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Rewriting Nested Graphs, through Term Graphs

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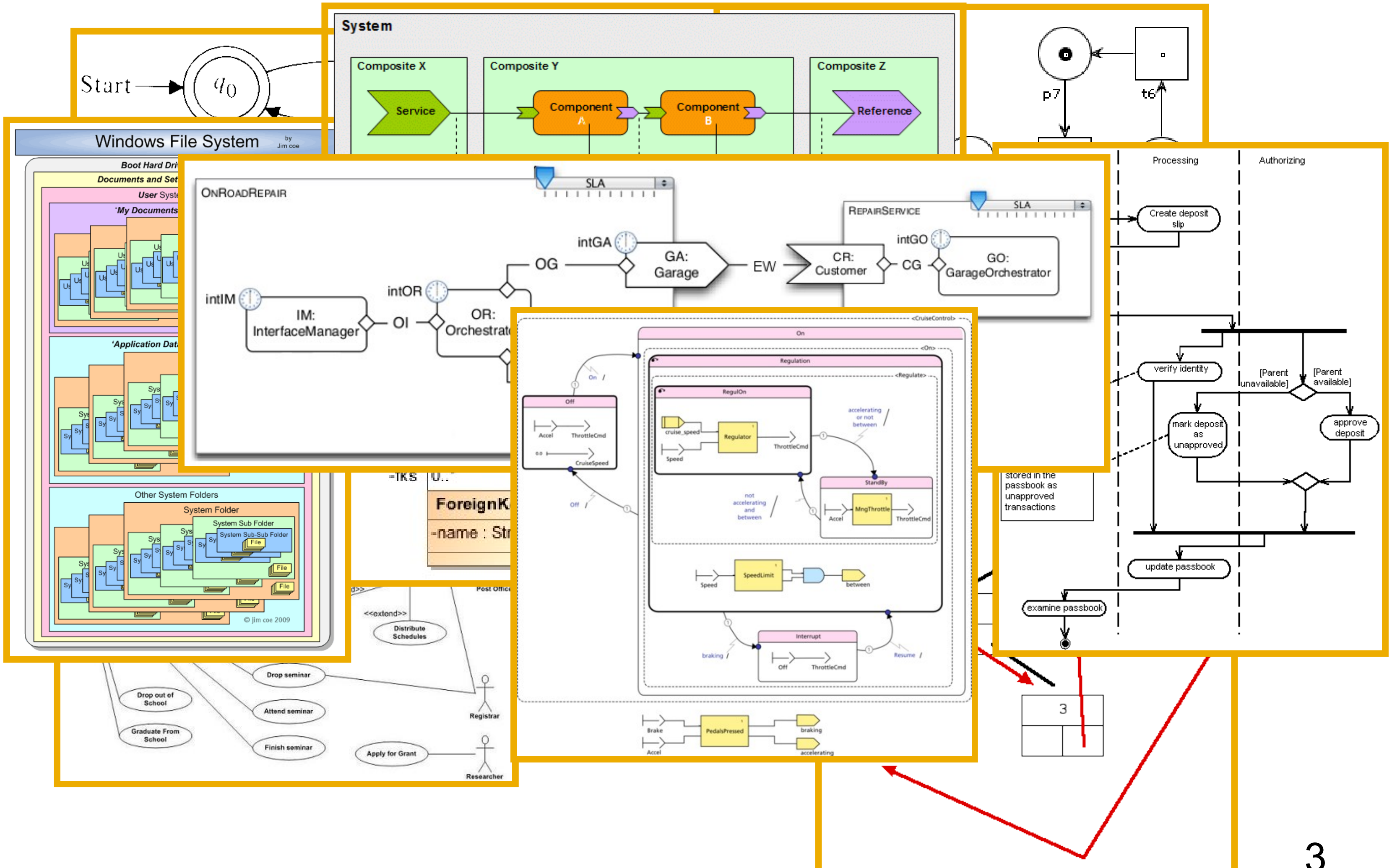
Outline

- Motivations: graphical modeling of process calculi (& other)
- A graph algebra as “intermediate language”
- Axiomatization of *NR-graphs* (graphs + nesting and restriction)
- Example: Encoding Ambient Calculus processes
  - This works for the *static part* of several calculi
- Extending the general approach to dynamics
- Encoding NR-graphs into Term Graphs: soundness, completeness and surjectivity on *well-scoped* term graphs
- Encoding Ambient Calculus rules as Term Graph rules
- What remains to be done...

Motivations: Graphs are everywhere

- Use of diagrams / graphs is pervasive to Computer Science



Graph-based approaches

- Some key features of graph-based approaches
  - help to convey ideas visually
  - ability to represent in a direct way relevant topological features
    - to make "links", "connection", "separation", ... explicit
  - ability to model systems at the “right” level of abstraction representing systems “up to isomorphism”
    - irrelevant details can be omitted (e.g. names of states in Finite State Automata, names of bound variables)
  - important body of theory available
    - Graph transformation approaches
      - DPO, SPO, SHR, ...
    - **Theory of parallelism/concurrency, unfolding, ...**
    - **Verification and analysis techniques**
  - Tools available

Encoding process calculi and the like: From algebraic to graph-based syntax

Algebraic

- Terms
  $\mathtt{a} \mid \mathtt{b}$
- Operations
  $\cdot|\cdot : \mathtt{W} \times \mathtt{W} \rightarrow \mathtt{W}$
- Axioms
  $\mathtt{x} \mid \mathtt{y} \equiv \mathtt{y} \mid \mathtt{x}$
- Rewrite rules
  $\mathtt{a} \longrightarrow \mathtt{b}$

elements

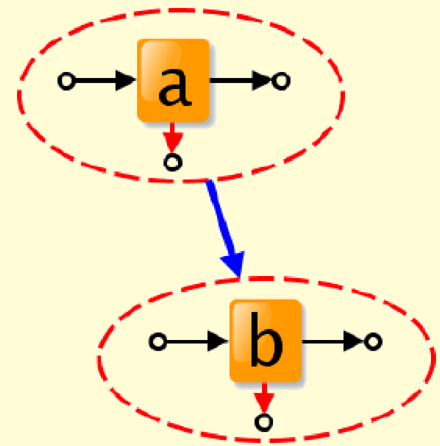
vocabulary

equivalence

dynamics

Graph-based

- Graphs (diagrams)
  flat, hierarchical, etc.
- Graph compositions
  Union, tensor, etc.
- Homomorphisms
  isomorphism, etc.
- Transformation rules



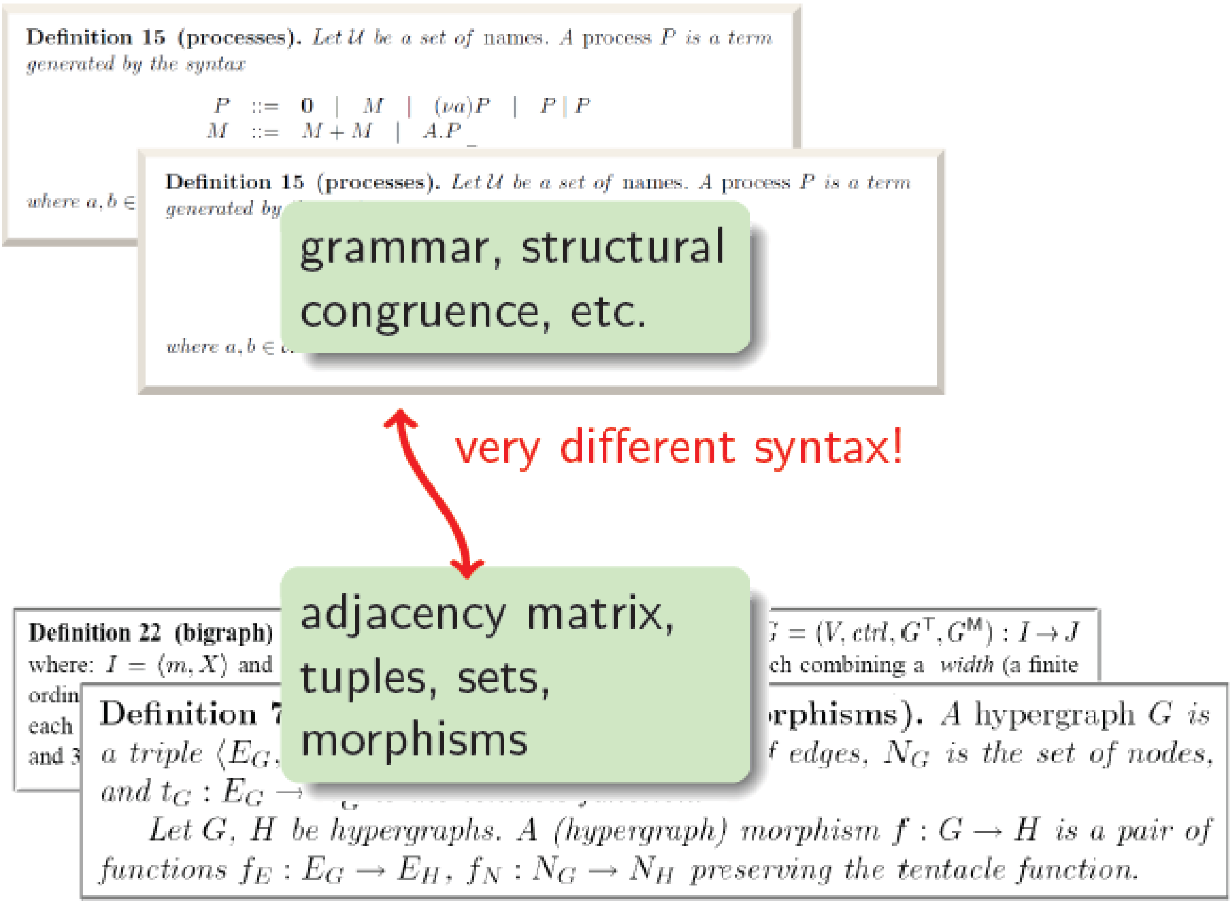
Goal: **sound and complete encoding:**
gven terms **t** and **s**,

[[t]] is isomorphic to **[[s]]**

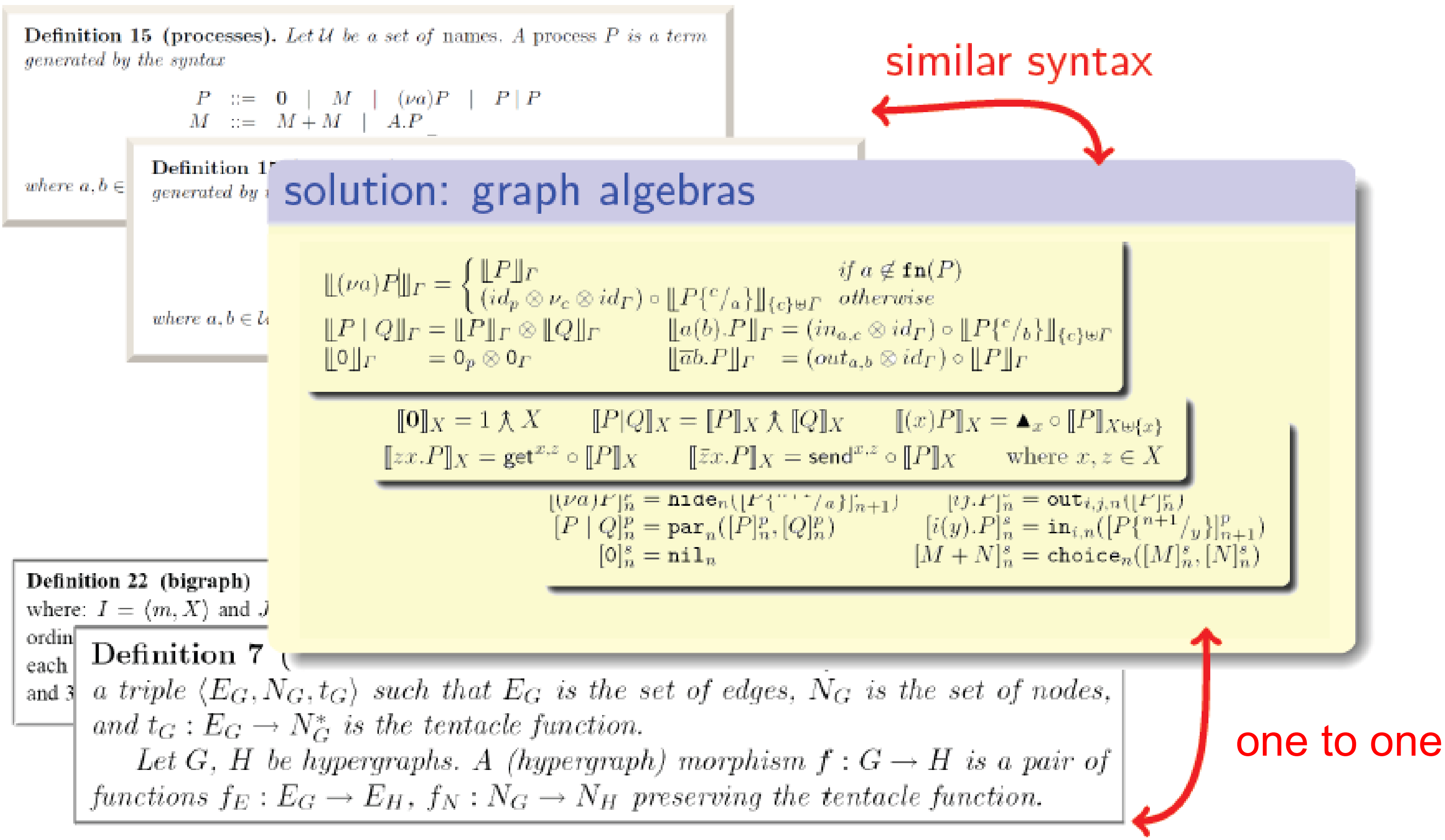
iff

t and **s** are congruent

Main complication: the representation gap



The proposed solution:
graph algebras as intermediate language



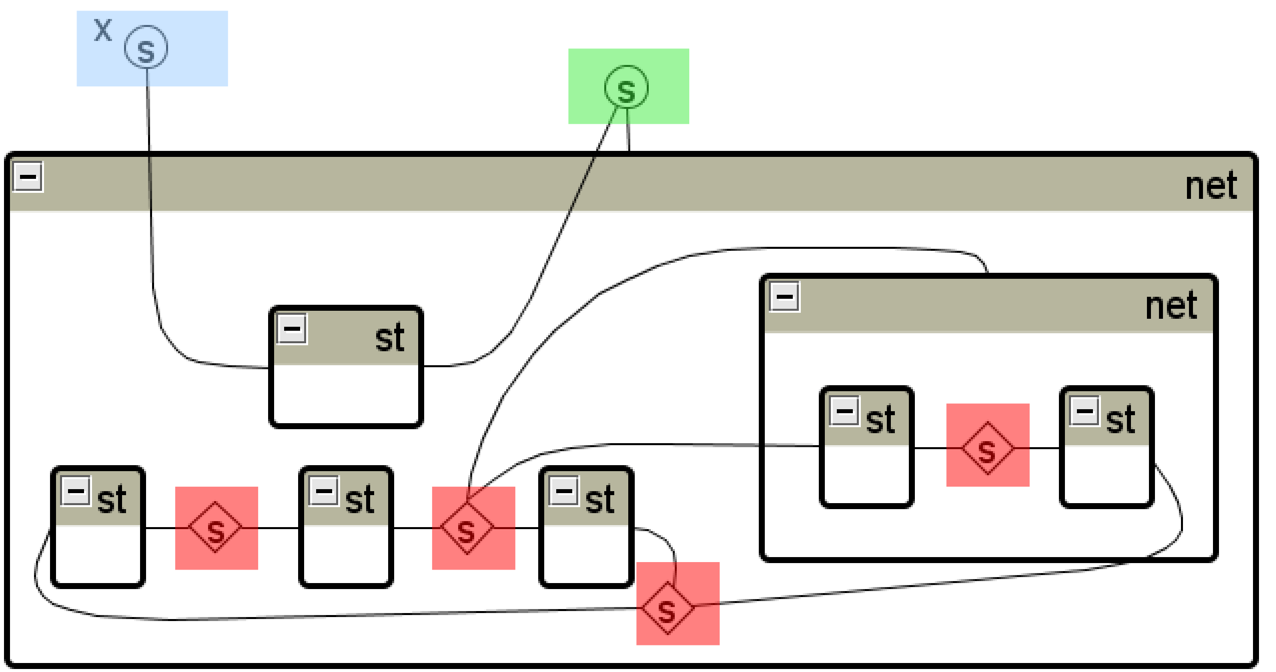
From graphs to graph algebras

- Start with a given class (category?) of graphs
- Define an equational signature,
  - operators correspond to operations on graphs
  - axioms describe their properties
- Prove *once and for all* soundness and completeness of the axioms with respect to the interpretation on graphs, as well as surjectivity
- Next, you can safely use the algebra as an alternative, more handy syntax for the graphs

Graphs with nesting and restriction (NR-graphs)

Hypergraphs where

- hyperedges may contain nested graphs
- nodes can be global, globally restricted, or locally restricted
- locally restricted nodes cannot be accessed “from outside”
- isomorphisms preserve names of free nodes



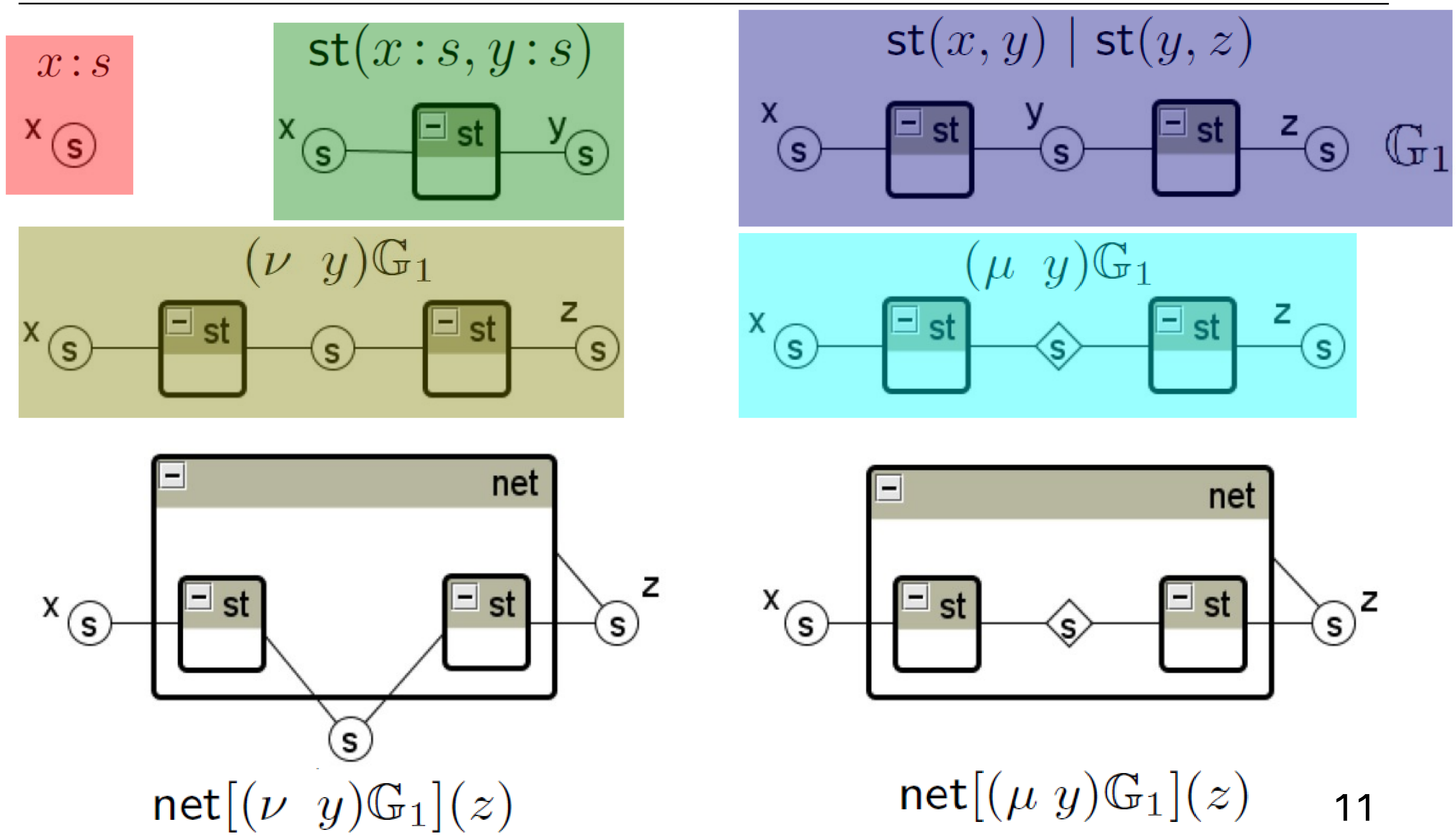
NR-graphs: the formal definition [for Fernando only...]

Definition 1 (NR-graphs). *An* NR-graph $G \in \textbf{NR-Graph}$ *is a tuple* $G = \langle FN, GR, H \rangle$, *where* FN *is a set of* free nodes, GR *is a set of* globally restricted nodes *with* $FN \cap GR = \emptyset$, *and* $H \in \textbf{NR-Graph}[FN \cup GR]$. *The set of* global nodes *of* G *is given by* $FN \cup GR$.

An NR-graph H with external nodes X, $H \in \textbf{NR-Graph}[X]$, *is a tuple* $H = \langle LR, E, l, c, \rho \rangle$, *where* LR *is a set of* locally restricted nodes *(satisfying* $LR \cap X = \emptyset$*),* E *is a set of (hyper-)edges,* $l \colon E \to B$ *labels each edge with an element of* B, $c \colon E \to (X \cup LR)^*$ *is the* connection function *(satisfying* $\tau(c(e)) = rnk(l(e))$ *for all* $e \in E$*), and* $\rho \colon E \to \textbf{NR-Graph}[X \cup LR]$ *maps each edge to a graph nested within it.*

The Algebra of Graphs with Nesting - AGN: syntax, some terms, and the denoted graphs

$$\mathbb{G} \ ::= \ \mathbf{0} \ \mid \ x:s \ \mid \ b[\mathbb{G}](\overline{y}) \ \mid \ \mathbb{G} \mid \mathbb{G} \ \mid \ (\nu\ x:s)\mathbb{G} \ \mid \ (\mu\ x:s)\mathbb{G}$$



The Algebra of Graphs with Nesting: Axioms

$$\mathbb{G} \;::=\; \mathbf{0} \;\mid\; x:s \;\mid\; b[\mathbb{G}](\overline{y}) \;\mid\; \mathbb{G} \mid \mathbb{G} \;\mid\; (\nu\, x:s)\mathbb{G} \;\mid\; (\mu\, x:s)\mathbb{G}$$

$$\mathbb{G} \mid \mathbb{H} \equiv \mathbb{H} \mid \mathbb{G} \qquad \text{(A1)}$$

$$\mathbb{G} \mid (\mathbb{H} \mid \mathbb{I}) \equiv (\mathbb{G} \mid \mathbb{H}) \mid \mathbb{I} \qquad \text{(A2)}$$

$$\mathbb{G} \mid \mathbf{0} \equiv \mathbb{G} \qquad \text{(A3)}$$

$$(\omega_1\ x:s)(\omega_2\ y:t)\mathbb{G} \equiv (\omega_2\ y:t)(\omega_1\ x:s)\mathbb{G} \quad \textit{if } x:s \neq y:t \qquad \text{(A4)}$$

$$(\omega\ x:s)\mathbf{0} \equiv (\omega\ x:s)x:s \qquad \text{(A5)}$$

$$\mathbb{G} \mid (\omega\ x:s)\mathbb{H} \equiv (\omega\ x:s)(\mathbb{G} \mid \mathbb{H}) \quad \textit{if } x:s \notin fn(\mathbb{G}) \qquad \text{(A6)}$$

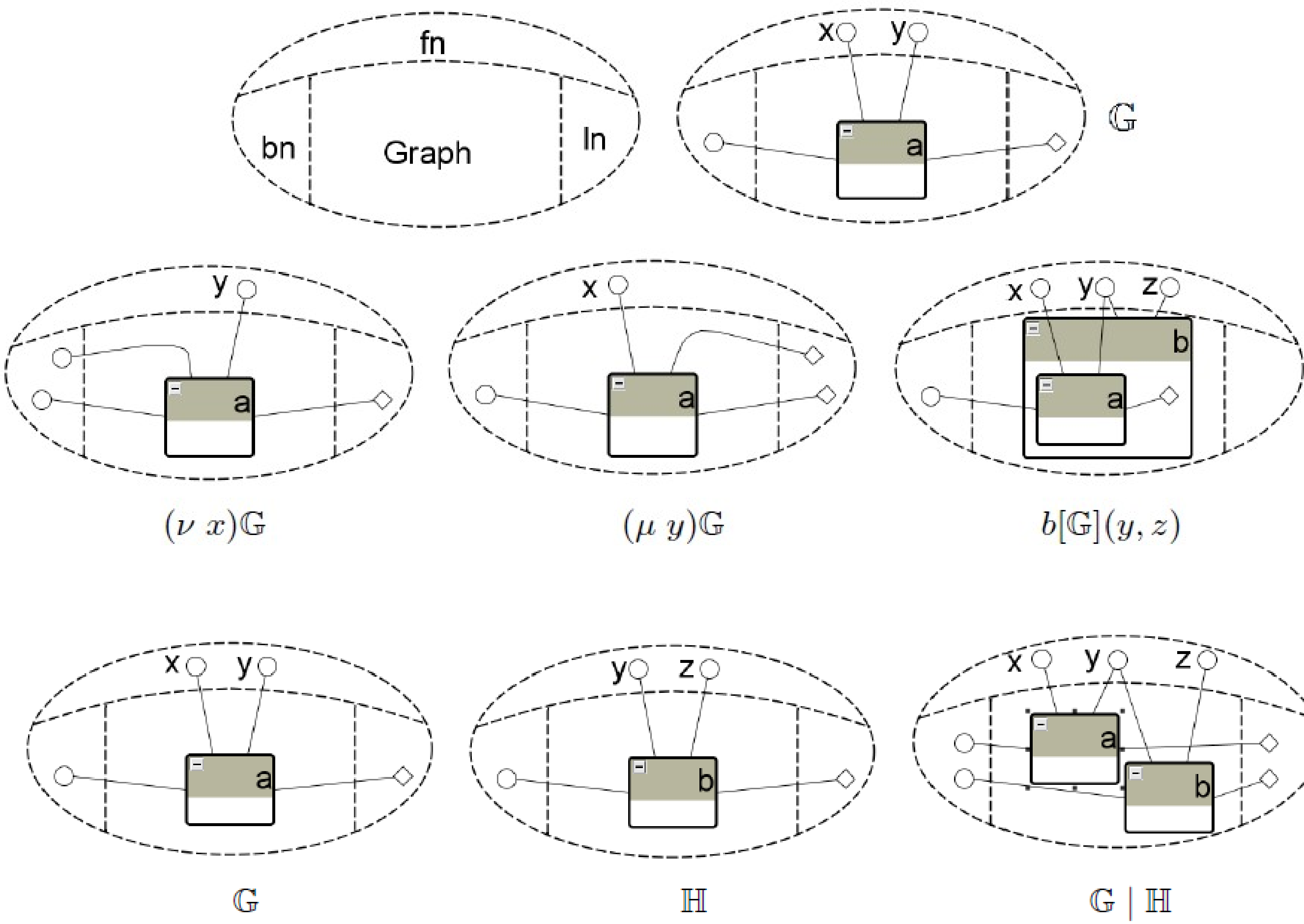
$$(\omega\ x:s)\mathbb{G} \equiv (\omega\ y:s)(\mathbb{G}\{^{y:s}/_{x:s}\}) \quad \textit{if } y:s \notin fn(\mathbb{G}) \qquad \text{(A7)}$$

$$b[(\nu\ x:s)\mathbb{G}](\overline{y}) \equiv (\nu\ x:s)b[\mathbb{G}](\overline{y}) \quad \textit{if } x:s \notin |\overline{y}| \qquad \text{(A8)}$$

$$x:s \mid \mathbb{G} \equiv \mathbb{G} \quad \textit{if } x:s \in fn(\mathbb{G}) \qquad \text{(A9)}$$

$$b[x:s \mid \mathbb{G}](\overline{y}) \equiv x:s \mid b[\mathbb{G}](\overline{y}) \qquad \text{(A10)}$$

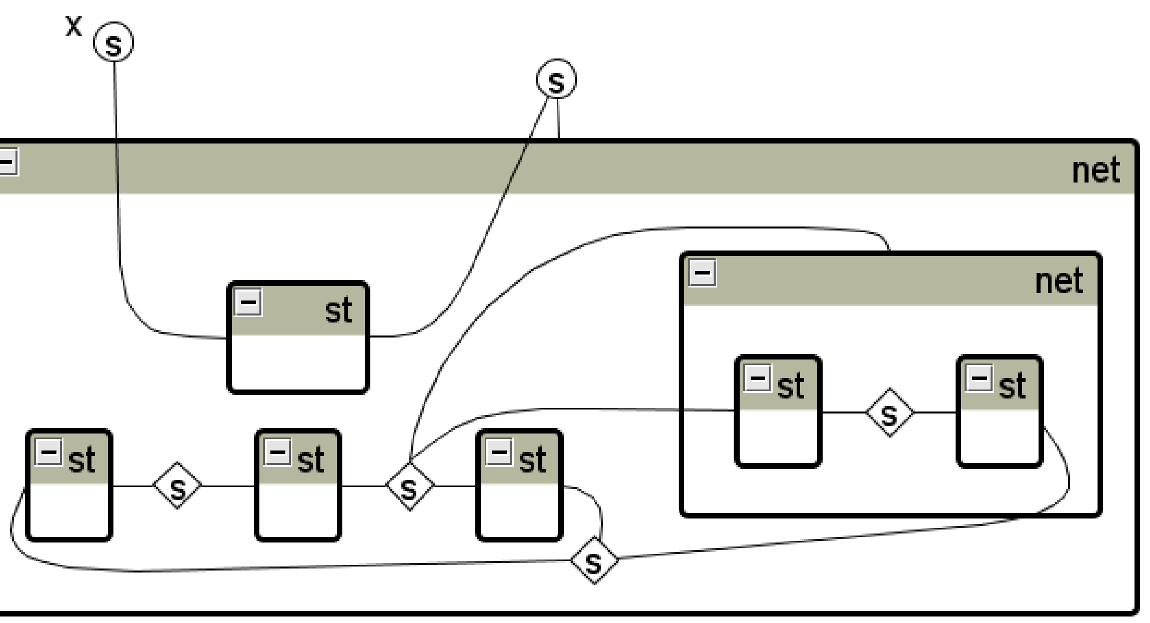
From terms of AGN to NR-graphs, informally



Properties of the axiomatization

- The axiomatization of NR-graphs is ***sound***, ***complete*** and **surjective**
- An AGN term and the corresponding NR-graph:

$$(\nu\ y)\mathsf{net}[\ \mathsf{st}(x, y)\ \mid\ (\mu\ z_1, z_2, z_3)(\ \mathsf{st}(z_3, z_1)\ \mid\ \mathsf{st}(z_1, z_2)\ \mid\ \mathsf{st}(z_2, z_3)\ \mid\ \mathsf{net}[\ (\mu\ z_4)(\ \mathsf{st}(z_2, z_4)\ \mid\ \mathsf{st}(z_4, z_3)\ )\ ](z_2)\ )\ ](y)$$



The simplest example: encoding the Ambient Calculus as AGN terms

- The syntax of the Ambient Calculus:

$$P ::= \mathbf{0} \mid (\nu n)P \mid P \mid P \mid M.P \mid n[P]$$
$$M ::= \mathsf{in}\ n \mid \mathsf{out}\ n \mid \mathsf{open}\ n$$

- box labels: "**[ ]**" for ambients; ***<u>M.P</u>*** for each process *M.P*

$$[\![\mathbf{0}]\!] \stackrel{\text{def}}{=} \mathbf{0}$$
$$[\![M.P]\!] \stackrel{\text{def}}{=} \underline{M.P}[\mathbf{0}](\overrightarrow{fn(M.P)})$$
$$[\![n[P]]\!] \stackrel{\text{def}}{=} [\,][\,[\![P]\!]\,](n)$$
$$[\![P \mid Q]\!] \stackrel{\text{def}}{=} [\![P]\!] \mid [\![Q]\!]$$
$$[\![(\nu w)P]\!] \stackrel{\text{def}}{=} (\nu w)[\![P]\!]$$

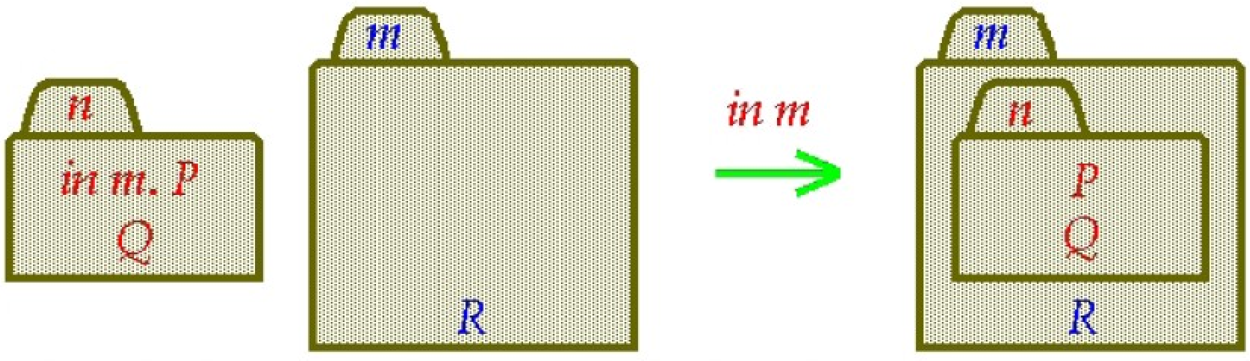
- We get automatically a representation of processes as NR-graphs

But what about the dynamics?

Reduction rules for the Ambient Calculus

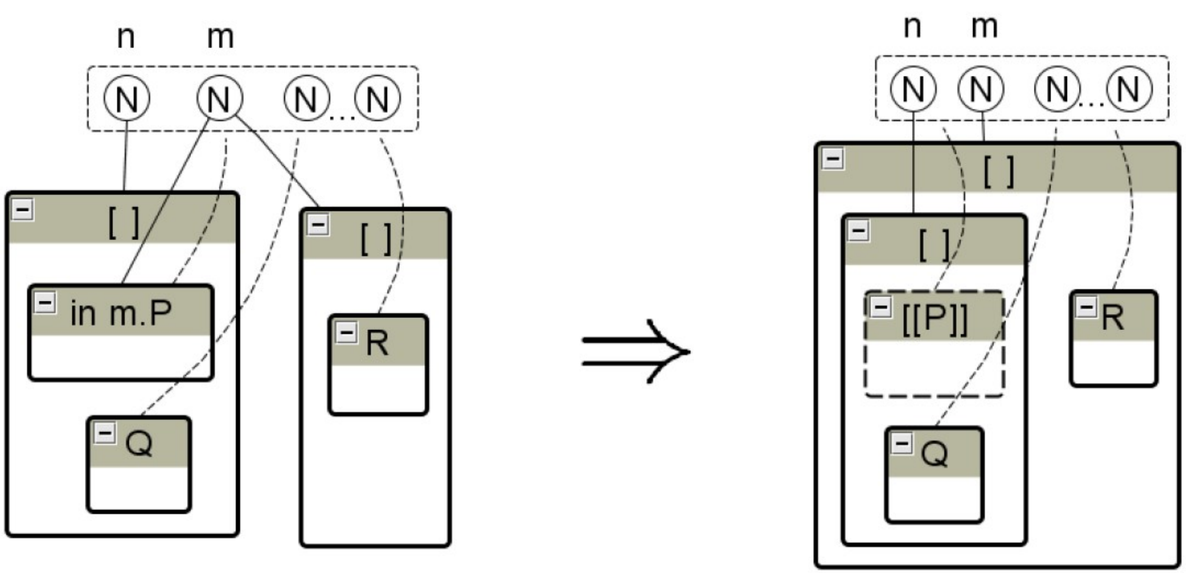
$$
\begin{array}{lll}
(1) & n[\mathsf{in}\ m.P \mid Q] \mid m[R] \Rightarrow m[n[P \mid Q] \mid R] & \textit{(Red In)} \\
(2) & m[n[\mathsf{out}\ m.P \mid Q] \mid R] \Rightarrow n[P \mid Q] \mid m[R] & \textit{(Red Out)} \\
(3) & \mathsf{open}\ n.P \mid n[Q] \Rightarrow P \mid Q & \textit{(Red Open)}
\end{array}
$$

A graphical intuition:



The in-rule, seen as pair of NR-graphs

$$n[\mathsf{in}\ m.P \mid Q] \mid m[R] \Rightarrow m[n[P \mid Q] \mid R] \qquad (Red\ In)$$



NR-graph rewriting needs to be formalized:

- role of R, Q and P
- meaning of [[P]]
- rule or *rule schema*?

- definition of matching?
- what is preserved?
- applicability?

Defining NR-graph rewriting: possible approaches

- Define from scratch rules, matches, rewriting (e.g. according to DPO approach), identify conditions for parallel/sequential independence, prove results about parallelism...
- Show that NR-graphs, equipped with suitable morphism, form an adhesive category (or a variation of it) and exploit general results.
- Embed NR-graphs into a known category of graphs, and work there, exploiting the existing results...
  - we embed NR-graphs into **Term Graphs**
    - many-sorted terms with sharing
    - acyclic hypergraphs (edges labeled by operators) with node indegree <= 1
  - it is a **quasi-adhesive** category, but the interesting results are not very interesting...

Encoding NR-graphs into Term Graphs

- Basic idea: add a new node sort for *locations*
  - every hyperedge and locally restricted node is attached to a location
  - every hyperedge *offers* a location (its interior)
  - locations form a tree
- We exploit an existing axiomatization of Term Graphs, as arrows of gs-monoidal theories.

$$\begin{array}{ccc}
\mathbf{AGN}(S,B)/_{\equiv_{\mathcal{A}}} & \Longleftrightarrow & \text{NR-Graphs over } (S,B) \\
{\scriptstyle Sec.\ 4}\ \big\downarrow & & {\scriptstyle Sec.\ 5}\ \big\downarrow \\
\mathbf{GS}(\Sigma_B^{\bullet}) & \xleftrightarrow{[10]} & \text{Term Graphs over } \Sigma_B^{\bullet}
\end{array}$$

GS-monoidal theory: an axiomatization of term graphs

$$(\text{op})\ \frac{f \in \Sigma_{u,s}}{f : u \to s} \qquad (\text{id})\ \frac{u \in S^*}{id_u : u \to u} \qquad (\text{bang})\ \frac{u \in S^*}{!_u : u \to \epsilon} \qquad (\text{dup})\ \frac{u \in S^*}{\nabla_u : u \to uu}$$

$$(\text{sym})\ \frac{u, v \in S^*}{\rho_{u,v} : uv \to vu} \qquad (\text{seq})\ \frac{t : u \to v \quad t' : v \to w}{t; t' : u \to w} \qquad (\text{par})\ \frac{t : u \to v \quad t' : u' \to v'}{t \otimes t' : uu' \to vv'}$$

Definition 11 (gs-monoidal theory). *Given a signature Σ over a set of sorts S, the associated* gs-monoidal theory $\mathbf{GS}(\Sigma)$ *is the category whose objects are the elements of S^*, and whose arrows are equivalence classes of* gs-monoidal terms, *i.e., terms generated by the inference rules in Fig. 6 subject to the following con*

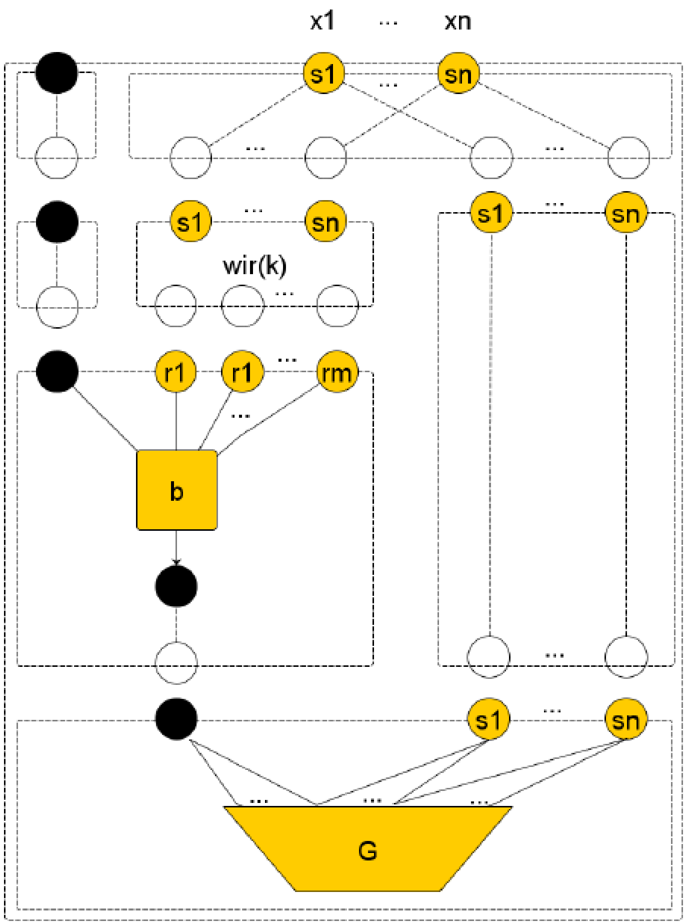
- *identities and sequential composition satify the axioms of categories:*
  [identity] $id_u ; t = t = t ; id_v$ *for all* $t : u \to v$*;*
  [associativity] $t_1 ; (t_2 ; t_3) = (t_1 ; t_2) ; t_3$ *whenever any side is defined,*
- $\otimes$ *is a monoidal functor with unit* id_ϵ*, i.e. it satisfies:*
  [functoriality] $id_{uv} = id_u \otimes id_v$*, and*
  $(t_1 \otimes t_2) ; (t'_1 \otimes t'_2) = (t_1 ; t'_1) \otimes (t_2 ; t'_2)$ *whenever both sides are defined,*
  [monoid] $t \otimes id_\epsilon = t = id_\epsilon \otimes t \qquad t_1 \otimes (t_2 \otimes t_3) = (t_1 \otimes t_2) \otimes t_3$
- ρ *is a natural transformation, i.e. it satisfies:*
  [naturality] $(t \otimes t') ; \rho_{v,v'} = \rho_{u,u'} ; (t' \otimes t)$ *for all* $t : u \to v$ *and* $t' : u' \to v'$
  and furthermore it satisfies:
  [symmetry] $(id_u \otimes \rho_{v,w}) ; (\rho_{u,w} \otimes id_v) = \rho_{u \otimes v, w} \qquad \rho_{u,v} ; \rho_{v,u} = id_{u \otimes v}$
  $\rho_{\epsilon,u} = \rho_{u,\epsilon} = id_u$
- ∇ *and* $!$ *satisfy the following axioms:*
  [unit] $!_\epsilon = \nabla_\epsilon = id_\epsilon$
  [duplication] $\nabla_u ; (id_u \otimes \nabla_u) = \nabla_u ; (\nabla_u \otimes id_u) \qquad \nabla_u ; (id_u \otimes !_u) = id_u$
  $\nabla_u ; \rho_{u,u} = \nabla_u$
  [monoidality] $\nabla_{uv} ; (id_u \otimes \rho_{v,u} \otimes id_v) = \nabla_u \otimes \nabla_v \qquad !_{uv} = !_u \otimes !_v$

Encoding AGN into Term Graphs

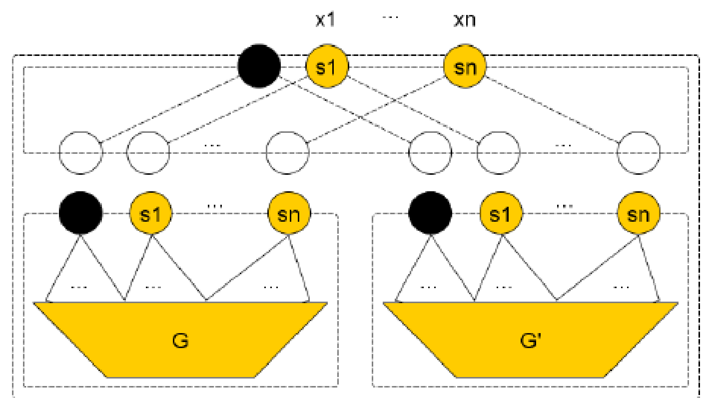
Inductive encoding from AGN terms to gs-monoidal terms

- $[\![\mathbf{0}]\!]_\sigma = [!_{\bullet,\tau(\sigma)}] : \bullet, \tau(\sigma) \to \epsilon$
- $[\![x : s]\!]_\sigma = [!_{\bullet,\tau(\sigma)}] : \bullet, \tau(\sigma) \to \epsilon$
- $[\![\mathbb{G}|\mathbb{G}']\!]_\sigma = [\nabla_{\bullet,\tau(\sigma)}] \ ; [\![\mathbb{G}]\!]_\sigma \otimes [\![\mathbb{G}']\!]_\sigma : \bullet, \tau(\sigma) \to \epsilon$
- $[\![(\nu\ x : s)\mathbb{G}]\!]_\sigma = [id_{\bullet,\tau(\sigma)} \otimes \nu_s] \ ; [\![\mathbb{G}\{^{y:s}/_{x:s}\}]\!]_{\sigma,y:s} : \bullet, \tau(\sigma) \to \epsilon$ where $y : s = fresh_{\mathbb{G}}(x : s, \sigma)$
- $[\![(\mu\ x : s)\mathbb{G}]\!]_\sigma = [(\nabla_\bullet \ ; id_\bullet \otimes \mu_s) \otimes id_{\tau(\sigma)}] \ ; [\![\mathbb{G}\{^{y:s}/_{x:s}\}]\!]_{y:s,\sigma} : \bullet, \tau(\sigma) \to \epsilon$ where $y : s = fresh_{\mathbb{G}}(x : s, \sigma)$

Encoding AGN into Term Graphs, graphically



$[\![b[\mathbb{G}](\overline{y})]\!]_\sigma$



$[\![\mathbb{G}|\mathbb{G}']\!]_\sigma$

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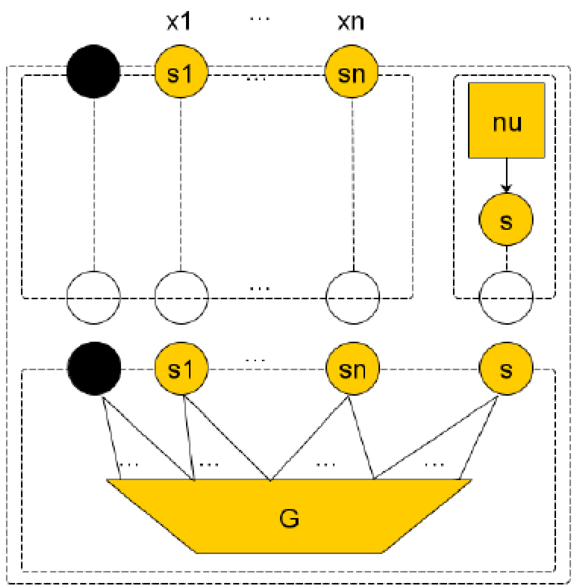
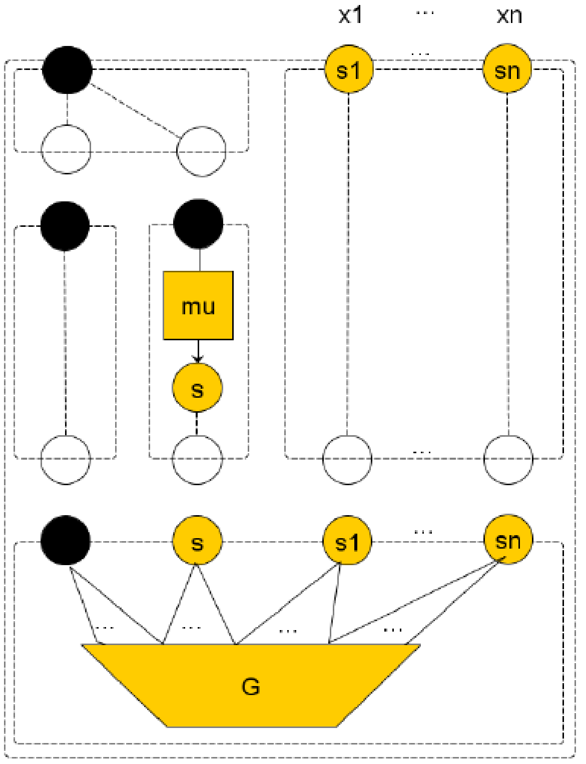


$[\![(\nu\ x : s)\mathbb{G}]\!]_\sigma$



$[\![(\mu\ x : s)\mathbb{G}]\!]_\sigma$

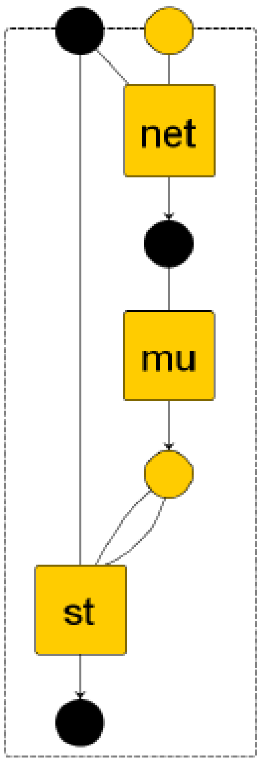
Properties of the encoding

Correct

Complete

Surjective onto **well-scoped** term graphs

A badly scoped term graph:
edge **st** accesses a node locally restricted in a sibling edge **net**



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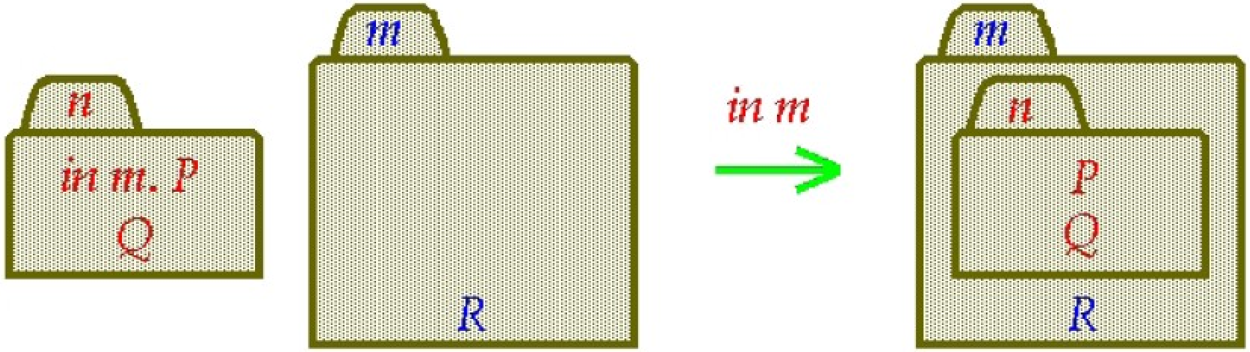
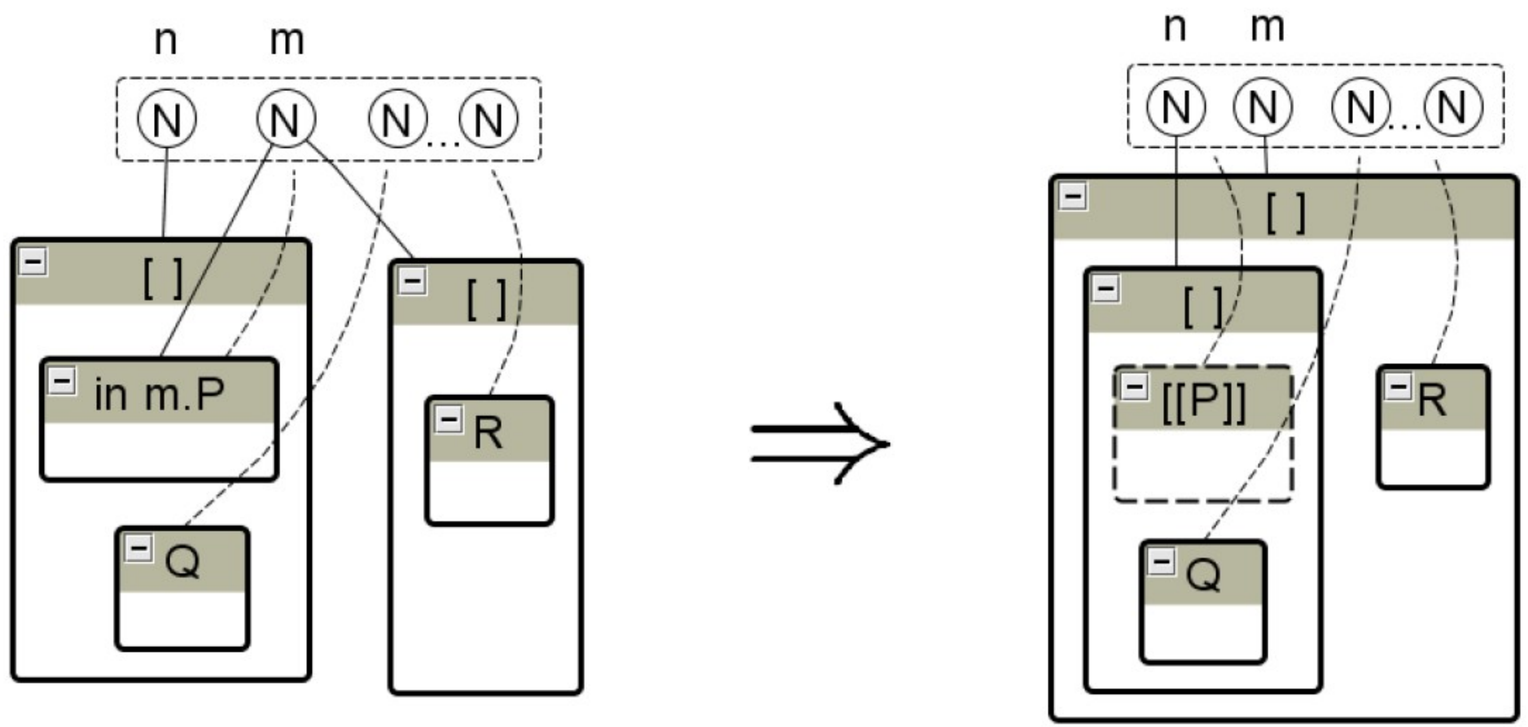


Back to the Ambient Calculus in-rule

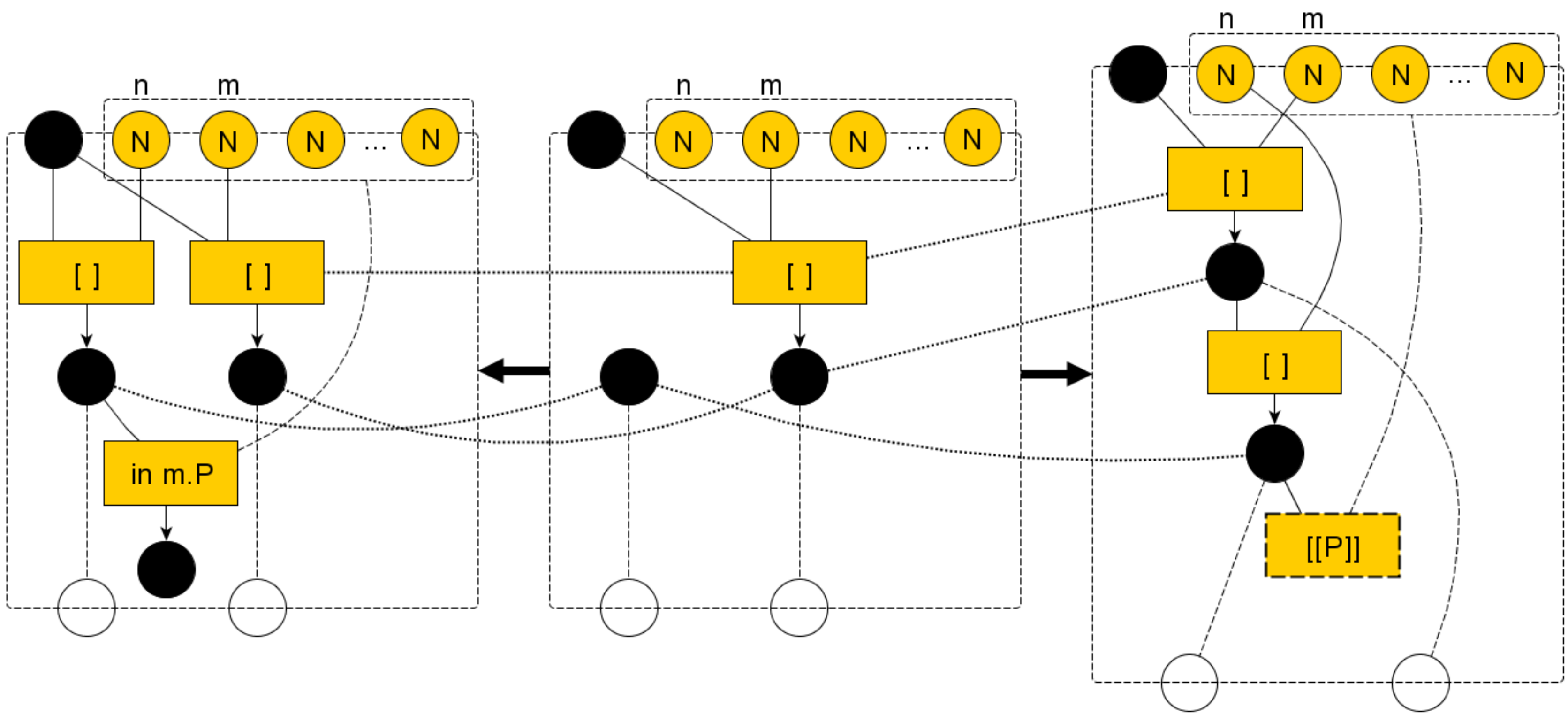
$$n[\text{in } m.P \mid Q] \mid m[R] \Rightarrow m[n[P \mid Q] \mid R] \qquad (Red\ In)$$



Let us translate it into term graphs

The in-rule, seen as Term Graph rule

$$n[\text{in } m.P \mid Q] \mid m[R] \Rightarrow m[n[P \mid Q] \mid R] \qquad (Red\ In)$$



The more formalized framework allows to

- identify the parts of the state that are preserved
- give a precise meaning to R and Q

Ongoing work

- Prove that the encoding of Ambient Calculus rules is correct
  - well-scopedness is preserved
  - rewrite steps are one-to-one with reductions
- Identify conditions on rules/matches that allow for the parallel application of rules, and thus for unfolding...
  - known results are too weak
    - Term Graphs are quasi-adhesive, but regular monos
      - are monos which preserve “variables”
      - you cannot even model rule **a** $\Rightarrow$ **b**
  - look for weaker conditions of applicability of Church-Rosser theorem
    - characterization of Van Kampen squares in Term Graphs

Conclusions

- A methodological approach for the graphical representation of process calculi and other computational formalisms
- **Static part:** *Using graph algebras as intermediate language*
  - Correct and complete axiomatization of class of graphs with nesting and restriction
  - Encoding of process terms into the graph algebra
  - Applied to **π-calculus**, **Sagas**, **CaSPiS**,...

DONE

- **Dynamics:** *Encode NR-graphs into Term Graphs*
  - Characterize conditions for parallel application of rules [existing ones are too weak
  - Exploit concurrent semantics of graph rewriting
    - unfolding techniques
    - analysis and verification

????