$. Possibly $M$ is a model of the full theory $\mathcal{T} = (\mathcal{T}_d, \mathcal{T}_c)$, namely if $M$ satisfies the FOL axioms $\mathcal{T}_c$.

OLP-FOL has a *possible state semantics*, that is, a model corresponds to a state in which the problem domain might occur according to the (incomplete) expert knowledge. At the level of the semantics, uncertainty on the definition of a concept is modeled by allowing models which give to the open predicates an arbitrary interpretation which satisfies the set of FOL axioms $\mathcal{T}_c$ (and not e.g. by having the third truth value **u** for these open predicates as in a belief set semantics). The occurrence of an undefined ground atom (i.e. an atom with truth value **u**) in a model of $\mathcal{T}_d$ reveals an ambiguity or a local inconsistency in the definition. This motivates the following definition.

**Definition 2.2** *An open normal logic program $\mathcal{T}_d$ is called a* correct definition *(or* correct*) iff each model of $\mathcal{T}_d$ is 2-valued. A theory $\mathcal{T} = (\mathcal{T}_d, \mathcal{T}_c)$ is* correct *iff $\mathcal{T}_d$ is a correct definition.*

We will use the notion of *dependency graph* of the predicates in a logic program. For the definition, we refer to [13]. A predicate $q$ is said to *depend on* a predicate $p$ iff there is a path from $q$ to $p$ in the dependency graph. A predicate $q$ is said to depend *only oddly* (resp. *only evenly*) on a predicate $p$ iff in each path from $q$ to $p$ in the dependency graph, there is an odd (resp. even) number of negative edges.

## 2.2 Some notations and assumptions

We consider a fixed alphabet $\mathcal{A}$, with a finite number of predicate symbols . For each predicate symbol $q/n$ of $\mathcal{A}$, we introduce a new and unique predicate symbol $q^*/n$. This results in an extension $\mathcal{A}^* = \mathcal{A} \cup \{q^*/n \mid q/n$ is a predicate symbol of $\mathcal{A}\}$ of the alphabet $\mathcal{A}$.
With $\rho_{r\to s}$ we denote the *renaming operator*, which replaces each occurrence of the predicate symbol $r$ by the predicate symbol $s$. Renaming operators can be applied to theories, program clauses, conjunctions of literals or literals.
We will consider theories whose definition parts are based on the alphabet $\mathcal{A}$. A predicate symbol $p^*$ of $\mathcal{A}^* \setminus \mathcal{A}$ can only occur in the definition part of a theory as a consequence of the application of the $p$-opening operator $\Theta_p$ or the conditional $p$-opening operator $\Theta_p^{c,j}$, which we'll introduce in subsections 3.1 and 4.1.
We will assume that the FOL part of each theory contains the axioms $p(X) \Longleftrightarrow p^*(X)$, for each predicate symbol $p$ of $\mathcal{A}$ (and that predicates of $\mathcal{A}^* \setminus \mathcal{A}$ don't appear in another axiom of $\mathcal{T}_c$). Because of this assumption, in each model of a theory, the truth values of the instantiations of $p$ and $p^*$ are the same. Hence, in all the following examples, when writing down a model of a theory, we will not explicitily write down the interpretation of the predicates $p^* \in \mathcal{A}^* \setminus \mathcal{A}$. Note also that adding these axioms imposes the restriction on the models of a theory to be 2-valued. Since we will only consider correct theories, and since the definition parts of the theories, where we start from, are based on the alphabet $\mathcal{A}$, these axioms don't add any relevant information.
We want to remark that the axioms of the form $p(X) \Longleftrightarrow p^*(X)$ will become relevant once we have applied the $p$-opening or conditional $p$-opening operator on a theory (which has as a consequence that $p^*$ appears in the bdy of the definition part of the theory). More precisely, these axioms are added to deal in a clear way with theory composition (see subsections 3.2 and 4.2). We will suppose for the rest of this paper that the FOL parts of all the theories contain these axioms $p(X) \Longleftrightarrow p^*(X)$, without explicitily writing these axioms down.

## 2.3 The compositionality criterion

We recall the compositionality criterion of [19]. In [19], a situation is considered where two (or more) experts have more or less disjunct subdomains of expertise and represent their knowledges independently in two theories, $\mathcal{T}_1$ and $\mathcal{T}_2$, with non-intersecting sets of defined predicate symbols, i.e. $\mathcal{D}ef(\mathcal{T}_1) \cap \mathcal{D}ef(\mathcal{T}_2) = \emptyset$. The problem considered in [19], is how to combine these two theories in one united theory $\mathcal{T}$. They argue that the *composition* $\mathcal{T}$ of two theories $\mathcal{T}_1$ and $\mathcal{T}_2$ should contain exactly the sum of the knowledges of the separate theories. Because OLP-FOL has a *possible state semantics*, the compositionality criterion has the following natural formulation: that the set of models of $\mathcal{T}$ is precisely the intersection of the sets of models of $\mathcal{T}_1$ and $\mathcal{T}_2$:

$$\mathcal{M}od(\mathcal{T}) = \mathcal{M}od(\mathcal{T}_1) \cap \mathcal{M}od(\mathcal{T}_2).$$

In [19], several conditions on $\mathcal{T}_1$ and $\mathcal{T}_2$ are given such that their composition can be obtained by simply taking the union, $\mathcal{T}_1 \cup \mathcal{T}_2 = (\mathcal{T}_{1d} \cup \mathcal{T}_{2d}, \mathcal{T}_{1c} \cup \mathcal{T}_{2c})$, that is, such that $\mathcal{M}od(\mathcal{T}_1 \cup \mathcal{T}_2) = \mathcal{M}od(\mathcal{T}_1) \cap \mathcal{M}od(\mathcal{T}_2)$. We give a slightly modified version of [19, theorem 5.2] (taking into account the assumptions we made in subsection 2.2).

**Theorem 2.1** *Given two theories $\mathcal{T}_1$ and $\mathcal{T}_2$, such that $\mathcal{D}ef(\mathcal{T}_1) \cap \mathcal{D}ef(\mathcal{T}_2) = \emptyset$. Let $D$ be the dependency graph of the defined predicates of $\mathcal{T}_1 \cup \mathcal{T}_2$. With $C$ we denote a cycle in $D$.*
*If $\forall C$, $\exists i \in \{1, 2\}$, such that all the predicates in $C$ are defined in $\mathcal{T}_i$,*
*then*

$$\mathcal{M}od(\mathcal{T}_1) \cap \mathcal{M}od(\mathcal{T}_2) = \mathcal{M}od(\mathcal{T}_1 \cup \mathcal{T}_2).$$

In general, the restriction $\mathcal{D}ef(\mathcal{T}_1) \cap \mathcal{D}ef(\mathcal{T}_2) = \emptyset$, which is imposed on the theories in order to compose them, is necessary. This is because a definition for a predicate $p$ in a theory, expresses the fact that the expert, who wrote the theory, has complete knowledge about $p$. If he wasn't sure about $p$, he wouldn't have written a definition for it. Consider for instance the following two programs, $\mathcal{T}_1 = \{p(a) \leftarrow \top\}$ and $\mathcal{T}_2 = \{p(b) \leftarrow \top\}$. Then, $\mathcal{M}od(\mathcal{T}_1) = \{\{p(a)^{\mathbf{t}}, p(b)^{\mathbf{f}}\}\}$ and $\mathcal{M}od(\mathcal{T}_2) = \{\{p(a)^{\mathbf{f}}, p(b)^{\mathbf{t}}\}\}$. There is no natural way to compose these two definitions of $p$, since one expert says that only $p(a)$ is true and no more, and the other says that only $p(b)$ is true and no more. This is because both definitions determine completely the interpretation of $p$.

Here, we will investigate the possibility of composing two theories, which both have a definition for the same predicate $p$. As the above example shows, this is not always possible. But, it is important to note that a definition for a predicate $p$ can depend on open predicates, and hence, the interpretation of $p$ remains open in a sense: the expert has complete knowledge about $p$ *w.r.t. the interpretation of the open predicates.* In the next two sections (3 and 4), two special forms of definitions are considered, namely *open* definitions (subsection 3.1) and *conditional open* definitions (subsection 4.1). These definitions are introduced via two operators: the $p$-opening operator and the conditional $p$-opening operator. An open definition for a predicate $p$ leaves $p$ completely open, while a conditional open definition for $p$ leaves $p$ open only if some condition is satisfied. We will show that we can compose two theories which both have an open definition for $p$ (subsection 3.2) or which both have a conditional open definition for $p$, with non-overlapping conditions (subsection 4.2).

# 3 Open definitions

## 3.1 The opening operator

An open definition for a predicate $p$ can be seen as the result of the application of the $p$-opening operator $\Theta_p$ on a theory $\mathcal{T}$, which defines the predicate $p$: the theory $\Theta_p(\mathcal{T})$ has an open definition for $p$. In the next definition, this $p$-opening operator is introduced.

**Definition 3.1** *Given is a theory* $\mathcal{T} = (\mathcal{T}_d, \mathcal{T}_c)$ *in which the predicate symbol* $p/n$ *is defined. Applying the* $p$-opening operator $\Theta_p$ *to* $\mathcal{T}$ *results in the theory* $\Theta_p(\mathcal{T})$*:*

$$\mathcal{T} = \begin{cases} \mathcal{T}_d \setminus \mathcal{T}_d^p \\ p(t_1) \leftarrow D_1 \\ \ldots \\ p(t_r) \leftarrow D_r \\ FOL : \begin{cases} \mathcal{T}_c \\ p^*(X) \iff p(X) \end{cases} \end{cases} \longrightarrow \quad \Theta_p(\mathcal{T}) = \begin{cases} \rho_{p\to p^*}(\mathcal{T}_d \setminus \mathcal{T}_d^p) \\ p(t_1) \leftarrow \rho_{p\to p^*}(D_1) \\ \ldots \\ p(t_r) \leftarrow \rho_{p\to p^*}(D_r) \\ p(X) \leftarrow p^*(X) \\ FOL : \begin{cases} \mathcal{T}_c \\ p^*(X) \iff p(X) \end{cases} \end{cases}$$

*where* $D_i$*,* $i = 1,..,r$*, is a conjunction of literals,* $t_i$*,* $i = 1,..,r$*, is an* $n$*-tuple of terms and* $X = (X_1,..,X_n)$*. We say that* $\Theta_p$ *makes the definition of* $p$ *in* $\mathcal{T}$ open*. The definition of* $p$ *(or briefly* $p$*) in* $\Theta_p(\mathcal{T})$ *is called* open*.*

We see that, when we apply the $p$-opening operator on a theory $\mathcal{T}$, we apply the renaming operator $\rho_{p\to p^*}$ to the bodies of all the clauses of $\mathcal{T}_d$, and add the clause $p(X) \leftarrow p^*(X)$ to $\mathcal{T}_d$. Note that the theory $\Theta_p(\mathcal{T})$ defines exactly the same predicate symbols as the theory $\mathcal{T}$, i.e. $\mathcal{D}ef(\Theta_p(\mathcal{T})) = \mathcal{D}ef(\mathcal{T})$. Note also that the FOL parts of both theories are equal. To clarify definition 3.1, we give a simple example.

**Example 3.1** *Let* $\mathcal{T}$ *be the following theory:*

$$\begin{cases} bird(tweety) & \leftarrow & \top \\ fly(X) & \leftarrow & bird(X), \neg abn(X) \\ abn(X) & \leftarrow & \neg has\_wings(X) \end{cases}$$

*Then,* $\mathcal{M}od(\mathcal{T}) = \{M_1 = \{has\_wings(tweety)^{\mathbf{f}},\ abn(tweety)^{\mathbf{t}},\ bird(tweety)^{\mathbf{t}},\ fly(tweety)^{\mathbf{f}}\},$ $M_2 = \{has\_wings(tweety)^{\mathbf{t}},\ abn(tweety)^{\mathbf{f}},\ bird(tweety)^{\mathbf{t}},\ fly(tweety)^{\mathbf{t}}\}\}$*.*
*When we apply* $\Theta_{abn}$ *to* $\mathcal{T}$*, we obtain the theory* $\mathcal{T}^* = \Theta_{abn}(\mathcal{T})$*:*

$$\begin{cases} bird(tweety) & \leftarrow & \top \\ fly(X) & \leftarrow & bird(X), \neg abn^*(X) \\ abn(X) & \leftarrow & \neg has\_wings(X) \\ abn(X) & \leftarrow & abn^*(X) \end{cases}$$

$\mathcal{M}od(\mathcal{T}^*) = \{M_1, M_2, M_3 = \{has\_wings(tweety)^{\mathbf{t}}, abn(tweety)^{\mathbf{t}}, bird(tweety)^{\mathbf{t}}, fly(tweety)^{\mathbf{f}}\}\}$.
*We see that* $M_1$ *and* $M_2$ *are also model of* $\mathcal{T}^*$*, but* $\mathcal{T}^*$ *has a third model, namely* $M_3$*, in which tweety is abnormal, and hence doesn't fly, although it has wings.*

The following proposition gives us a necessary and sufficient condition for the correctness of the theory $\Theta_p(\mathcal{T})$.

**Proposition 3.1** $\Theta_p(\mathcal{T})$ *is correct iff* $\mathcal{T}_d \setminus \mathcal{T}_d^p$ *is correct.*

Hence, checking whether the theory $\Theta_p(\mathcal{T})$ is correct, can be done by checking whether $\mathcal{T}_d \setminus \mathcal{T}_d^p$ is correct. Note that the correctness of $\mathcal{T}_d$ doesn't imply the correctness of $\mathcal{T}_d \setminus \mathcal{T}_d^p$.
An open definition of a predicate $p$ in a theory can be closed again, by applying the *p-closing operator* $\mathcal{C}_p$, which undoes the effect of $\Theta_p$. Syntactically, closing the definition of $p$ again, is obtained by removing the clause $p(X) \leftarrow p^*(X)$ and, by applying the renaming operator $\rho_{p^*\to p}$ to the bodies of all clauses of the definition part. It is obvious that the following equality holds: $\mathcal{M}od(\mathcal{T}) = \mathcal{M}od(\mathcal{C}_p(\Theta_p(\mathcal{T})))$.

The $p$-opening operator is defined as a syntactical operation on theories. What happens semantically when we apply the $p$-opening operator to a theory? The next theorem gives the meaning of a theory in which the definition of a predicate $p$ is open (we also refer to [17] for a motivation of the introduction of the $p$-opening operator). The theorem states that, opening the definition of a predicate $p$ in a theory $\mathcal{T}$ is equivalent to putting the definition of $p$, $\mathcal{T}_d^p$, in the FOL part of the theory, and hence, making the predicate $p$ open. This justifies the name of the operator $\Theta_p$: *p-opening* operator.

**Theorem 3.1** *Let $\mathcal{T}$ be a theory which defines the predicate $p$. Then,*

$$\mathcal{M}od(\Theta_p(\mathcal{T})) = \mathcal{M}od((\mathcal{T}_d \setminus \mathcal{T}_d^p, \mathcal{T}_c \cup \mathcal{T}_d^p)).$$

**Example 3.2** *Recall theory $\mathcal{T}$ of example 3.1. Consider the theory obtained from $\mathcal{T}$ by putting the definition of $abn/1$ in the FOL part, $(\mathcal{T}_d \setminus \mathcal{T}_d^{abn}, \mathcal{T}_c \cup \mathcal{T}_d^{abn})$:*

$$\left\{\begin{array}{lcl} bird(tweety) & \leftarrow & \top \\ fly(X) & \leftarrow & bird(X), \neg abn(X) \\ FOL: & & \\ abn(X) & \leftarrow & \neg has\_wings(X) \end{array}\right.$$

*It is easy to see that the set of models of this theory is $\{M_1, M_2, M_3\}$ (see example 3.1), which is exactly the set of models of $\Theta_{abn}(\mathcal{T})$.*

Because of theorem 3.1, the theory $\Theta_p(\mathcal{T})$ can be seen as a kind of revision of the theory $\mathcal{T}$, in which the status of the predicate $p$ is revised from defined to open.
The last two propositions of this subsection study the relation between the models of a theory and the models of the $p$-opening of that theory. We already noticed in example 3.1 that the models of the theory $\mathcal{T}$ are also models of $\Theta_{abn}(\mathcal{T})$. This holds in general. Namely, if we open the definition of a predicate $p$ in a theory $\mathcal{T}$, the models of $\mathcal{T}$ remain models of $\Theta_p(\mathcal{T})$. We just enlarge the set of models.

**Proposition 3.2** *Let $\mathcal{T}$ be a theory which defines the predicate $p$. Then,*

$$\mathcal{M}od(\mathcal{T}) \subseteq \mathcal{M}od(\Theta_p(\mathcal{T})).$$

The next proposition gives us a more detailed relation between $\mathcal{M}od(\mathcal{T})$ and $\mathcal{M}od(\Theta_p(\mathcal{T}))$. It tells us something about the possible truth values of atoms in the models of $\Theta_p(\mathcal{T})$, in comparison with their truth values in the models of $\mathcal{T}$. Before going into any details, let's first return to example 3.1.

**Example 3.3** *Recall the theory $\mathcal{T}$ of example 3.1, with $\mathcal{M}od(\mathcal{T}) = \{M_1, M_2\}$ and $\mathcal{M}od(\Theta_{abn}(\mathcal{T}))$ $= \{M_1, M_2, M_3\}$.*
*Consider the partial HI $I = \{has\_wings(tweety)^{\mathbf{t}}\}$, of the open predicate $has\_wings/1$, then there is only one model of $\mathcal{T}$ extending this partial HI, namely $M_2$. Whereas $\Theta_{abn}(\mathcal{T})$ has two models extending this partial HI, $M_2$, and also $M_3$. If we look at the truth values of $abn(tweety)$ and $fly(tweety)$ in the model $M_3$ and compare them with their truth values in $M_2$, we see that the truth value of $abn(tweety)$ in $M_3$ is greater than its truth value in $M_2$, whereas the truth value of $fly(tweety)$ in $M_3$ is smaller that its truth value in $M_2$. Note that the predicate $fly/1$ depends negatively on $abn/1$.*

**Proposition 3.3** *Let $\mathcal{T}$ be a theory which defines the predicate $p$.*
*Let $I$ be a partial 2-valued HI of the open predicates of $\mathcal{T}$ which belong to $\mathcal{A}$. Suppose there exists a model $M$ of $\mathcal{T}$ extending $I$. Then, $M$ is the unique model of $\mathcal{T}$ extending $I$. Denote the set of models of $\Theta_p(\mathcal{T})$ which extend $I$ with $\mathcal{M}$ (then $M \in \mathcal{M}$).*
*Consider the dependency graph of $\mathcal{T}_d$. Suppose $p$ depends only evenly on itself. Let $t$ be an arbitrary element of the Herbrand universe. Then,*

- $\forall M' \in \mathcal{M} : \mathcal{H}_{M'}(p(t)) \geq \mathcal{H}_M(p(t))$ ,
- $\forall q$ *defined in $\mathcal{T}$, which depends only evenly on $p$:* $\forall M' \in \mathcal{M} : \mathcal{H}_{M'}(q(t)) \geq \mathcal{H}_M(q(t))$ ,
- $\forall r$ *defined in $\mathcal{T}$, which depends only oddly on $p$:* $\forall M' \in \mathcal{M} : \mathcal{H}_{M'}(r(t)) \leq \mathcal{H}_M(r(t))$ ,
- $\forall s$ *defined in $\mathcal{T}$, which doesn't depend on $p$:* $\forall M' \in \mathcal{M} : \mathcal{H}_{M'}(s(t)) = \mathcal{H}_M(s(t))$ .

## 3.2 Composing open definitions

In this subsection, we prove that, as a direct consequence of theorem 3.1, we can compose two theories which both have an open definition for $p$. More formally, we consider the following situation. There are two experts, representing their knowledges independently in distinct theories, $\mathcal{T}_1^*$ and $\mathcal{T}_2^*$, such that $\mathcal{D}ef(\mathcal{T}_1^*) \cap \mathcal{D}ef(\mathcal{T}_2^*) = \{p\}$, and such that the theories are of the following form: $\mathcal{T}_1^* = \Theta_p(\mathcal{T}_1)$ and $\mathcal{T}_2^* = \Theta_p(\mathcal{T}_2)$. That is, both theories have an open definition for $p$. The theory $\mathcal{T}_i^*$ ($i \in \{1,2\}$) can be seen either as the result of a revision of the theory $\mathcal{T}_i$ (the expert revised his knowledge on the status of the predicate $p$ from defined to open, and he opened the definition of $p$), or as directly written in this way by the expert. Because of theorem 3.1, we can actually consider the predicate $p$ as open in $\mathcal{T}_1^*$ and $\mathcal{T}_2^*$. Hence, the knowledges of these two theories can be composed. The *composition* of $\mathcal{T}_1^*$ and $\mathcal{T}_2^*$ is given by the intersection of their sets of models:

$$\mathcal{M}od(\mathcal{T}_1^*) \cap \mathcal{M}od(\mathcal{T}_2^*).$$

The next theorem is a direct consequence of theorem 2.1. It gives a condition on $\mathcal{T}_1^*$ and $\mathcal{T}_2^*$, such that the composition is simply the union of both theories.

**Theorem 3.2** *Given two theories $\mathcal{T}_1^* = \Theta_p(\mathcal{T}_1)$ and $\mathcal{T}_2^* = \Theta_p(\mathcal{T}_2)$, such that $\mathcal{D}ef(\mathcal{T}_1^*) \cap \mathcal{D}ef(\mathcal{T}_2^*) = \{p\}$. Let $D$ be the dependency graph of the defined predicates of $(\mathcal{T}_{1d} \setminus \mathcal{T}_{1d}^p) \cup (\mathcal{T}_{2d} \setminus \mathcal{T}_{2d}^p)$. With $C$ we denote a cycle in $D$.*
*If $\forall C$, $\exists i \in \{1,2\}$, such that all the predicates of $C$ are defined in $\mathcal{T}_{id} \setminus \mathcal{T}_{id}^p$,*
*then*

$$\mathcal{M}od(\mathcal{T}_1^*) \cap \mathcal{M}od(\mathcal{T}_2^*) = \mathcal{M}od(\mathcal{T}_1^* \cup \mathcal{T}_2^*).$$

**Example 3.4** *Let $\mathcal{T}_1^* = \Theta_{abn}(\mathcal{T}_1)$ and $\mathcal{T}_2^* = \Theta_{abn}(\mathcal{T}_2)$ be the following theories:*

$$\mathcal{T}_1^* = \left\{ \begin{array}{lll} bird(tweety) & \leftarrow & \top \\ bird(fleety) & \leftarrow & \top \\ fly(X) & \leftarrow & bird(X), \neg abn^*(X) \\ abn(X) & \leftarrow & \neg has\_wings(X) \\ abn(X) & \leftarrow & abn^*(X) \end{array} \right. \qquad \mathcal{T}_2^* = \left\{ \begin{array}{lll} penguin(tweety) & \leftarrow & \top \\ abn(X) & \leftarrow & penguin(X) \\ abn(X) & \leftarrow & abn^*(X) \end{array} \right.$$

*The fact that $\mathcal{T}_1^*$ and $\mathcal{T}_2^*$ have an open definition for $abn/1$, means that the experts, who designed these theories, both know that they don't have complete knowledge about abnormal birds w.r.t. flying and, that they leave their theories open for adding knowledge about the $abn/1$ predicate. To obtain the sum of the knowledges of these two theories, we take the intersection of the models* *of the two theories:* $\mathcal{M}od(\mathcal{T}_1^*) \cap \mathcal{M}od(\mathcal{T}_2^*)$. *This intersection contains six models. In all the models tweety will not fly, because it's a penguin. In four of the six models, fleety will not fly. In two of them because fleety doesn't have wings, but in the other two, fleety will not fly, because it's abnormal, although it has wings.* $\mathcal{T}_1^*$ *and* $\mathcal{T}_2^*$ *satisfy the conditions of theorem 3.2, hence,* $\mathcal{M}od(\mathcal{T}_1^*) \cap \mathcal{M}od(\mathcal{T}_2^*) = \mathcal{M}od(\mathcal{T}_1^* \cup \mathcal{T}_2^*)$.

Note that, when the experts of example 3.4, think that there is no more knowledge about $abn/1$ to add, they can close, by means of the closing operator, the definition of $abn/1$ in the composition of their theories. All this has strong connections with *theory updating*, more precisely, with inserting rules in logic programs, see for instance [11]. That is, inserting a set of rules $\mathcal{T}_2$ for the predicate $p$ in a theory $\mathcal{T}_1$, can be done by opening in both $\mathcal{T}_1$ and $\mathcal{T}_2$ the predicate $p$, composing $\Theta_p(\mathcal{T}_1)$ and $\Theta_p(\mathcal{T}_2)$ and closing the predicate $p$ in this composition again. By the lack of space, we don't give a more elaborate example here, but we hope that the idea is made clear by the previous small example 3.4.
In a natural way, we can extend the results of subsection 3.1 and this subsection, by considering the case in which we open the definition of more than one predicate symbol in a theory and compose theories which have several open definitions. By the lack of space, we omit these results and refer to [18] instead.

# 4 Conditional open definitions

## 4.1 The conditional opening operator

A conditional open definition for a predicate $p$ can be seen as the result of the application of the conditional $p$-opening operator $\Theta_p^{c,j}$ on a theory $\mathcal{T}$, which defines the predicate $p$: the theory $\Theta_p^{c,j}(\mathcal{T})$ has a conditional open definition for $p$. The next definition introduces this conditional $p$-opening operator. For a term $t$, we denote with $Var(t)$ the set of variables occuring in $t$.

**Definition 4.1** *Let* $\mathcal{T} = (\mathcal{T}_d, \mathcal{T}_c)$ *be a theory in which the predicate* $p/n$ *is defined.*
*Let* $c/m$ *be a predicate, which is either open or defined in* $\mathcal{T}$, *such that, in the dependency graph of* $\mathcal{T}_d$, $c/m$ *doesn't depend on* $p/n$.
*Let* $j = (j_1, \ldots, j_m)$ *be a mapping from* $T^n$ *to* $T^m$, *with* $T$ *the set of terms, such that for each* $i \in \{1, \ldots, m\}$, $j_i(X_1, .., X_n) = t_i$, *with* $Var(t_i) \subseteq \{X_1, .., X_n\}$, *and for all substitutions* $\phi = \{X_1/s_1, .., X_n/s_n\}$, $j_i(s_1, .., s_n) = t_i\phi$.
*The application of the* conditional $p$-opening operator $\Theta_p^{c,j}$ *on the theory* $\mathcal{T}$ *yields the theory* $\Theta_p^{c,j}(\mathcal{T})$:

$$\mathcal{T} = \begin{cases} \mathcal{T}_d \setminus \mathcal{T}_d^p \\ p(t_1) \leftarrow D_1 \\ \ldots \\ p(t_r) \leftarrow D_r \\ FOL: \begin{cases} \mathcal{T}_c \\ p^*(X) \iff p(X) \end{cases} \end{cases} \longrightarrow \quad \Theta_p^{c,j}(\mathcal{T}) = \begin{cases} \mathcal{T}_d \setminus \mathcal{T}_d^p \\ p_h(t_1) \leftarrow D_1 \\ \ldots \\ p_h(t_r) \leftarrow D_r \\ p(X) \leftarrow p_h(X), c(j(X)) \\ p(X) \leftarrow p^*(X), \neg c(j(X)) \\ FOL: \begin{cases} \mathcal{T}_c \\ p^*(X) \iff p(X) \end{cases} \end{cases}$$

*where* $p_h/n$ *is a new predicate symbol depending on the theory* $\mathcal{T}$, $D_i$, $i = 1, .., r$, *is a conjunction of literals,* $t_i$, $i = 1, .., r$, *is an* $n$*-tuple of terms and* $X = (X_1, .., X_n)$. *We say that the predicate* $p$ *has a* conditional open definition *in the theory* $\Theta_p^{c,j}(\mathcal{T})$. *The definition of* $p$ *(or briefly* $p$*) in* $\Theta_p^{c,j}(\mathcal{T})$ *is called* conditional open.

**Remark 1:** Note that, in contrast to the $p$-opening operator $\Theta_p$, the application of the conditional $p$-opening operator $\Theta_p^{c,j}$, introduces a new defined predicate symbol $p_h/n$: $Def(\Theta_p^{c,j}(\mathcal{T})) = Def(\mathcal{T}) \cup \{p_h/n\}$. But, the introduction of the predicate symbol $p_h/n$ is not really necessary. Namely, we can replace the clause $p(X) \leftarrow p_h(X), c(j(X))$ in $\Theta_p^{c,j}(\mathcal{T})$ by the clauses:

$$\begin{cases} p(t_1) \leftarrow D_1, c(j(t_1)) \\ \dots \\ p(t_r) \leftarrow D_r, c(j(t_r)) \end{cases}$$

and obtain a theory which has exactly the same models as $\Theta_p^{c,j}(\mathcal{T})$ (this follows for instance from [2, Lemma 7.19]). So, actually, the use of the predicate $p_h/n$ can be seen as syntactic sugar. We use it here for the sake of clearness.

**Remark 2:** Note that we impose the condition that, in the dependency graph of $\mathcal{T}_d$, $c/m$ doesn't depend on $p/n$. This condition has as a consequence that we can write $\mathcal{T}$ as a disjoint union $\mathcal{T}_{cond} \cup \mathcal{T}'$, such that $c/m$ is defined in $\mathcal{T}_{cond}$ and the defined predicates of $\mathcal{T}_{cond}$ don't depend on the defined predicates of $\mathcal{T}'$. More precisely, let $\mathcal{T}_{cond}$ be the set of definitions of the predicate $c/m$ and of the predicates on which $c/m$ depends, and let $\mathcal{T}'$ be the theory containing all the other definitions of $\mathcal{T}_d$ (i.a. the definition of $p$). Concerning the FOL part, let $[\mathcal{T}_{cond}]_c = \mathcal{T}_c = \mathcal{T}'_c$. If $c/m$ is an open predicate in $\mathcal{T}$, then $\mathcal{T}_{cond} = (\emptyset, \mathcal{T}_c)$. We see that, $\Theta_p^{c,j}(\mathcal{T}) = \Theta_p^{c,j}(\mathcal{T}') \cup \mathcal{T}_{cond}$ and, because of theorem 2.1, the equality $\mathcal{M}od(\Theta_p^{c,j}(\mathcal{T})) = \mathcal{M}od(\Theta_p^{c,j}(\mathcal{T}')) \cap \mathcal{M}od(\mathcal{T}_{cond})$ holds. We return to this remark in subsection 4.2, where we compose two theories which both have a conditional open definition for $p$.

In the theory $\Theta_p^{c,j}(\mathcal{T})$, the definition for $p$ is restricted to the case that $c(j(X))$ holds. That is, in case $c(j(t))$ holds, $p(t)$ can only be true if the corresponding $p_h$-atom, $p_h(t)$, is true. In case $\neg c(j(t))$ holds, the predicate $p$ is completely open (and via $p^*$ more true $p$-atoms, $p(t)$, can be added).

**Example 4.1** *Consider the following situation. A museum decides to lower the entrance fee for children. The rules for the previous entrance fee are contained in the theory $\mathcal{T} \supseteq \{fee(X,Y) \leftarrow D_1, .., fee(X,Y) \leftarrow D_r\}$. The fee for children is still under discussion. The conditional fee-opening operator $\Theta_{fee}^{adult,\pi_1^2}$ is applied to the theory $\mathcal{T}$, with $\pi_1^2 : T^2 \to T : (t,s) \mapsto t$:*

$$\Theta_{fee}^{adult,\pi_1^2}(\mathcal{T}) = \begin{cases} \mathcal{T}_d \setminus \mathcal{T}_d^{fee} \\ fee_h(X,Y) \leftarrow D_1 \\ \dots \\ fee_h(X,Y) \leftarrow D_r \\ fee(X,Y) \leftarrow fee_h(X,Y), adult(X) \\ fee(X,Y) \leftarrow fee^*(X,Y), \neg adult(X) \\ FOL : \mathcal{T}_c \end{cases}$$

*The theory $\Theta_{fee}^{adult,\pi_1^2}(\mathcal{T})$ has several models, since the predicate $fee/2$ is open in the case of children.*

The next proposition gives a relation between the sets $\mathcal{M}od(\mathcal{T})$ and $\mathcal{M}od(\Theta_p^{c,j}(\mathcal{T}))$. Note that, as opposed to proposition 3.2, it is not the case that the models of $\mathcal{T}$ are also models of $\Theta_p^{c,j}(\mathcal{T})$. But, informally, if $M$ is a model of $\mathcal{T}$ such that, if a $p$-atom, $p(t)$, is true then $c(j(t))$ is true, then $M$ is also a model of $\Theta_p^{c,j}(\mathcal{T})$.

**Proposition 4.1**
*If $M \in \mathcal{M}od(\mathcal{T})$, such that $M \models \forall X(p(X) \to c(j(X)))$ and $M \models \forall X(p(X) \Longleftrightarrow p_h(X))$, then $M \in \mathcal{M}od(\Theta_p^{c,j}(\mathcal{T}))$.*

### 4.2 Composing conditional open definitions

In the previous subsection 4.1, we introduced the conditional $p$-opening operator $\Theta_p^{c,j}$. Different as in the case of the $p$-opening operator $\Theta_p$, the predicate $p$ can not be seen as completely open in a theory with a conditional open definition for $p$. Hence, in general, we can not compose two theories $\Theta_p^{c_1,j_1}(\mathcal{T}_1)$ and $\Theta_p^{c_2,j_2}(\mathcal{T}_2)$, because they both contain a definition for $p$ (by means of the predicate $p_{h_1}$, resp. $p_{h_2}$) which is applied if the condition $c_1(j_1(X))$, resp. $c_2(j_2(X))$, holds. But, if the conditions $c_1(j_1(X))$ and $c_2(j_2(X))$ don't overlap, that is, the definitions for $p$ in the two theories describe a different case, we can compose $\Theta_p^{c_1,j_1}(\mathcal{T}_1)$ and $\Theta_p^{c_2,j_2}(\mathcal{T}_2)$. We will show how this can be done. Let $\Theta_p^{c_1,j_1}(\mathcal{T}_1)$ and $\Theta_p^{c_2,j_2}(\mathcal{T}_2)$ be the following two theories:

$$\Theta_p^{c_1,j_1}(\mathcal{T}_1) = \begin{cases} \mathcal{T}_{1d} \setminus \mathcal{T}_{1d}^p \\ p_{h_1}(t_1) \leftarrow D_1 \\ \dots \\ p_{h_1}(t_r) \leftarrow D_r \\ p(X) \leftarrow p_{h_1}(X), c_1(j_1(X)) \\ p(X) \leftarrow p^*(X), \neg c_1(j_1(X)) \\ FOL : \begin{cases} \mathcal{T}_{1c} \\ p(X) \iff p^*(X) \end{cases} \end{cases} \qquad \Theta_p^{c_2,j_2}(\mathcal{T}_2) = \begin{cases} \mathcal{T}_{2d} \setminus \mathcal{T}_{2d}^p \\ p_{h_2}(t_1^{'}) \leftarrow D_1^{'} \\ \dots \\ p_{h_2}(t_{r^{'}}^{'}) \leftarrow D_{r^{'}}^{'} \\ p(X) \leftarrow p_{h_2}(X), c_2(j_2(X)) \\ p(X) \leftarrow p^*(X), \neg c_2(j_2(X)) \\ FOL : \begin{cases} \mathcal{T}_{2c} \\ p(X) \iff p^*(X) \end{cases} \end{cases}$$

First, we will make more formal what is meant by *the two conditions don't overlap.* We already noticed in remark 2 after definition 4.1, that we can write a theory $\Theta_p^{c,j}(\mathcal{T})$ as a union $\Theta_p^{c,j}(\mathcal{T}^{'})$ $\cup \mathcal{T}_{cond}$, such that $p$ is defined in $\Theta_p^{c,j}(\mathcal{T}^{'})$, $c$ is defined in $\mathcal{T}_{cond}$ (or $\mathcal{T}_{cond} = (\emptyset, \mathcal{T}_c)$), and the defined predicates of $\mathcal{T}_{cond}$ don't depend on the defined predicates of $\Theta_p^{c,j}(\mathcal{T}^{'})$. In this section, where we deal with two theories $\Theta_p^{c_1,j_1}(\mathcal{T}_1)$ and $\Theta_p^{c_2,j_2}(\mathcal{T}_2)$, we will suppose that $c_1$ and $c_2$ are open in $\Theta_p^{c_1,j_1}(\mathcal{T}_1)$ and in $\Theta_p^{c_2,j_2}(\mathcal{T}_2)$, and that we have a third theory $\mathcal{T}_{cond}$ in which both predicates $c_1$ and $c_2$ are (possibly) defined. More formally, we will make the following assumptions:

**Ass1:** $c_1$ and $c_2$ are open in $\Theta_p^{c_1,j_1}(\mathcal{T}_1)$ and in $\Theta_p^{c_2,j_2}(\mathcal{T}_2)$.

**Ass2:** $\mathcal{D}ef(\Theta_p^{c_1,j_1}(\mathcal{T}_1)) \cap \mathcal{D}ef(\Theta_p^{c_2,j_2}(\mathcal{T}_2)) = \{p\}$,
$\mathcal{D}ef(\Theta_p^{c_i,j_i}(\mathcal{T}_i)) \cap \mathcal{D}ef(\mathcal{T}_{cond}) = \emptyset$, for $i = 1, 2$.

**Ass3:** The defined predicates of $\mathcal{T}_{cond}$ don't depend on the predicates which are defined in $\Theta_p^{c_1,j_1}(\mathcal{T}_1)$ or in $\Theta_p^{c_2,j_2}(\mathcal{T}_2)$.

The fact that the definitions for $p$ in $\Theta_p^{c_1,j_1}(\mathcal{T}_1)$ and $\Theta_p^{c_2,j_2}(\mathcal{T}_2)$ describe a different case, i.e. $c_1$ and $c_2$ don't overlap, can now be expressed as follows:

**Ass4:** $\mathcal{T}_{cond} \models \forall X \neg (c_1(j_1(X)) \wedge c_2(j_2(X)))$.

If these four assumptions hold, we can *compose* $\Theta_p^{c_1,j_1}(\mathcal{T}_1)$ *and* $\Theta_p^{c_2,j_2}(\mathcal{T}_2)$ *w.r.t.* $\mathcal{T}_{cond}$. This is because, although $p$ is defined in $\Theta_p^{c_1,j_1}(\mathcal{T}_1)$ and $\Theta_p^{c_2,j_2}(\mathcal{T}_2)$, the definitions of $p$ describe a different case, namely $c_1(j_1(X))$, resp. $c_2(j_2(X))$, and $p$ is open in $\Theta_p^{c_1,j_1}(\mathcal{T}_1)$, resp. $\Theta_p^{c_2,j_2}(\mathcal{T}_2)$, in case $c_2(j_2(X))$, resp. $c_1(j_1(X))$, holds. The meaning of the composition is given by the intersection of the models of $\Theta_p^{c_1,j_1}(\mathcal{T}_1)$, $\Theta_p^{c_2,j_2}(\mathcal{T}_2)$ and $\mathcal{T}_{cond}$:

$$\mathcal{M}od(\Theta_p^{c_1,j_1}(\mathcal{T}_1)) \quad \cap \quad \mathcal{M}od(\Theta_p^{c_2,j_2}(\mathcal{T}_2)) \quad \cap \quad \mathcal{M}od(\mathcal{T}_{cond}).$$

In subsection 3.2, we asked the question when the composition of $\Theta_p(\mathcal{T}_1)$ and $\Theta_p(\mathcal{T}_2)$ is given by the union of these two theories. Here, we ask the question when the composition of $\Theta_p^{c_1,j_1}(\mathcal{T}_1)$ and

$\Theta_p^{c_2,j_2}(\mathcal{T}_2)$ w.r.t. $\mathcal{T}_{cond}$ is obtained by taking the union of $\mathcal{T}_{cond}$, $\Theta_p^{c_1,j_1}(\mathcal{T}_1)$ and $\Theta_p^{c_2,j_2}(\mathcal{T}_2)$, but in which we replaced the two clauses $p(X) \leftarrow p^*(X), \neg c_1(j_1(X))$ and $p(X) \leftarrow p^*(X), \neg c_2(j_2(X))$ by one clause $p(X) \leftarrow p^*(X), \neg c_1(j_1(X)), \neg c_2(j_2(X))$ ($p$ is only open in case $\neg c_1(j_1(X))$ *and* $\neg c_2(j_2(X))$ hold). With $\Theta_p^{(c_1,j_1),(c_2,j_2)}(\mathcal{T}_1, \mathcal{T}_2)$, we denote the following theory:

$$\Theta_p^{(c_1,j_1),(c_2,j_2)}(\mathcal{T}_1, \mathcal{T}_2) = \begin{cases} \mathcal{T}_{1d} \setminus \mathcal{T}_{1d}^p \\ \mathcal{T}_{2d} \setminus \mathcal{T}_{2d}^p \\ p_{h_1}(t_1) \leftarrow D_1 \\ \ldots \\ p_{h_1}(t_r) \leftarrow D_r \\ p_{h_2}(t_1') \leftarrow D_1' \\ \ldots \\ p_{h_2}(t_{r'}') \leftarrow D_{r'}' \\ p(X) \leftarrow p_{h_1}(X), c_1(j_1(X)) \\ p(X) \leftarrow p_{h_2}(X), c_2(j_2(X)) \\ p(X) \leftarrow p^*(X), \neg c_1(j_1(X)), \neg c_2(j_2(X)) \\ FOL : \begin{cases} \mathcal{T}_{1c} \cup \mathcal{T}_{2c} \\ p(X) \iff p^*(X) \end{cases} \end{cases}$$

The next theorem gives conditions on $\Theta_p^{c_1,j_1}(\mathcal{T}_1)$, $\Theta_p^{c_2,j_2}(\mathcal{T}_2)$ and $\mathcal{T}_{cond}$, such that the composition of $\Theta_p^{c_1,j_1}(\mathcal{T}_1)$ and $\Theta_p^{c_2,j_2}(\mathcal{T}_2)$ w.r.t. $\mathcal{T}_{cond}$ is given by $\Theta_p^{(c_1,j_1),(c_2,j_2)}(\mathcal{T}_1, \mathcal{T}_2)$.

**Theorem 4.1** *Given are the theories $\Theta_p^{c_1,j_1}(\mathcal{T}_1)$, $\Theta_p^{c_2,j_2}(\mathcal{T}_2)$ and $\mathcal{T}_{cond}$, satisfying* **Ass1, Ass2, Ass3** *and* **Ass4**. *Let $D$ be the dependency graph of the defined predicates of $\Theta_p^{c_1,j_1}(\mathcal{T}_1) \cup \Theta_p^{c_2,j_2}(\mathcal{T}_2)$. With $C_{\overline{p}}$, we denote a cycle in $D$ not containing the predicate $p$. With $C_p$, we denote a cycle in $D$ containing the predicate $p$.*
*If*

1. *$\forall C_{\overline{p}}$, $\exists i \in \{1, 2\}$, such that all the predicates in $C_{\overline{p}}$ are defined in $\Theta_p^{c_i,j_i}(\mathcal{T}_i)$,*

2. *$\exists i \in \{1, 2\}$, $\forall C_p$, such that all the predicates in $C_p$ are defined in $\Theta_p^{c_i,j_i}(\mathcal{T}_i)$,*

*then*

$$\mathcal{M}od(\Theta_p^{c_1,j_1}(\mathcal{T}_1)) \cap \mathcal{M}od(\Theta_p^{c_2,j_2}(\mathcal{T}_2)) \cap \mathcal{M}od(\mathcal{T}_{cond})$$
$$=$$
$$\mathcal{M}od(\Theta_p^{(c_1,j_1),(c_2,j_2)}(\mathcal{T}_1, \mathcal{T}_2)) \cap \mathcal{M}od(\mathcal{T}_{cond})$$
$$=$$
$$\mathcal{M}od(\Theta_p^{(c_1,j_1),(c_2,j_2)}(\mathcal{T}_1, \mathcal{T}_2) \cup \mathcal{T}_{cond}).$$

The second equality is a direct consequence of theorem 2.1 and assumption **Ass3**. Note that the second condition in the proposition statement can alternatively be formulated as follows: either $p_{h_1}$ depends on $p$, or $p_{h_2}$ depends on $p$, buth not both.

**Example 4.2** *We give a very simple example in genetics. There is a disease D, which is under research by a number of experts. One researcher studies male persons with disease D, and he finds out that a man has disease D if one of his parents has disease D. He wants to write down his knowledge about disease D in a theory. Let the predicate $disD/1$ be such that $disD(X)$ is true* *if X has disease D. Because the researcher has only complete knowledge about the occurrence of disease D in the male population, he restricts the predicate disD w.r.t. men, and he opens disD in the other case (since he has no knowledge about that). He writes down his knowledge in the theory* $\Theta_{disD}^{male,I}(\mathcal{T}_1)$*, with I the identity function on the set of terms T:*

$$\Theta_{disD}^{male,I}(\mathcal{T}_1) = \begin{cases} disD_{h_1}(X) & \leftarrow & parent(Y,X), disD(Y) \\ disD(X) & \leftarrow & disD_{h_1}(X), male(X) \\ disD(X) & \leftarrow & disD^*(X), \neg male(X) \end{cases}$$

*Another researcher studies women with disease D. He finds out that for women, disease D is a consequence of disease E. His results are expressed in the following theory:*

$$\Theta_{disD}^{female,I}(\mathcal{T}_2) = \begin{cases} disD_{h_2}(X) & \leftarrow & disE(X) \\ disD(X) & \leftarrow & disD_{h_2}(X), female(X) \\ disD(X) & \leftarrow & disD^*(X), \neg female(X) \end{cases}$$

*Suppose that there is a theory* $\mathcal{T}_{cond}$ *such that* $\mathcal{T}_{cond} \models \forall X \neg(male(X) \wedge female(X))$*. Then, the composition of* $\Theta_{disD}^{male,I}(\mathcal{T}_1)$ *and* $\Theta_{disD}^{female,I}(\mathcal{T}_2)$ *w.r.t.* $\mathcal{T}_{cond}$ *is given by:*

$$\mathcal{M}od(\Theta_{disD}^{male,I}(\mathcal{T}_1)) \cap \mathcal{M}od(\Theta_{disD}^{female,I}(\mathcal{T}_2)) \cap \mathcal{M}od(\mathcal{T}_{cond}).$$

*In this composition, it is for instance the case that if a woman has disease E (and hence also disease D), and she is the mother of a boy, then this boy has disease D. Since the conditions of theorem 4.1 are satisfied, the composition w.r.t.* $\mathcal{T}_{cond}$ *is given by:*

$$\Theta_{disD}^{(male,I),(female,I)}(\mathcal{T}_1,\mathcal{T}_2) = \begin{cases} disD_{h_1}(X) & \leftarrow & parent(Y,X), disD(Y) \\ disD_{h_2}(X) & \leftarrow & disE(X) \\ disD(X) & \leftarrow & disD_{h_1}(X), male(X) \\ disD(X) & \leftarrow & disD_{h_2}(X), female(X) \\ disD(X) & \leftarrow & disD^*(X), \neg female(X), \neg male(X) \end{cases}$$

A special case to consider is when $\mathcal{T}_{cond} \models \forall X(c_1(j_1(X)) \iff \neg c_2(j_2(X)))$. That is, both cases don't overlap and togheter, they cover the whole space. In the composition, the predicate $p^*$ becomes useless, because the case $\neg c_1(j_1(X)) \wedge \neg c_2(j_2(X))$ never occurs. If the composition is given by $\Theta_p^{(c_1,j_1),(c_2,j_2)}(\mathcal{T}_1,\mathcal{T}_2)$, then we can just as wel remove the clause $p(X) \leftarrow p^*(X), \neg c_1(j_1(X)), \neg c_2(j_2(X))$. For instance, this can be done in example 4.2, if also $\mathcal{T}_{cond} \models \forall X(male(X) \vee female(X))$.
We mention that all this can be extended to the case we want to compose $n$ theories, all with a conditional open definition for the predicate $p$, and each describing a different case. By the lack of space, we will not go into details about that here.
Also here, there is a connection with theory updating. Consider for example the Law. A law (a set of rules for one predicate, e.g. the right to vote, the schoolage), which is applicable to one group $A$ of people, can be restricted to a smaller group $B \subseteq A$ of people and opened in the other case $\overline{B}$. The result can be composed with another law (a set of rules for the same predicate) applicable to a group $C \subseteq \overline{B}$ of people and open in the other case $\overline{C}$. The result is a law, applicable to the group $B \cup C$ and open in the other case $\overline{B \cup C}$.

# 5 Conclusion

We studied knowledge representation and knowledge composition in the logic OLP-FOL [8], [7]. Concerning *knowledge representation*, we investigated two special forms of definitions for a predicate $p$, namely *open* and *conditional open* definitions. These were introduced via two operators: the $p$-opening operator and the conditional $p$-opening operator. A dual approach

to the $p$-opening operator, is to consider the $p$-closing operator (mentioned in subsection 3.1), which closes the open definition of the predicate $p$. The $p$-closing operator puts the set of clauses with the predicate $p$ in the head, from the FOL part in the definition part of a theory (theorem 3.1). When we close the definition of a predicate in a theory, we reduce the set of models of the theory (proposition 3.2), i.e. we consider only a subset of the set of models of the theory. Actually, in general, a semantics for logic programs can be seen as a closing operator. In [3], these closing operators are studied. There, each normal logic program is denoted by the set of Herbrand models of its positive version. Mappings (closing operators) are given from this set to some existing semantics of normal logic programs. Note that here, we consider the case in which we close (or dually, open), the definition of only one predicate (and possibly more).
Concerning *theory composition*, we argued that, two theories, which both have an open definition for the predicate $p$, can be composed in the sense of [19]. Also in [3], and further in [4], the compositionality of normal logic programs is studied. In [4], a compositional model-theoretic semantics for positive logic programs is presented, where the composition of programs is modelled by the composition of the admissible Herbrand models of the programs. These results, together with the results of [3] (which we mentioned above), provide normal logic programs with a compositional semantics. We also studied the composition of two theories which both have a conditional open definition for the same predicate, in case the conditions don't overlap. This can be seen as a form of reasoning by cases (both theories give a definition for $p$ in a different case). The compositionality issue is also studied in [1], [9], [15], [5], [10] and [12]. For more references and a detailed discussion on theory composition, we refer to [19]. A survey of the compositionality issue in logic programming is given in [6].
As we already mentioned, applying the $p$-opening operator or the conditional $p$-opening operator on a theory, can be seen as a revision technique. The theory $\Theta_p(\mathcal{T})$ can be seen as the result of a revision of the status of the predicate $p$ in the theory $\mathcal{T}$: $p$ is made open. Whereas, when we apply the conditional $p$-opening operator $\Theta_p^{c,j}$ on the theory $\mathcal{T}$, we restrict the definition for the predicate $p$ in $\mathcal{T}$ to a certain case and open $p$ otherwise. Actually, we retract some cases in the theory $\mathcal{T}$ for which we don't know the predicate $p$, and make $p$ open in these cases.
We mentioned at the end of subsection 3.2, that there is a strong connection with *theory updating*, more precisely with inserting rules in logic programs, see for instance [11]. A topic for future work is to investigate the problem of retracting (as opposed to adding) knowledge about a predicate $p$ in a theory.


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