Electronic Notes in Theoretical Computer Science 44 No. 3 (2001)
URL: http://www.elsevier.nl/locate/entcs/volume44.html 18 pages

Some algebraic laws for spans (and their connections with multirelations) [1]

Roberto Bruni and Fabio Gadducci

Dipartimento di Informatica, Università di Pisa
{bruni,gadducci}@di.unipi.it

Abstract

This paper investigates some key algebraic properties of the categories of spans and cospans (up to isomorphic supports) over the category **Set** of (small) sets and functions, analyzing the monoidal structures induced over both spans and cospans by cartesian product and disjoint union of sets. Our results find analogous counterparts in (and are partly inspired by) the theory of relational algebras, thus our paper also sheds some light on the relationship between (co)spans and the categories of (multi)relations and of equivalence relations. And, since (co)spans yield an intuitive presentation of dynamical systems with input and output interfaces, our results introduce an expressive, two-fold algebra that can serve as a specification formalism for rewriting systems and for composing software modules.

Key words: spans, relations, monoidal categories, modular specification

Introduction

The use of spans [1,6] (and of the dual notion of cospans) have been quite ubiquitous in recent years. They have been used as a syntactical device for system specification [4,18,19]; as a foundational tool for graphs and graph rewriting [14,15]; or as a semantical domain for the study of partial and multi-algebras [9,26], and of predicate transformers [17].

Their presentation is deceptively simple: given a category $\mathcal{C}$ with pullbacks, the associated category $\mathsf{Span}(\mathcal{C})$ has the same objects as $\mathcal{C}$, and as a generic arrow from a to b a *span*, i.e., an object $c \in \mathcal{C}$ (called the *support*) together with two arrows $f\colon c \to a$ and $g\colon c \to b$ in $\mathcal{C}$ (called *legs*) having c as common source. Intuitively, a span can be viewed as a sort of input/output system, where each system module c is equipped with two "views" [19], which can be used to simulate system interaction, i.e., through which different modules may be interfaced.

[1] Research partly supported by the EC TMR Network GETGRATS and by the Italian MURST Project TOSCA.

Despite being straightforward, the span construction presents some subtleties. First of all, for a given category $\mathcal{C}$, the resulting $\mathsf{Span}(\mathcal{C})$ would be better described as a (bi)category, since composition is not really associative (if not up-to isomorphism); and, moreover, an intuitive notion of morphism between spans is easily given, making the resulting structure a *2-category* at least. Second, various restriction on the class of admissible spans may heavily restrict the class of derived arrows (e.g., requiring that at least one of the leg is a mono, see [26]) and the resulting structure may greatly vary, from the point of view of the algebraic properties.

The goal of this paper is rather modest: we consider the category **Set** of (small) sets and functions, and study some algebraic properties of $\mathsf{Span}(\mathbf{Set})$ (and of the dual $\mathsf{CoSpan}(\mathbf{Set})$). More precisely, we investigate the key features of the monoidal structures induced on these categories by lifting the cartesian product and the disjoint union in **Set**, obtaining a preliminary taxonomy for these properties. The relevance of such a taxonomy lies in the uses of (co)spans as a general-purpose framework. In fact, from one side it offers an abstract view of programs with interfaces (respectively the support and the legs of the span) and therefore the algebraic laws of the category express the way of composing programs by connecting their interfaces; from the other side, having an algebra of spans yields an abstract data type that can be used in rewriting systems for modeling distributed system via interconnected modules that can change dynamically.

As a suitable example, we illustrate a quite straightforward application for our framework as a semantic domain for Petri nets [24], a well-known specification mechanism for representing program schemata and distributed systems, which are built out of connecting, e.g., software modules and architecture components (i.e., the *transitions* of the net). In our paper we tackle *contextual nets* [23], i.e., nets offering read-only arcs (also called positive contexts, which allow to model multiple concurrent readings of the same resource), and we show how modeling each net computation as a suitable cospan, which is able to recover causal dependencies between fetched and produced resources (i.e., the *tokens* of the net), so that our result may seem to recast the well-known correspondence result for *processes* of ordinary nets [22]. Furthermore, we show that by modeling, instead, each net computation as a span, we can recover information about the persistence of resources in a computation, namely, those tokens which are may have been read but have not been consumed. In this setting, system reconfiguration can be viewed as a rule-driven reduction system that allows for simplifying program schemata and software architectures, and therefore, one can prototype automated normalizations of processes by using languages based on rewrite rules (e.g., MAUDE and ELAN, cfr. the documentation available at the URL's http://maude.csl.sri.com and http://elan.loria.fr) on the underlying span algebra.

The restriction to **Set** is limiting, and we leave as future work the task of lifting our taxonomy to other categories. Nevertheless, $\mathsf{Span}(\mathbf{Set})$ represents an important instance of the paradigm, which might be generalized to e.g. complete or regular categories. Furthermore, it better drives our intuition, since its correspondence with (multi)relations allows us to cast our properties against analogous results already presented in e.g. the theory of relational algebras [2]. In fact, even if the overall justification for the study of the categories of spans have been their connections with relations [5,6] (and it was already clear since [3] the one-to-one correspondence between *unitary pretabular allegories* [13] and cartesian bicategories), a careful analysis of these connections may result in a few surprises.

In Section 2 we make the correspondence precise by exploiting the notion of *multirelation*. As a suitable introduction, the relationship is fully worked out, even if it could be considered as an established fact in the literature, after Lawvere's work [21] (but see [29] for an up-to date presentation of the connection between categories of matrices and relations). What seems to be original, to the best of our knowledge, is the negative result in Proposition 2.2, stating the non-existence of object-preserving functors between the category of relations and $\mathsf{Span}(\mathbf{Set})$.

In Section 3 we provide the relevant results of the paper, analyzing the equational properties of the monoidal structures induced on $\mathsf{Span}(\mathbf{Set})$ by cartesian product and disjoint union in **Set**. As we will see, the resulting axiomatizations fully coincide with two different proposals present in the flownomial calculus on the one side, and in cartesian bicategories on the other.

In Section 4 we consider the "opposite" of the category of spans, namely, the category $\mathsf{CoSpan}(\mathbf{Set}) = \mathsf{Span}(\mathbf{Set}^{\mathrm{op}})$ of sets and cospans. We sketch the correspondence between cospans and *equivalence relations*, analyzing the equational properties of the structures induced on $\mathsf{CoSpan}(\mathbf{Set})$ by cartesian product and disjoint union in **Set**: While the latter category shares many of the properties of the monoidal category induced on spans by the cartesian product, the former fails to be monoidal, yielding a *pre-monoidal category* [25]. These correspondence results are, to the best of our knowledge, new.

Finally, in Section 5 we illustrate an application of spans to the modeling of Petri nets. We consider in particular contextual nets, an extension of the classical model including read-only arcs, and we prove that both categories of spans and cospans can be used to recover information about the causal dependencies between resources, as well as their persistence, in a computation.

1 Preliminaries

We introduce now the terminology and the notation that will be used in the rest of the paper, assuming the reader familiar with the basic concepts of category theory.

Abusing the notation we will often denote the identity arrow id_a by the object name a itself. Sequential composition of $f\colon a \to b$ and $g\colon b \to c$ is written $f;g\colon a \to c$, i.e., the symbol $\_;\_$ composes in the diagrammatic order.

We shall refer to the *opposite* category $\mathcal{C}^{\mathrm{op}}$ of a category $\mathcal{C}$ as the category having the same objects as $\mathcal{C}$, but where the direction of the arrows is reversed. For two categories $\mathcal{C}$ and $\mathcal{D}$ we let $\mathcal{C} \times \mathcal{D}$ denote their cartesian product in **Cat**, the category of (small) categories and functors.

1.1 Monoidal Categories

As standard in functorial semantics, we restrict to categories *strictly* preserving the additional structure; and, in particular, to monoidal structures where the tensor product is *strictly* associative and with unit, instead of being so only up to a canonical isomorphism. This allows us to focus on the overall presentation of the monoidal structure itself as a suitable data type, at the same time easing the notation, even if all the relevant structure is indeed preserved only up to iso on concrete categories such as **Set**. We refer the reader for details to the recent [9].

Definition 1.1 A (strict) *monoidal category* (MC) is a triple $\langle \mathcal{C}, \otimes, e\rangle$, where $\mathcal{C}$ is the underlying category, the *tensor product* $\otimes\colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is a functor satisfying the associative law $(t_1 \otimes t_2) \otimes t_3 = t_1 \otimes (t_2 \otimes t_3)$, and the *unit* e is an object of $\mathcal{C}$ satisfying the identity law $t_1 \otimes e = t_1 = e \otimes t_1$, for all arrows t_1, t_2 and t_3.

A *symmetric monoidal category* (SMC) is a 4-tuple $\langle \mathcal{C}, \otimes, e, \gamma\rangle$, where $\langle \mathcal{C}, \otimes, e\rangle$ is a MC, and $\gamma\colon {\_}_1 \otimes {\_}_2 \Rightarrow {\_}_2 \otimes {\_}_1$ is a natural transformation satisfying the coherence axioms $\gamma_{a\otimes b,c} = (a \otimes \gamma_{b,c});(\gamma_{a,c} \otimes b)$ and $\gamma_{a,b};\gamma_{b,a} = a \otimes b$, for all a, b, c.

As a matter of terminology, given two functors $F, G\colon \mathcal{C} \to \mathcal{D}$, we use the term *transformation* (from F to G) to denote a family η of arrows in $\mathcal{D}$ indexed by the objects of $\mathcal{C}$, such that $\eta_a\colon F(a) \to G(a)$ for all objects a in $\mathcal{C}$. It is a *natural transformation* if $F(f);\eta_b = \eta_a;G(f)$ for any arrow $f\colon a \to b$ in $\mathcal{C}$. For example, the naturality of symmetries in a SMC $\langle \mathcal{C}, \otimes, e, \gamma\rangle$ amounts to say that for all $f\colon a \to b$, $g\colon c \to d$ we have $(f \otimes g);\gamma_{b,d} = \gamma_{a,c};(g \otimes f)$.

Additional structures may live in SMC's, and suitable arrows such as *duplicators* and *dischargers* (and also their co-versions) do play significant roles. In particular, cartesian categories are monoidal categories such that duplicators and dischargers automatically exist, and are uniquely determined by the universal property of terminal object and products. A duplicator ∇ is a transformation from the identity on $\mathcal{C}$ to the functor $\mathsf{D};\otimes = {\_}_1 \otimes {\_}_1$ obtained by precomposing the tensor by the *diagonal functor* $\mathsf{D}\colon \mathcal{C} \to \mathcal{C} \times \mathcal{C}$ (with $\mathsf{D}(f) = (f, f)$ for all $f \in \mathcal{C}$). A discharger ! is a transformation between the identity on $\mathcal{C}$ and the constant functor that sends everything into the unit e. Moreover, the following coherence axioms must be satisfied:

$$\nabla_{a\otimes b} = (\nabla_a \otimes \nabla_b);(a \otimes \gamma_{a,b} \otimes b) \qquad !_{a\otimes b} = !_a \otimes !_b \qquad \nabla_e = !_e = id_e$$

$$\nabla_a;(\nabla_a \otimes a) = \nabla_a;(a \otimes \nabla_a) \qquad \nabla_a;\gamma_{a,a} = \nabla_a \qquad \nabla_a;(!_a \otimes a) = a$$

This situation is sometimes referred to in the literature by saying that each object is equipped with a comonoid structure [12], while, after [7], we denote such a structured category as *gs-monoidal*. If ∇ and ! are natural (i.e., if they satisfy $f;\nabla_b = \nabla_a;(f \otimes f)$ and $f;!_b = !_a$ for all $f\colon a \to b$), then the category is cartesian.

Similarly, co-duplicators $\Delta_a\colon a \otimes a \to a$ and co-dischargers $¡_a\colon e \to a$ must satisfy dual coherence axioms, and we denote the resulting category as *cogs-monoidal*; when naturality is also satisfied, it yields *co-cartesianity*. There are various ways for the gs-monoidal and the cogs-monoidal structures to interact. In particular, the following laws have been pointed out in the literature (see, e.g., [6,10]).

(1)	$\nabla_a;\Delta_a = a$	(5)	$¡_a;!_a = e$
(2)	$\Delta_a;\nabla_a = (\nabla_a \otimes \nabla_a);(a \otimes \gamma_{a,a} \otimes a);(\Delta_a \otimes \Delta_a)$	(6)	$!_a;¡_a = a$
(3)	$\Delta_a;\nabla_a = (\nabla_a \otimes a);(a \otimes \Delta_a)$	(7)	$¡_a;\nabla_a = ¡_a \otimes ¡_a$
(4)	$\Delta_a;\nabla_a = (a \otimes \nabla_a);(\Delta_a \otimes a)$	(8)	$\Delta_a;!_a = !_a \otimes !_a$

As a set, these laws are of course redundant. For example, laws (3) and (4) are equivalent, and they subsume law (2). At the same time, they are linked with naturality: The law (2) is e.g. satisfied when the monoidal product is both cartesian and co-cartesian, i.e., if the underlying category has *biproducts*.

Several subsets of the eight laws listed above have been studied in the literature, usually as a model for various algebras of systems. Case studies range from the *flownomials calculus* proposed by Căzănescu and Ştefănescu [10,28], to the *bicategories of relations* introduced by Carboni and Walters [5,6]. As stated in the Introduction, we plan to show which algebraic laws are satisfied by (co)span categories over **Set**, thus offering a taxonomy for those laws.

1.2 On spans (and cospans)

Definition 1.2 [Span] Given a category $\mathcal{C}$, a *span* on $\mathcal{C}$ is an ordered pair of arrows with common source $f\colon a \to b$ and $g\colon a \to c$ in $\mathcal{C}$, and it is denoted by

$$\langle f,g\rangle\colon b \leftarrow a \to c.$$

The object a is called the *support* of the span, while f and g are called respectively the *left* and the *right leg* of the span.

Two spans $\langle f_1,g_1\rangle\colon b \leftarrow a_1 \to c$ and $\langle f_2,g_2\rangle\colon b \leftarrow a_2 \to c$ are *equivalent* if an isomorphism $\phi\colon a_1 \to a_2 \in \mathcal{C}$ exists such that $\phi;f_2 = f_1$ and $\phi;g_2 = g_1$. Other choices of equivalence are available, but this is enforced by our understanding of spans as abstract *modules* of a distributed system: The arrows represent the interfaces offered for connecting to other components, and the support implements the functionality of the module. Morphisms between supports could be considered as refinement operations, as in [18]; thus, our equivalence modulo isomorphism abstracts away from the choice of the support, without altering its internal structure. At the same time, the equivalence allows for an associative operation of sequential composition: A span $\langle f,g\rangle$ is seen as an arrow from the target of f to the target of g.

Definition 1.3 [Composition of spans] Given any two composable spans $\langle f,g\rangle\colon b \leftarrow a \to c$ and $\langle h,k\rangle\colon c \leftarrow d \to e$, their sequential composition $\langle f,g\rangle;\langle h,k\rangle$ is the span $\langle p_1;f,p_2;k\rangle$, where p_1 and p_2 are the projections associated with the pullback of g and h in $\mathcal{C}$, if it exists (see Figure 1).

Since pullbacks are unique only up to isomorphism, it is essential to work on equivalence classes of spans; otherwise, either a choice of pullbacks would be required, or suitable *bicategories of spans* [1] should be considered. Summarizing, by slightly abusing the notation we use the terminology "spans" to denote equivalence classes rather than concrete diagrams.

$$\begin{tikzcd}
 & & g \times_0 h \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & & \\
b & a \arrow[l, "f"'] \arrow[dr, "g"'] & & d \arrow[dl, "h"] \arrow[r, "k"'] & e \\
 & & c & &
\end{tikzcd}$$

Figure 1. Sequential composition of spans.

Similarly, we will ignore the obvious 2-categorical enrichment [1,20]: Each hom-set has a natural pre-order, where $\langle f_1, g_1\rangle : b \leftarrow a_1 \rightarrow c \leq \langle f_2, g_2\rangle : b \leftarrow a_2 \rightarrow c$ if there exists a function $\phi : a_1 \rightarrow a_2$ such that $\phi; f_2 = f_1$ and $\phi; g_2 = g_1$. As we argued before, this enrichment falls outside the scope of the paper, which is interested only in a preliminary analysis of the equational properties of the framework.

Definition 1.4 [Identities] For each object $a \in \mathcal{C}$, we let $\mathbf{id}_a = \langle id_a, id_a\rangle$.

Definition 1.5 [Category of spans on $\mathcal{C}$] Given a category $\mathcal{C}$ with all pullbacks, the category $\mathsf{Span}(\mathcal{C})$ has the objects of $\mathcal{C}$ as objects and the spans on $\mathcal{C}$ as arrows. Arrow composition and identities are defined as in Definition 1.3 and Definition 1.4.

We focus on the structure of categories $\mathsf{Span}(\mathbf{Set})$ and $\mathsf{Span}(\mathbf{Set}^{\mathrm{op}}) = \mathsf{CoSpan}(\mathbf{Set})$. In the latter case, objects are sets $A, B, C, \ldots$ and arrows are *cospans* $[f, g] : B \rightarrow A \leftarrow C$, with identities $\overline{\mathbf{id}}_A = [id_A, id_A]$, and composition of cospans is computed via pushout in **Set**.

2 Spans and relations

In this section we focus on the category **Rel** of (small) sets and relations, highlightening its relationship with the category $\mathsf{Span}(\mathbf{Set})$. While a functor $\mathscr{R} : \mathsf{Span}(\mathbf{Set}) \rightarrow \mathbf{Rel}$ is easily defined (see Section 2.1), a straightforward definition for an inverse mapping from **Rel** to $\mathsf{Span}(\mathbf{Set})$ fails to be functorial, because spans retain more information about the *multiplicity* of components. Moreover, under reasonable assumptions no such functor exists, while a more precise correspondence can be drawn between $\mathsf{Span}(\mathbf{Set})$ and the category **MRel** of *multiset relations*.

2.1 Spans are more concrete than relations

There is an obvious full functor $\mathscr{R} : \mathsf{Span}(\mathbf{Set}) \rightarrow \mathbf{Rel}$: It is the identity on objects and maps each span $\langle f, g\rangle : B \leftarrow A \rightarrow C$ to the relation

$$\mathscr{R}(\langle f, g\rangle) = \{(b, c) \in B \times C \mid \exists a \in A. f(a) = b \wedge g(a) = c\}.$$

It can be easily verified that $\mathscr{R}$ preserves identities and composition, e.g., that $\mathscr{R}(\langle f, g\rangle; \langle h, k\rangle) = \mathscr{R}(\langle f, g\rangle); \mathscr{R}(\langle h, k\rangle)$ for spans as in Figure 1.

Remark 2.1 We recall that in **Set** the pullback of g and h can be defined as the triple $(g \times_0 h, p_1, p_2)$, where $g \times_0 h = \{(a, d) \in A \times D \mid g(a) = h(d)\}$, with p_1 and p_2 the obvious projections.

Let us consider the function $\mathscr{S}$ from **Rel** to $\mathsf{Span}(\mathbf{Set})$ that is the identity on objects and maps a relation $r \subseteq B \times C$ to the span $\langle q_1, q_2\rangle : B \leftarrow r \rightarrow C$, with q_i the obvious restriction, to the subset r, of the projection $\pi_i^{B\times C}$ from $B \times C$ to the ith component (B if $i = 1$, C if $i = 2$), later denoted $\pi_i^{B\times C}{}_{|r}$.

The mapping $\mathscr{S}$ models the intuitive way of regarding a relation as a span, but can it be extended to a functor? It preserves identities, because A is isomorphic to the set $\{(a,a) \mid a \in A\}$. To check if $\mathscr{S}$ preserves composition, we must show that for any relations $r: B \rightarrow C$ and $s: C \rightarrow E$, then $\mathscr{S}(r;s)$ is equivalent to $\mathscr{S}(r);\mathscr{S}(s)$. But this is not the case, because $\mathscr{S}(r);\mathscr{S}(s) = \langle \pi_1^{B\times C}{}_{|r}, \pi_2^{B\times C}{}_{|r}\rangle ; \langle \pi_1^{C\times E}{}_{|s}, \pi_2^{C\times E}{}_{|s}\rangle$ $= \langle \pi_1^{B\times C\times C\times E}{}_{|t}, \pi_4^{B\times C\times C\times E}{}_{|t}\rangle$, where $t = \{(b,c,c,e) \mid (b,c) \in r \wedge (c,e) \in s\}$. Since $r;s = \{(b,e) \mid \exists c \in C.(b,c) \in r \wedge (c,e) \in s\}$, it follows that $\mathscr{S}(r;s) = \mathscr{S}(r);\mathscr{S}(s)$ holds only if t and $r;s$ are isomorphic. This happens if for all $b \in B$ and $e \in E$, at most one $c \in C$ exists, with $(b,c) \in r$ and $(c,e) \in s$; otherwise the cardinality of t is greater than that of $r;s$. Thus, the obvious lluf (i.e., bijective on objects) functor from **Rel** to $\mathsf{Span}(\mathbf{Set})$ does not work. Also the "dual" solution (i.e., saturating the span with infinite multiplicities for each pair in the relation by letting $\mathscr{S}(r) = \langle p_1, p_2\rangle : B \leftarrow r \times \mathbb{N} \rightarrow C$ and taking advantage of the fact that $\mathbb{N}\times\mathbb{N}$ is iso to $\mathbb{N}$) fails, since the induced functor would preserve composition, but not identities.

Proposition 2.2 *No lluf functor $\mathscr{S}: \mathbf{Rel} \rightarrow \mathsf{Span}(\mathbf{Set})$ exists s.t. $\mathscr{S};\mathscr{R} = 1$.*

Proof *Let $A = \{a\}$ and $B = \{b_1, b_2\}$. It is straightforward that the identity relation on A coincides with $r;s$, for $r = \{(a,b_1),(a,b_2)\}$ and $s = \{(b_1,a),(b_2,a)\}$. Thus, let $\mathscr{S}(r) = \langle f, g\rangle : A \leftarrow X \rightarrow B$ and $\mathscr{S}(s) = \langle h, k\rangle : B \leftarrow Y \rightarrow A$ for suitable X and Y. Since we expect that $\mathscr{R}(\langle f, g\rangle) = r$, the set X must contain at least two elements x_1 and x_2 that are mapped by g in different elements of B, and similarly for Y. Thus, the pullback object $g \times_0 h$ also contains two elements (at least) and therefore $\mathscr{S}(r);\mathscr{S}(s)$ cannot be equivalent to the identity span on the singleton A.* $\square$

2.2 Spans define multirelations

We denote by $\mathbb{N}^\omega$ the semiring $\langle \mathbb{N}, +, \cdot\rangle$ extended with the top element $\top$ and the obvious infinitary operations (assuming that $0 \cdot \top = 0$).

Definition 2.3 [Multirelations] Let A and B be sets. A *multiset relation* (*multirelation*) on A and B is a multiset $M: A \times B \rightarrow \mathbb{N}^\omega$, where $M(a,b)$ defines the *multiplicity* of the pair $(a,b) \in A \times B$. As for relations, a multirelation $M: A \times B \rightarrow \mathbb{N}^\omega$ can be viewed as an arrow $M: A \rightarrow B$.

Definition 2.4 Given two multirelations $M: A \rightarrow B$ and $N: B \rightarrow C$, their composition is the multirelation $M;N$ such that

$$(M;N)(a,c) = \sum_{b\in B} M(a,b) \cdot N(b,c)$$

for any $(a,c) \in A \times C$, assuming the sum being $\top$ for B countable. Sets and multirelations form a category, denoted **MRel**.

With the previous definition, **MRel** results the Kleisli category of the monad induced by the functor $\_ \times \mathbb{N}^{\omega}$ from **Set** to itself.

Given a multirelation M, its underlying relation r_M is such that $(a,b) \in r_M$ iff $M(a,b) > 0$. It is immediate to note that a relation is just a multirelation $M{:}A \times B \to \{0,1\}$, however **Rel**, the category of sets and relations, is not a subcategory of **MRel**, as it follows from the results in the previous section. Instead, it is then possible to define a functor from $\mathsf{Span}(\mathbf{Set})$ to **MRel**. The function $\mathscr{M}$ is the identity on objects and maps each span $\langle f,g\rangle{:}B \leftarrow A \to C$ to the multirelation $\mathscr{M}(\langle f,g\rangle)$ such that for all $b \in B$ and $c \in C$ we have

$$\mathscr{M}(\langle f,g\rangle)(b,c) = |\{a \in A \mid f(a) = b \wedge g(a) = c\}|$$

Proposition 2.5 *The function $\mathscr{M}$ can be extended to a functor.*

The functor $\mathscr{P}{:}\mathbf{MRel} \to \mathsf{Span}(\mathbf{Set})$, inverse to $\mathscr{M}$, can be defined as follows: $\mathscr{P}$ is the identity on objects and maps any multirelation $M{:}B \to C$ to the span $\langle p_1,p_2\rangle{:}B \leftarrow \widehat{M} \to C$, where $\widehat{M} = \{(b,c,i) \in B \times C \times \mathbb{N} \mid 0 < i \leq M(b,c)\}$ and p_1 and p_2 are the obvious projections. For identities, the set $\widehat{A} = \{(a,a,1) \mid a \in A\}$ is of course isomorphic to the set A. Moreover, given any two multirelations $M{:}B \to C$ and $N{:}C \to E$, then $\mathscr{P}(M;N) = \mathscr{P}(M);\mathscr{P}(N)$.

It is easy to verify that $\mathscr{P};\mathscr{M}$ is the identity functor on **MRel**, since

$$\mathscr{M}(\mathscr{P}(M))(b,c) = |\{(b,c,i) \mid 0 < i \leq M(b,c)\}| = M(b,c).$$

Taken a generic span $\langle f,g\rangle{:}B \leftarrow A \to C$, then:

$$\mathscr{P}(\mathscr{M}(\langle f,g\rangle)) = \langle p_1,p_2\rangle{:}B \leftarrow \tilde{A} \to C,$$

where $\tilde{A} = \{(f(a),g(a),i) \mid 0 < i \leq \mathscr{M}(\langle f,g\rangle)(f(a),g(a))\}$, and p_1 and p_2 are the obvious projections. Then, any isomorphism between A and $\tilde{A}$ that maps a into $(f(a),g(a),i_a)$ for suitable index i_a proves the equivalence between $\mathscr{P}(\mathscr{M}(\langle f,g\rangle))$ and $\langle f,g\rangle$.

Proposition 2.6 *The categories* $\mathsf{Span}(\mathbf{Set})$ *and* **MRel** *are equivalent.*

2.3 Partial functions, minimal spans and relations

A further abstraction has to be required on spans, in order to discard multiplicities, but "syntactical" restrictions on span components do not suffice. For example, requesting left legs to be mono is a property closed under composition (pullbacks preserve monos) and thus yields a category, but in $\mathsf{Span}(\mathbf{Set})$ this just captures *partial functions* [26]. In fact if in $\langle f,g\rangle$, $f{:}A \to B$ is mono, then at most one pair $(b,g(a))$ can be assigned to $b \in B$ with $f(a) = b$. For the converse correspondence, given a partial function $k{:}B \to C$ we just take the span k_{p} given by $\langle \mathrm{in}, k_{|\mathrm{dom}(k)}\rangle{:}B \leftarrow \mathrm{dom}(k) \to C$, with $\mathrm{in}{:}\mathrm{dom}(k) \to B$ the obvious inclusion. Indeed, given a second partial function $h{:}C \to E$, then the composition $k_{\mathrm{p}};h_{\mathrm{p}}$ has as pullback object the domain of $k;h$, as expected.

Similarly, the restriction to mono right legs still yields a subcategory (dual to the previous one), while no category exists if restricting to *jointly mono* spans, because they are not closed under composition. Nevertheless, **Rel** can be recovered either redefining the composition for the category of jointly mono spans (namely, considering a suitable epi-mono decomposition component of the resulting span), or equivalently by collapsing spans whose supports are related by a surjective function (an epimorphism in **Set**.)

Definition 2.7 Two spans $\langle f_1, g_1\rangle{:}B \leftarrow A_1 \rightarrow C$ and $\langle f_2, g_2\rangle{:}B \leftarrow A_2 \rightarrow C$ are *kernel ordered*, written $\langle f_1, g_1\rangle \preceq_{\mathrm{e}} \langle f_2, g_2\rangle$, if there is an epi $e{:}A_2 \rightarrow A_1$ such that $f_2;e = f_1$ and $g_2;e = g_1$.

Composition of spans is monotonic with respect to such an ordering, and thus $\preceq_{\mathrm{e}}$ induces a partial order over $\mathsf{Span}(\mathbf{Set})$. We denote by $\cong_{\mathrm{e}}$ the symmetric closure of $\preceq_{\mathrm{e}}$, and by $\mathsf{Span}_{\mathrm{e}}(\mathbf{Set})$ the category of spans modulo the equivalence $\cong_{\mathrm{e}}$. Each $\cong_{\mathrm{e}}$-equivalence class has a minimal element (unique up to iso) that gives the minimal representation of the underlying relation.

Proposition 2.8 *The categories* $\mathsf{Span}_{\mathrm{e}}(\mathbf{Set})$ *and* **Rel** *are isomorphic.*

3 Two monoidal structures for spans

In this section we study the equational properties of the two different monoidal structures over $\mathsf{Span}(\mathbf{Set})$, induced by product and disjoint union in **Set**. While both structures result gs-monoidal *and* cogs-monoidal (since they are self-dual), the laws satisfied by the interaction of their respective operators are quite different.

3.1 Spans and product

We first analyze the case in which the tensor product is induced by the cartesian product in **Set**, denoting the resulting structure as $\langle\mathsf{Span}(\mathbf{Set}), \otimes\rangle$. Given two objects A_1 and A_2, their tensor product is the set $A_1 \times A_2$, and given two spans $\langle f_1, g_1\rangle{:}B_1 \leftarrow A_1 \rightarrow C_1$ and $\langle f_2, g_2\rangle{:}B_2 \leftarrow A_2 \rightarrow C_2$, their product is the span $\langle f_1 \times f_2, g_1 \times g_2\rangle{:}B_1 \times B_2 \leftarrow A_1 \times A_2 \rightarrow C_1 \times C_2$. The unit for the tensor product is the singleton $1 = \{\bullet\}$ (e.g., we assume $A \times 1 = A = 1 \times A$ for all sets A).

The symmetry at A and B is the span $\langle id_A \times id_B, \mathsf{X}_{A,B}\rangle{:}A \times B \leftarrow A \times B \rightarrow B \times A$, where $\mathsf{X}_{A,B}(a,b) = (b,a)$ for all $a \in A$ and $b \in B$. Note that the symmetry could have been defined as well as being the span $\langle \mathsf{X}_{B,A}, id_B \times id_A\rangle$, which is in fact equivalent to the previous one. We denote $\langle id_A \times id_B, \mathsf{X}_{A,B}\rangle$ by the symbol $\gamma^{\otimes}_{A,B}$. Coherence and naturality of $\gamma^{\otimes}_{A,B}$ rely on the properties of $\mathsf{X}_{A,B}$ in **Set**.

The duplicator at A is given by $\nabla^{\otimes}_A = \langle id_A, \nabla_A\rangle{:}A \leftarrow A \rightarrow A \times A$, where the function $\nabla_A{:}A \rightarrow A \times A$ is defined as $\nabla_A(a) = (a,a)$ for all $a \in A$. Although the duplicator ∇ is natural in **Set**, in general the duplicator $\nabla^{\otimes}$ is not natural in $\langle\mathsf{Span}(\mathbf{Set}), \otimes\rangle$.

Proposition 3.1 *Given a span $\langle f, g\rangle{:}B \leftarrow A \rightarrow C$, the compositions $\langle f, g\rangle;\nabla^{\otimes}_C$ and $\nabla^{\otimes}_B;\langle f \times f, g \times g\rangle$ are equivalent iff f is mono.*

This result is in fact pivotal in the characterization of spans as partial functions (see e.g. [26], and the recent [9] for a general discussion on gs-monoidal categories and varieties of partial algebras).

The discharger at A is given by $!^{\otimes}_A = \langle id_A, !_A \rangle : A \leftarrow A \rightarrow 1$, where $!_A : A \rightarrow 1$ is the unique constant function with $!_A(a) = \bullet$ for all $a \in A$. Although the discharger $!$ is natural in **Set**, the discharger $!^{\otimes}$ is not natural in $\langle \mathsf{Span}(\mathbf{Set}), \otimes \rangle$.

Proposition 3.2 *Given a span $\langle f, g \rangle : B \leftarrow A \rightarrow C$, the composition $\langle f, g \rangle ; !^{\otimes}_C$ is equivalent to $!^{\otimes}_B$ iff f is iso.*

Hence, this amounts to say that the discharger in $\langle \mathsf{Span}(\mathbf{Set}), \otimes \rangle$ is not natural because 1 is not terminal. In fact, for each set A there are infinitely many arrows to 1, which are differentiated by their left component. The coherence axioms can be easily verified due to the particular nature of the spans $\gamma^{\otimes}$, $\nabla^{\otimes}$ and $!^{\otimes}$ that have an iso as the left leg, and due to the fact that this property is preserved by span composition. Thus all coherence axioms trivially reduce to the analogous ones in **Set**, which are of course valid.

The self-dual nature of $\mathsf{Span}(\mathbf{Set})$ allows for a straightforward definition of the dual of duplicators and dischargers. Hence, we have the co-duplicator $\Delta^{\otimes}_A = \langle \nabla_A, id_A \rangle : A \times A \leftarrow A \rightarrow A$, and the co-discharger $¡^{\otimes}_A = \langle !_A, id_A \rangle : 1 \leftarrow A \rightarrow A$. It is obvious that co-duplicators and co-dischargers are not natural and satisfy the coherence axioms. The interesting question concerns the validity of laws (1)-(8).

Proposition 3.3 *The category $\langle \mathsf{Span}(\mathbf{Set}), \otimes \rangle$ satisfies the laws (1)-(4).*

The other laws are a different matter. For example, by composing dischargers with co-dischargers in $\langle \mathsf{Span}(\mathbf{Set}), \otimes \rangle$ in general we get $¡^{\otimes}_A ; !^{\otimes}_A = \langle !_A, !_A \rangle \neq \mathbf{id}_1$ and $!^{\otimes}_A ; ¡^{\otimes}_A = \langle \pi_1^{A \times A}, \pi_2^{A \times A} \rangle \neq \mathbf{id}_A$. In the first case, the equality holds iff A is either iso to 1, or to $\emptyset$; in the second case, iff A is iso to 1.

Likewise, the composition of duplicators and co-dischargers usually yields $¡^{\otimes}_A ; \nabla^{\otimes}_A = \langle !_A, \nabla_A \rangle \neq ¡^{\otimes}_A \times ¡^{\otimes}_A$ and $\Delta^{\otimes}_A ; !^{\otimes}_A = \langle \nabla_A, !_A \rangle \neq !^{\otimes}_A \times !^{\otimes}_A$. Also in these cases, the equalities hold iff A is either iso to 1, or to $\emptyset$.

Note that the same set of equations holds for relations. Thus, considering $\mathsf{Span_e}(\mathbf{Set})$, and denoting by $\langle \mathsf{Span_e}(\mathbf{Set}), \otimes \rangle$ the monoidal category induced by the cartesian product of sets, we can state the following result.

Proposition 3.4 *The category $\langle \mathsf{Span_e}(\mathbf{Set}), \otimes \rangle$ satisfies the laws (1)-(4).*

3.2 Spans and disjoint union

The disjoint union in **Set** induces a different gs-monoidal structure on $\mathsf{Span}(\mathbf{Set})$, denoted by $\langle \mathsf{Span}(\mathbf{Set}), \oplus \rangle$. Given two sets A and B, their tensor product is the disjoint union $A \uplus B = \{(0,a) \mid a \in A\} \cup \{(1,b) \mid b \in B\}$, with the empty set $\emptyset$ as unit. Given two spans $\langle f_1, g_1 \rangle$ and $\langle f_2, g_2 \rangle$, their tensor is $\langle f_1 \uplus f_2, g_1 \uplus g_2 \rangle$, where

$$(f \uplus g)(x) = \begin{cases} (0, f(a)) & \text{if } x = (0,a) \\ (1, g(b)) & \text{if } x = (1,b) \end{cases}$$

The symmetry at A and B is $\gamma^{\oplus}_{A,B} = \langle id_A \uplus id_B, \chi_{A,B} \rangle : A \uplus B \leftarrow A \uplus B \rightarrow B \uplus A$, where $\chi_{A,B}(i,x) = (i+1 \ \mathbf{mod}\ 2, x)$ (for $x \in A \cup B$). The duplicator at A is given by

$\nabla^{\oplus}_A = \langle \tau_A, id_{A\uplus A}\rangle : A \leftarrow A\uplus A \rightarrow A\uplus A$, where $\tau_A : A\uplus A \rightarrow A$ is defined as $\tau_A(i,a) = a$ for all $i \in [0,1]$ and $a \in A$, and it is natural. The discharger at A is defined by $!^{\oplus}_A = \langle \phi_A, id_{\emptyset}\rangle : A \leftarrow \emptyset \rightarrow \emptyset$, where $\phi_A : \emptyset \rightarrow A$ is the unique arrow from the initial object of **Set** to the set A, and it is also natural. Since also the coherence axioms are satisfied by $\gamma^{\oplus}$, $\nabla^{\oplus}$ and $!^{\oplus}$, we can conclude the following result.

Proposition 3.5 *Disjoint union makes* $\mathsf{Span}(\mathbf{Set})$ *a cartesian category.*

Dually, the natural co-duplicator is $\Delta^{\oplus}_A = \langle id_{A\uplus A}, \tau_A\rangle : A\uplus A \leftarrow A\uplus A \rightarrow A$ and the natural co-discharger is $¡^{\oplus}_A = \langle id_{\emptyset}, \phi_A\rangle : \emptyset \leftarrow \emptyset \rightarrow A$.

Proposition 3.6 *Disjoint union makes* $\mathsf{Span}(\mathbf{Set})$ *a co-cartesian category.*

However, the interplay between the two dual structures in $\langle \mathsf{Span}(\mathbf{Set}), \oplus\rangle$ satisfies only laws (2), (5), (7) and (8), whereas the other laws are trivially satisfied iff A is empty. Instead, relations satisfy one more law: We can summarize the situation with a result analogous to Proposition 3.3 and Proposition 3.4.

Proposition 3.7 *The category* $\langle \mathsf{Span}(\mathbf{Set}), \oplus\rangle$ *satisfies the laws (2), (5), (7) and (8); whereas the category* $\langle \mathsf{Span_e}(\mathbf{Set}), \oplus\rangle$ *additionally satisfies law (1).*

4 Cospans

Many analogies can be drawn between the categories $\mathsf{Span}(\mathbf{Set})$ and $\mathsf{CoSpan}(\mathbf{Set})$. In particular, *equivalence relations* play for cospans the role played by relations for spans: In a cospan $[f,g] : B \rightarrow A \leftarrow C$ all the elements of B and C that are mapped to the same element $a \in A$ are viewed as belonging to the same equivalence class. It is worth remarking that more information is stored in cospans, if the two arrows f,g are not *jointly epi*. Differently from $\mathsf{Span}(\mathbf{Set})$, $\mathsf{CoSpan}(\mathbf{Set})$ possesses only one monoidal structure, obtained by lifting disjoint union. The structure resulting from the lifting of product is just pre-monoidal: One of the few "natural" examples of such categories, to the best of our knowledge.

4.1 Cospans and equivalence relations

We try now to establish similar results between cospans and equivalence relations, as those holding between spans and relations. We first give the explicit definition of **ERel**, the category of sets and equivalence relations.

Definition 4.1 [Equivalence relations] Let A and B be sets. An *equivalence relation* (also, *partition*) e from A to B is a reflexive, transitive and symmetric relation over $A\uplus B$. A *redundant* partition e_p from A to B is a pair $\langle e, n_e\rangle$, for partition $e : A \rightarrow B$ and $n_e \in \mathbb{N}^{\omega}$. Given partitions $E : A \rightarrow B$ and $F : B \rightarrow C$, their composition is given by the pair $\langle \widehat{(e;f)^*}, n_e + n_f + n_B\rangle$, where $\widehat{(e;f)^*}$ is the restriction to $A\uplus C$ of the transitive closure of $e;f$, and n_B is the cardinality of the family of equivalence classes in $(e;f)^*$ containing only elements in B.

Sets and (redundant) equivalence relations form a category, denoted by **ERel** (respectively **RERel**).

There is an obvious full functor $\mathscr{P}$ that maps each cospan $[f,g]{:}B \to A \leftarrow C$ to the partition $P = \mathscr{P}([f,g])$, such that $P = \{P_1,\ldots,P_n \mid n = |f(B) \cup g(C)|\}$ and $d \in P_i$ iff, given a total ordering $x_1,...,x_n,...$ over $f(B) \cup g(C)$, then either $d \in B$ and $f(B) = x_i$, or $d \in C$ and $g(C) = x_i$. In the other direction, there is a lluf functor from **RERel** to $\mathsf{CoSpan}(\mathbf{Set})$. In addition, **ERel** can be identified with a suitable quotient category of $\mathsf{CoSpan}(\mathbf{Set})$.

Definition 4.2 Two cospans $[f_1,g_1]{:}B \to A_1 \leftarrow C$ and $[f_2,g_2]{:}B \to A_2 \leftarrow C$ are *image ordered*, written $\langle f_1,g_1\rangle \preceq_{\mathrm{m}} \langle f_2,g_2\rangle$, if there is a mono $m{:}A_2 \to A_1$ such that $f_2;e = f_1$ and $g_2;e = g_1$.

Composition of cospans is monotonic with respect to such an ordering, and thus $\preceq_{\mathrm{m}}$ induces a partial order over $\mathsf{CoSpan}(\mathbf{Set})$. We denote by $\cong_{\mathrm{m}}$ the symmetric closure of $\preceq_{\mathrm{m}}$, and by $\mathsf{CoSpan}_{\mathrm{m}}(\mathbf{Set})$ the category of spans modulo the equivalence $\cong_{\mathrm{m}}$. Each $\cong_{\mathrm{m}}$-equivalence class has a minimal element (unique up to iso) that gives the minimal representation of the underlying relation.

Proposition 4.3 *The categories* $\mathsf{CoSpan}(\mathbf{Set})$ *(*$\mathsf{CoSpan}_{\mathrm{m}}(\mathbf{Set})$*) and* **RERel** *(respectively* **ERel***) are equivalent (isomorphic).*

4.2 Cospans and disjoint union

A monoidal structure of $\mathsf{CoSpan}(\mathbf{Set})$ is given by taking as tensor product of A and B their disjoint union $A \uplus B$: We denote the resulting structure as $\langle \mathsf{CoSpan}(\mathbf{Set}), \oplus\rangle$. Given two cospans $[f_1,g_1]$ and $[f_2,g_2]$, their tensorial product is $[f_1 \uplus f_2, g_1 \uplus g_2]$ and the unit is the empty set $\emptyset$.

The symmetry at A and B is $\overline{\gamma}^{\oplus}_{A,B} = [id_A \uplus id_B, \chi_{B,A}]{:}A \uplus B \to A \uplus B \leftarrow B \uplus A$, and satisfies the naturality axiom for symmetries. The duplicator at A is defined by $\overline{\nabla}^{\oplus}_A = [id_A, \tau_A]{:}A \to A \leftarrow A \uplus A$; the discharger at A is $\overline{!}^{\oplus}_A = [id_A, \phi_A]{:}A \to A \leftarrow \emptyset$. In general, duplicators and dischargers are not natural.

Proposition 4.4 *Given* $[f,g]{:}B \to A \leftarrow C$*, then* $[f,g];\overline{\nabla}^{\oplus}_C = \overline{\nabla}^{\oplus}_B;[f \uplus f, g \uplus g]$ *iff* f *is surjective; whereas* $[f,g];\overline{!}^{\oplus}_C = \overline{!}^{\oplus}_B$ *iff* f *is iso.*

It is straightforward that the coherence axioms are satisfied by $\overline{\gamma}^{\oplus}$, $\overline{\nabla}^{\oplus}$ and $\overline{!}^{\oplus}$. The co-duplicator and co-discharger at A are $\overline{\Delta}^{\oplus}_A = [\tau_A, id_A]{:}A \uplus A \to A \leftarrow A$ and $\overline{\text{¡}}^{\oplus}_A = [\phi_A, id_A]{:}\emptyset \to A \leftarrow A$, respectively. The interplay between the two dual structures yields a monoidal structure very similar to the one of $\langle \mathsf{Span}(\mathbf{Set}), \otimes\rangle$.

Proposition 4.5 *The category* $\langle \mathsf{CoSpan}(\mathbf{Set}), \oplus\rangle$ *satisfies the laws (1)–(4).*

This analogy is partly supported by looking at laws (5)–(8): They are satisfied iff A is empty, as summarized in Table 1.

4.3 Cospans and products

As we already claimed, lifting the cartesian product of **Set** to $\mathsf{CoSpan}(\mathbf{Set})$ does not result into a monoidal category. A simple counterexample is given by considering the discharger $\overline{!}^{\otimes}_{A}$ and the co-discharger $\overline{¡}^{\otimes}_{A}$ of the monoidal structure induced by the disjoint union, and showing that the functoriality of product does not hold, namely, $(\overline{!}^{\otimes}_{A};\overline{¡}^{\otimes}_{A}) \otimes (\overline{!}^{\otimes}_{A};\overline{¡}^{\otimes}_{A}) \neq (\overline{!}^{\otimes}_{A} \otimes \overline{!}^{\otimes}_{A});(\overline{¡}^{\otimes}_{A} \otimes \overline{¡}^{\otimes}_{A})$. In fact, the structure $\langle \mathsf{CoSpan}(\mathbf{Set}), \otimes \rangle$ forms a pre-monoidal category: A relevant structure, albeit it lies ouside of the scope of the present paper, and its analysis is left for future work.

5 Spans, cospans and Petri nets

A *Petri net* N is a graph whose nodes are taken in the free commutative monoid $S^{\oplus}$ generated by a set of *places* and whose arcs are called *transitions*. We let $a, b, \ldots$ range over S and $t, s, \ldots$ range over the set of transitions T. The elements $u, v, \ldots$ of $S^{\oplus}$ are called *markings*. We let $u \cup v$, $u \subseteq v$, and $v \setminus u$ denote respectively multiset union, multiset inclusion (of u into v) and multiset difference (defined if $u \subseteq v$). Each marking $u = \oplus_{a \in S} n_a \cdot a$ defines a configuration of the system, i.e. a multiset of resources (*tokens*) typed over the places S; a transition $t\colon u_t \to v_t$ is *enabled* at u if $u_t \subseteq u$, and its execution (*firing*) leads to $(u \setminus u_t) \cup v_t$; a multiset of transitions $\tau = \{n_1 \cdot t_1, \ldots, n_k \cdot t_k\}$ with $t_i\colon u_i \to v_i$ for $i \in [1,k]$, is (concurrently) *enabled* at u if $\bigcup_{i \in [1,k]} n_i \cdot u_i \subseteq u$ and their firing leads to $(u \setminus \bigcup_{i \in [1,k]} n_i \cdot u_i) \cup \bigcup_{i \in [1,k]} n_i \cdot v_i$.

Nets with read arcs (or *positive contexts*) differ from ordinary Petri nets by distinguishing for a transition $t\colon u \to v$ a multiset w of 'read but not consumed' resources, with the obvious requirement that $u \supseteq w \subseteq v$. As a consequence, the concurrent firing of a multiset of transitions is allowed whenever they are concurrently enabled for what concerns their 'fetched' resources, but where 'contexts' can be instead shared, so that one 'contextual' token can enable more transitions at the same time. Let us now make some precise definition, restricting our attention, for the sake of simplicity, to nets where each arc has weight 1.

Definition 5.1 [contextual net] A *contextual net* N (also *c-net*) is a five-tuple $(S_N, T_N, pre_N, post_N, ctx_N)$ such that S_N and T_N are finite sets of *places* and *transitions*, respectively; and $pre_N, post_N, ctx_N : T_N \to S_N$ are jointly injective functions.

We plan to further restrict the class of nets under analysis. To this end, we need some additional definitions, concerning possible *sequences* of transitions.

Definition 5.2 [computations] Let N be a net and u a marking. A computation σ, starting from u, is a sequence of transition $t_1, \ldots, t_n$, such that for each $i \in [1,n]$, the transition t_i is enabled in u_i (i.e., $pre_N(t_i) \cup ctx_N(t_i) \subseteq u_i$), with $u_1 = u$ and $u_{i+1} = u_i \setminus pre_N(t_i) \cup post_N(t_i)$. The markings u_i are said to be *reachable* from u.

Spans can interpret computations in such a way that the information about non-consuming readings is maintained. For the sake of simplicity we illustrate the mapping for the class of *safe* nets, just sketching its extension to all nets.

Definition 5.3 [safeness] Given an initial marking $u_N \subseteq S_N$, the net N is *safe* (for u_N) if all the markings which can be reached starting from u_N are also just sets.

The notion of enabling for multisets of transitions can be easily given for safe contextual nets: simply, the tokens in the preset of each single transition are disjointly united, while those in the contexts are simply united.

It is possible to interpret each computation (eventually composed by a sequence of multisets of transitions) of a safe net N in $\langle \mathsf{Span}(\mathbf{Set}), \oplus \rangle$, by assigning to each transition t the span

$$\langle f_t, g_t \rangle : pre_N(t) \cup ctx_N(t) \leftarrow ctx_N(t) \rightarrow post_N(t) \cup ctx_N(t),$$

where f_t and g_t are the obvious injections.

The firing of t at marking $u \supseteq pre_N(t) \cup ctx_N(t)$ is then interpreted as the tensor product (in $\langle \mathsf{Span}(\mathbf{Set}), \oplus \rangle$) of $\langle f_t, g_t \rangle$ with the identity of $u \setminus (pre_N(t) \cup ctx_N(t))$; i.e., as $\langle f_t, g_t \rangle \oplus \mathbf{id}_{u \setminus (pre_N(t) \cup ctx_N(t))} : u \leftarrow u \setminus pre_N(t) \rightarrow (u \setminus pre_N(t)) \cup post_N(t)$. In fact, the support contains those resources which are only read by the transition (namely, $ctx_N(t)$) and those which are not checked at all in the transition itself (namely, $(u \setminus ctx_N(t)) \setminus pre_N(t)$).

Finally, the span associated with a computation $\sigma = t_1, ..., t_n$ is given by sequentially composing the spans associated with the n firings of the t_i's (at their respective u_i's, see Definition 5.2). Thus, to each computation a span is associated inductively, and the following claim may be safely stated.

Claim 5.4 *Let $\rho_\sigma : u \leftarrow w \rightarrow v$ be the span associated with a computation σ. The elements $a \in u$ which are in the image of the support w are those resources which are eventually read, but never consumed, in the computation. Moreover, for each element $a \in w$, its images through the left and right legs of the span ρ_σ represent the same (idle or read) resource of the computation σ.*

A more intriguing characterization may be obtained interpreting computations in $\langle \mathsf{CoSpan}(\mathbf{Set}), \oplus \rangle$, by assigning to each transition t the cospan

$$[f_t, g_t] : pre_N(t) \cup ctx_N(t) \rightarrow \{t\} \cup ctx_N(t) \leftarrow post_N(t) \cup ctx_N(t),$$

where f_t and g_t are the identities on $ctx_N(t)$, and the constant functions on $pre_N(t)$ and $post_N(t)$ (sending everything to the element t of the support). Thus, the support contains, in addition to those resources which are only read (namely, $ctx_N(t)$), also a token simulating the occurrence of a causal dependency among the other resources. Or, more appropriately, a relation of necessary consumption between them. While, to some extent, fetched and produced resources belong to the same thread (the element t in the support), read resources have associated one side-thread each.

This could be better understood by introducing two types in the support: $\mathtt{T}$ and $\mathtt{C}$, where $\mathtt{C} \leq \mathtt{T}$, with typing assignments $\tau(t) = \mathtt{T}$ and $\tau(a) = \mathtt{C}$ for all $a \in ctx_N(t)$. When cospans are sequentially composed, the types of the elements in the resulting support is given by the sup of the types of the elements in their counterimages, i.e., when composing $[f, g]$ with $[h, k]$ via the pushout $(D, q_1 : B \rightarrow D, q_2 : C \rightarrow D)$ of $g : A \rightarrow C$ and $h : A \rightarrow B$, then the type $\tau(d)$ of any element $d \in D$ is given by $\tau(d) = \bigsqcup_{b \in q_1^{-1}(d)} \tau(b) \sqcup \bigsqcup_{c \in q_2^{-1}(d)} \tau(c)$.

This reflects the intuition that a token read by the first step of a computation and then consumed in the second step becomes a consumed token (similarly for tokens that are first produced and then read). Of course, to each computation a cospan is associated inductively as before, and the following claim may be safely stated.

Claim 5.5 *Let $\eta_\sigma : u \to w \leftarrow v$ be the cospan associated with a computation σ. Given $a \in u$ and $b \in v$, if they are mapped by the legs of the span to the same element, say c, of the support, then in the computation the resource a has been fetched for producing the resource b. Moreover, if $\tau(c) = \mathtt{C}$, then a and b would correspond to the same contextual (or idle) token.*

It is by no chance that cospans may intuitively allow for a discussion on causal dependencies among the different *threads*, so to say, of a computation. In fact, the causal and concurrent semantics of a net N is usually expressed by means of so-called *deterministic processes*, a particular kind of acyclic safe nets, each one of them modeling a computation of N. In this case, the possible simultaneous execution of two transitions sharing the same resource has its counterpart in the causal independence between their representation as processes.

In [11] it has been shown that the deterministic processes of a generic net N form the arrows of a suitable symmetric monoidal category (symmetries have the task of eventually rearranging multiple tokens in the same place, whenever this is needed for composing processes), and in [16], the relationship has been extended to a fairly sophisticate embedding of processes of a contextual nets into *match-share* categories. These categories are essentially symmetric monoidal categories equipped with duplicators and coduplicators satisfying laws (1) and (3), and consequently, also laws (2) and (4). Reading without consuming is in fact modeled by first duplicating the contextual resources, then executing the transition with an idle copy of the original resources in parallel, finally matching (via a coduplicator) the idle copy with the corresponding resources in the postset of the step.

Thus, since $\mathsf{CoSpan}(\mathbf{Set})$ is a model of the match-share category generated by N, each contextual process of N can be interpreted in that category, by first defining the image of transitions (as we have illustrated above) and then exploiting the features of initial model semantics to lift the mapping to all the processes via the unique strict match-share functor that extends the interpretation of transitions.

In non-safe nets, markings can involve several tokens in the same place. Correspondingly, when composing computations, it is crucial not to mix up those tokens, as they can carry different causal (and persistence) information. Though the token types (i.e., the place where they belong) get lost in the interpretation, the correct typing would be preserved by the functorial interpretation of processes.

Remark 5.6 Alternatively, typed (co)spans could be employed. Formally, a span $\langle f, g \rangle : A \leftarrow C \to B$ is *typed* over S when it is equipped with two functions $f_S : A \to S$ and $g_S : B \to S$; moreover, it is composable with a span $\langle h, k \rangle : B \leftarrow E \to D$ typed over S by h_S and k_S, if $g_S = h_S$ (composition is defined as usual, with resulting typings given by f_S and k_S). Likewise, for typed cospans.

	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	nat. dup.	nat. dis.
$\mathsf{Span}(\mathbf{Set}), \otimes$	+	+	+	+	$\mid A \mid \leq 1$	$\mid A \mid \leq 1$	$A \simeq 1$	$A \simeq 1$	f mono	f iso
$\mathsf{Span}(\mathbf{Set}), \oplus$	$A = \emptyset$	+	$A = \emptyset$	$A = \emptyset$	+	$A = \emptyset$	+	+	+	+
$\mathsf{CoSpan}(\mathbf{Set}), \oplus$	+	+	+	+	$A = \emptyset$	$A = \emptyset$	$A = \emptyset$	$A = \emptyset$	f epi	f iso
$\mathbf{Rel}, \otimes$	+	+	+	+	$\mid A \mid \leq 1$	$\mid A \mid \leq 1$	$A \simeq 1$	$A \simeq 1$	f mono	f iso
$\mathbf{Rel}, \oplus$	+	+	$A = \emptyset$	$A = \emptyset$	+	$A = \emptyset$	+	+	+	+
$\mathbf{ERel}, \oplus$	+	+	+	+	$A = \emptyset$	$A = \emptyset$	$A = \emptyset$	$A = \emptyset$	f epi	f iso

Table 1
Summary of results for (co)spans and (equivalence) relations.

Concluding Remarks

The aim of our paper was to investigate some algebraic properties satisfied by the categories $\mathsf{Span}(\mathbf{Set})$ and $\mathsf{CoSpan}(\mathbf{Set})$. In particular, we analyzed the monoidal structures over those two categories, induced by cartesian product and disjoint union in **Set**: Our results are summarized in Table 1. Each row is dedicated to a particular symmetric monoidal category; each column is associated with a particular property (laws (1)–(8), naturality of (co)duplicators and (co)dischargers). Each entry describes whether the property holds in the category: The symbol + states that the axiom is valid, while the other entries describe the sufficient and necessary condition under which the axiom is satisfied. As for the naturality axioms, conditions refer to a generic span $\langle f, g \rangle$ or cospan $[f, g]$ with support A.

In particular, the table displays the similarity between $\mathsf{Span}(\mathbf{Set})$ with products and $\mathsf{CoSpan}(\mathbf{Set})$ with unions: Not too surprising, given the duality in their definition; more striking is the different behaviour over the alternative structures, which we were not able to pinpoint in a formal way. In fact, our results are still preliminary. As an example, we did not tackle at all the issue of the intuitive ordering over (co)spans, except in defining the categories $\mathsf{Span}_{\mathrm{e}}(\mathbf{Set})$ and $\mathsf{CoSpan}_{\mathrm{m}}(\mathbf{Set})$. This is a relevant topic, both semantically, as shown e.g. in the predicate transformer construction in [17]; and syntactically, since it would allow us to take further the connection between our work and e.g. the notion of direct product [2] in relational algebras, which is for now forbidden by our restriction with respect to the 2-dimensional aspects of the formalism. Nevertheless, the *flat* view we pursued in this paper seems to be enough for recasting and extending several other properties used in relational approaches (e.g., thanks to the correspondence we sketched in Section 4.1, one can easily generalize the notion of *difunctionality* [27] which is then preserved by composition, yielding a subcategory).

Moreover, we also plan to investigate if, and how, our taxonomy can be extended and generalized to (either complete or regular) categories other than **Set**. We are thinking in particular of **Graph**, given the importance of the resulting categories in the modeling of the operational behaviour of rewriting systems and of automata, as pointed out in [8,15] and [18,19], respectively.

References

[1] J. Bénabou. Introduction to bicategories. In *Midwest Category Seminar*, volume 47 of *Lectures Notes in Mathematics*, pages 1–77. Springer Verlag, 1967.

[2] R. Berghammer, A. Haeberer, G. Schmidt, and P. Veloso. Comparing two different approaches to products in abstract relational algebra. In M. Nivat, Ch. Rattray, T. Rus, and G. Scollo, editors, *Algebraic Methodology and Software Technology*, Workshops in Computing, pages 167–176. Springer Verlag, 1994.

[3] C. Brown and A. Jeffrey. Allegories of circuits. In A. Nerode and Y. Matiyasevich, editors, *Logic Foundations of Computer Science*, volume 813 of *Lect. Notes in Comp. Science*, pages 56–68. Springer Verlag, 1994.

[4] R. Bruni, F. Gadducci, and U. Montanari. Normal forms for partitions and relations. In J.L. Fiadeiro, editor, *Recent Trends in Algebraic Development Techniques*, volume 1589 of *Lect. Notes in Comp. Science*, pages 31–47. Springer Verlag, 1999.

[5] A. Carboni, S. Kasangian, and R.H. Street. Bicategories of spans and relations. *Journal of Pure and Applied Algebra*, 33:259–267, 1984.

[6] A. Carboni and R.F.C. Walters. Cartesian bicategories I. *Journal of Pure and Applied Algebra*, 49:11–32, 1987.

[7] A. Corradini and F. Gadducci. An algebraic presentation of term graphs, via gs-monoidal categories. *Applied Categorical Structures*, 7:299–331, 1999.

[8] A. Corradini and F. Gadducci. Rewriting on cyclic structures: Equivalence between the operational and the categorical description. *Informatique Théorique et Applications/Theoretical Informatics and Applications*, 33:467–493, 1999.

[9] A. Corradini and F. Gadducci. A functorial semantics for multi-algebras and partial algebras, with applications to syntax. *Theoret. Comput. Sci.*, 2001. To appear. Available at `http://www.di.unipi.it/~gadducci/papers`.

[10] V.-E. Căzănescu and Gh. Ştefănescu. Classes of finite relations as initial abstract data types I. *Discrete Mathematics*, 90:233–265, 1991.

[11] P. Degano, J. Meseguer, and U. Montanari. Axiomatizing the algebra of net computations and processes. *Acta Informatica*, 33:641–667, 1996.

[12] T. Fox. Coalgebras and cartesian categories. *Communications in Algebra*, 4:665–667, 1976.

[13] P. Freyd and A. Scedrov. *Categories, Allegories*. North-Holland, 1990.

[14] F. Gadducci and R. Heckel. An inductive view of graph transformation. In F. Parisi-Presicce, editor, *Recent Trends in Algebraic Development Techniques*, volume 1376 of *Lect. Notes in Comp. Science*, pages 219–233. Springer Verlag, 1998.

[15] F. Gadducci, R. Heckel, and M. Llabrés. A bi-categorical axiomatisation of concurrent graph rewriting. In M. Hofmann, D. Pavlovič, and G. Rosolini, editors,

Category Theory and Computer Science, volume 29 of *Electronic Notes in Theoretical Computer Science*. Elsevier Sciences, 1999. Available at `http://www.elsevier.nl/locate/entcs/volume29.html/`.

[16] F. Gadducci and U. Montanari. Axioms for contextual net processes. In K.G. Larsen, S. Skyum, and G. Winskel, editors, *Automata, Languages and Programming*, volume 1443 of *Lect. Notes in Comp. Science*, pages 296–308. Springer Verlag, 1998.

[17] P. Gardiner, C. Martin, and O. de Moor. An algebraic construction of predicate trasformers. In R.S. Bird, C.C. Morgan, and Woodcock J.C.P., editors, *Mathematics of Program Construction*, volume 669 of *Lect. Notes in Comp. Science*. Springer Verlag, 1992.

[18] P. Katis, N. Sabadini, and R.F.C. Walters. Bicategories of processes. *Journal of Pure and Applied Algebra*, 115:141–178, 1997.

[19] P. Katis, N. Sabadini, and R.F.C. Walters. SPAN(Graph): A categorical algebra of transition systems. In M. Johnson, editor, *Algebraic Methodology and Software Technology*, volume 1349 of *Lect. Notes in Comp. Science*, pages 307–321. Springer Verlag, 1997.

[20] G.M. Kelly and R.H. Street. Review of the elements of 2-categories. In G.M. Kelly, editor, *Sydney Category Seminar*, volume 420 of *Lecture Notes in Mathematics*, pages 75–103. Springer Verlag, 1974.

[21] F.W. Lawvere. Metric spaces, generalized logic, and closed categories. *Rendiconti del Seminario Matematico e Fisico di Milano*, 43:135–166, 1973.

[22] J. Meseguer and U. Montanari. Petri nets are monoids. *Information and Computation*, 88:105–155, 1990.

[23] U. Montanari and F. Rossi. Contextual nets. *Acta Informatica*, 32:545–596, 1995.

[24] C.A. Petri. *Kommunikation mit Automaten*. PhD thesis, Institut für Instrumentelle Matematik, Bonn, 1962.

[25] J. Power and E. Robinson. Premonoidal categories and notions of computation. *Mathematical Structures in Computer Science*, 7:453–468, 1998.

[26] E. Robinson and G. Rosolini. Categories of partial maps. *Information and Computation*, 79:95–130, 1988.

[27] G. Schmidt and T. Ströhlein. *Relations and Graphs, Discrete Mathematics for Computer Scientists*. EATCS-Monographs on Theoretical Computer Science. Springer Verlag, 1993.

[28] Gh. Ştefănescu. *Network Algebra*. Springer Verlag, 2000.

[29] M. Winter. A pseudo representation theorem for various categories of relations. *Theory and Application of Categories*, 7:23–37, 2000. Available at `http://www.tac.mta.ca/tac/`.