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Some algebraic laws for spans (and their connections with multirelations)

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Plan of the talk

- **A few words on motivation**

i.e. what spans can be used for

- **An application to Petri nets with read arcs**

i.e. an abstract domain for software architectures

- **Span(Set), (multi)relations, and some algebraic properties**

i.e. a closer look to spans

- **CoSpan(Set), difunctionality, and some algebraic properties**

i.e. a closer look to cospans

- **Causality and concurrency via (co)spans**

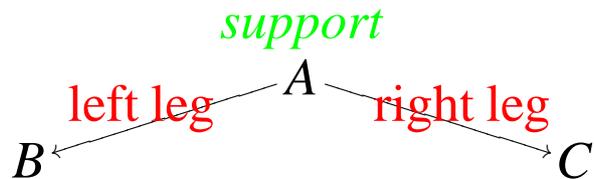
i.e. back to Petri nets

- **Summary and conclusion**

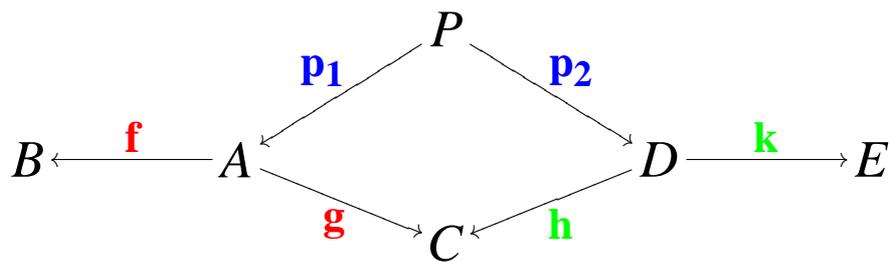
i.e. a tableau of algebraic laws

Spans

Scenario: A category \mathcal{C} with pullbacks

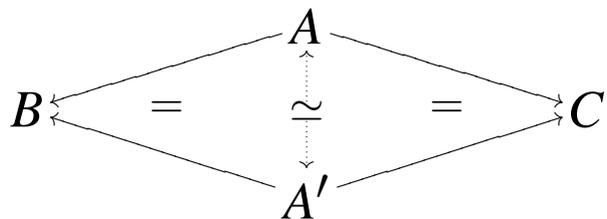


Sequential composition via pullback:



(In **Set**, $P = \{(a, d) \in A \times D \mid g(a) = h(d)\}$, with p_1 and p_2 the obvious projections)

we consider supports taken up-to-iso



Motivation

In recent years, (co)spans used for:

- **system specification;**
- **graph rewriting;**
- **predicate transformers;**
- **semantical domain for partial and multi-algebras;**
- **from the Sw Arch. point of view:**
 - abstract sw modules = programs with interfaces;
 - hidden support, plug-in composition.

⇒ **(co)spans as a general purpose framework**

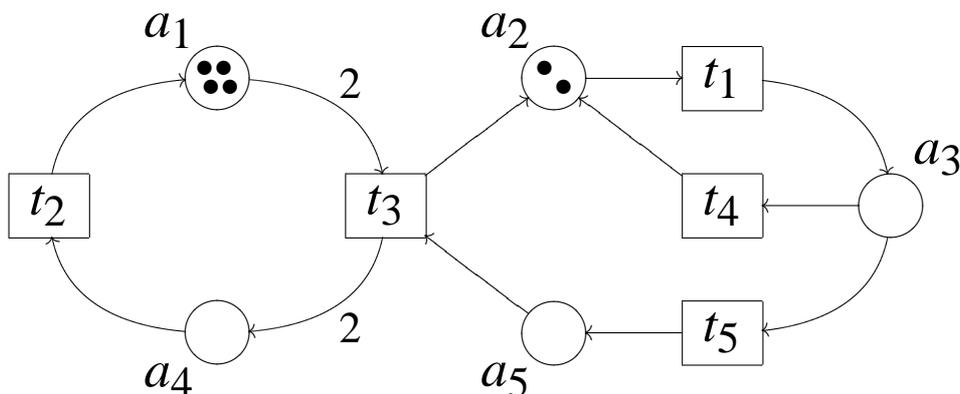
Petri nets

Well-known model for concurrent/distributed systems

Terminology and notation: places \circ , transitions \square , weighted arcs \xrightarrow{n} , tokens \bullet , markings, firings, steps

Key concepts: multiset rewriting, (non)deterministic processes, event structures

Issues: concurrency, causality, conflict, deadlock-freedom, liveness, boundedness, reachability, coverability, place/transition invariants

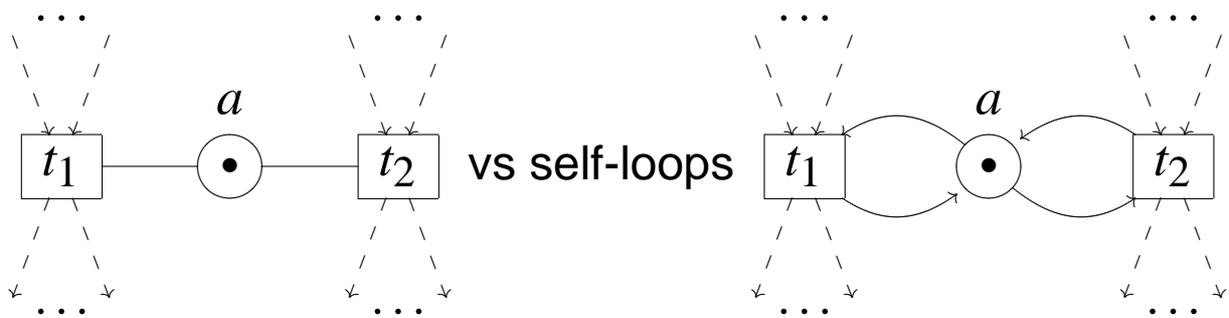


Read arcs

Multiple accesses (in reading) to the same resource

Terminology and notation: read arcs $\circ \xrightarrow{n} \square$,

Key concepts: concurrent reading, contextual processes, asymmetric event structures



concurrent firing

vs

sequentialization

Algebraic models for processes

Petri nets: (symmetric) monoidal categories [MM90]

$$\langle \mathcal{C}, -; -, - \otimes -, e, \gamma \rangle$$

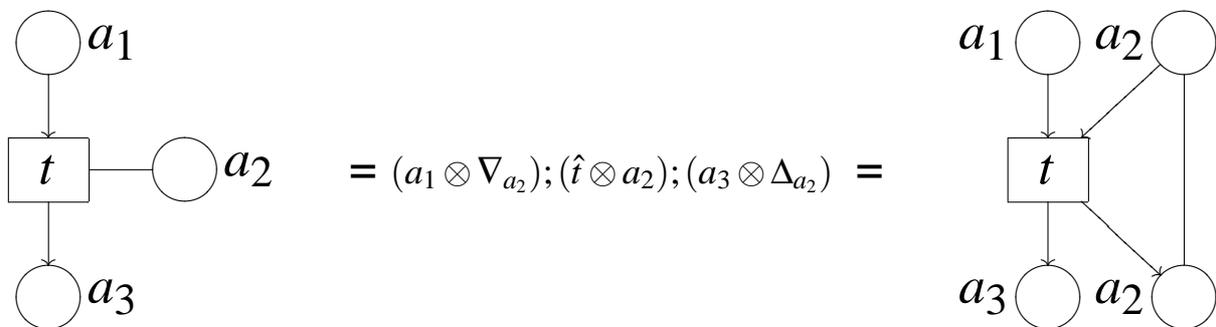
markings are objects, processes are arrows, process concatenation is $-; -$, parallel composition is $- \otimes -$, the empty marking is the unit e , token permutations are expressed via symmetries γ

key law: $(p_1; p_2) \otimes (q_1; q_2) = (p_1 \otimes q_1); (p_2 \otimes q_2)$

(functoriality of tensor product)

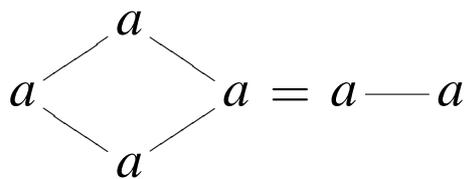
Read arcs: match-share categories [GM98]

$$\langle \mathcal{C}, -; -, - \otimes -, 0, \nabla, \gamma, \Delta \rangle$$

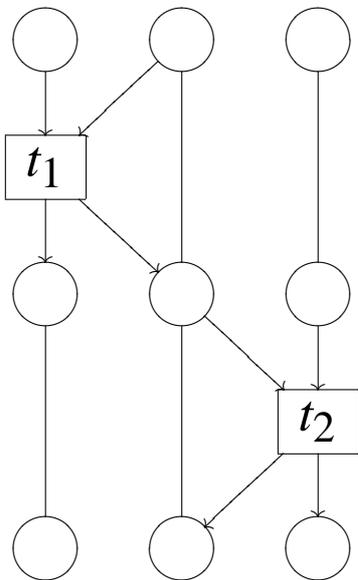
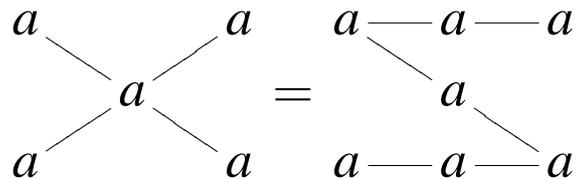


Key laws for contextual processes

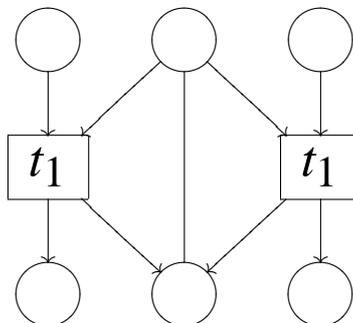
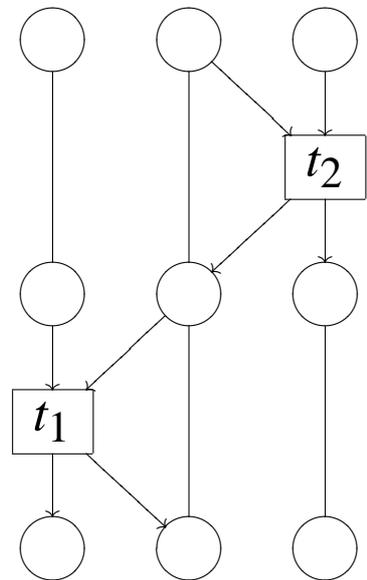
$$\nabla_a; \Delta_a = a$$



$$\Delta_a; \nabla_a = (\nabla_a \otimes a); (a \otimes \Delta_a)$$



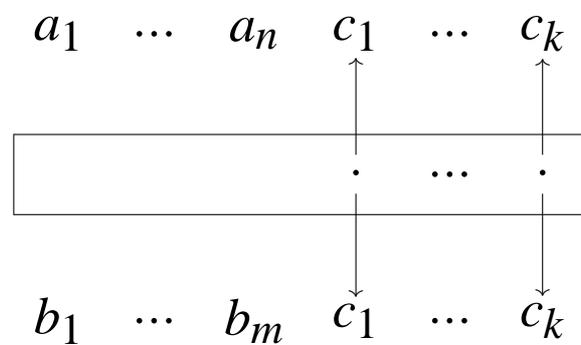
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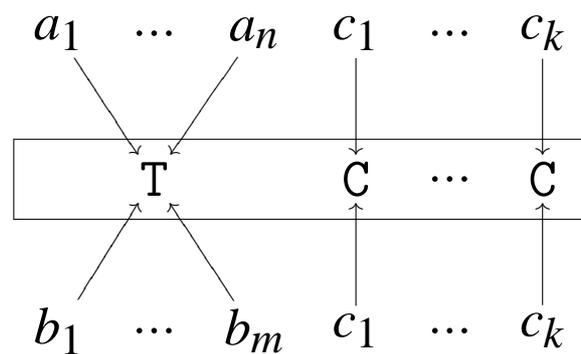
Visual representation

$$pre(t) = \bigoplus_i a_i \qquad post(t) = \bigoplus_i b_i \qquad ctx(t) = \bigoplus_i c_i$$

THE SPAN VIEW

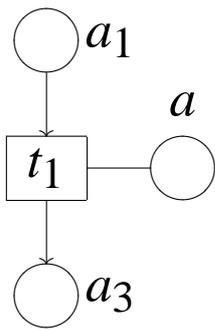
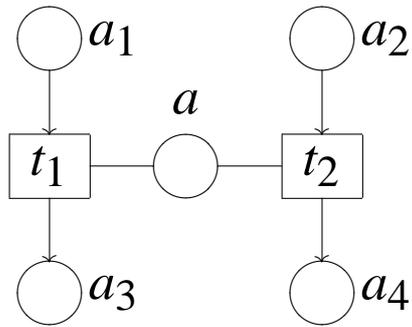


THE COSPAN VIEW

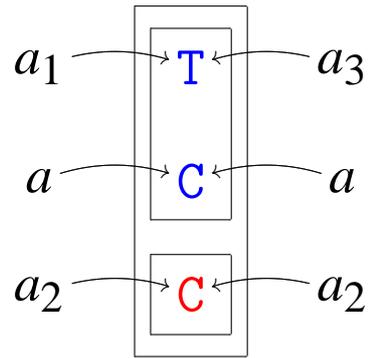
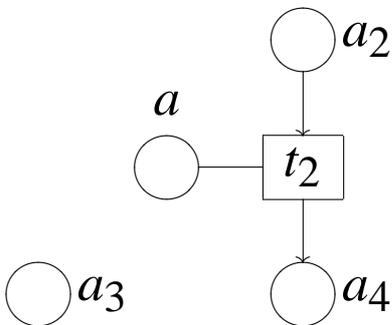


$C \leq T$, taking the sup when composing via pushout

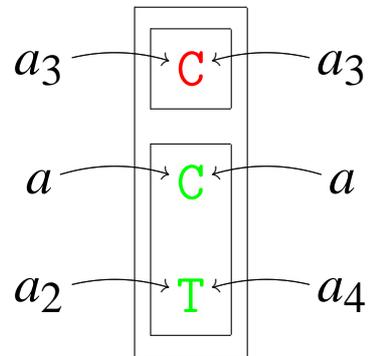
A closer look at the example



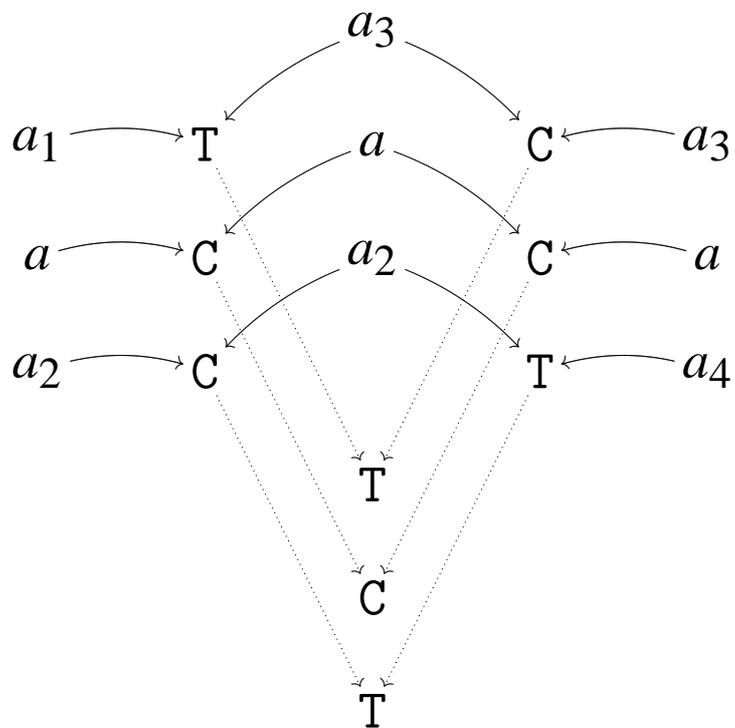
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Putting things together



Different flavors of monoidal categories

INGREDIENTS

symmetries: $\gamma_{a,b}: a \otimes b \rightarrow b \otimes a$

natural transformation plus coherence axioms

$$\begin{aligned} \gamma_{a \otimes b, c} &= (a \otimes \gamma_{b,c}); (\gamma_{a,c} \otimes b) & \gamma_{a,b}; \gamma_{b,a} &= a \otimes b \\ (t_1 \otimes t_2); \gamma_{b_1, b_2} &= \gamma_{a_1, a_2}; (t_2 \otimes t_1) & & \text{(for } t_1: a_1 \rightarrow b_1, t_2: a_2 \rightarrow b_2) \end{aligned}$$

duplicators, dischargers: $\nabla_a: a \rightarrow a \otimes a, !_a: a \rightarrow e$

coherence axioms (but not naturality!)

$$\begin{aligned} \nabla_{a \otimes b} &= (\nabla_a \otimes \nabla_b); (a \otimes \gamma_{a,b} \otimes b) & !_{a \otimes b} &= !_a \otimes !_b & \nabla_e &= !_e = id_e \\ \nabla_a; (\nabla_a \otimes a) &= \nabla_a; (a \otimes \nabla_a) & \nabla_a; \gamma_{a,a} &= \nabla_a & \nabla_a; (!_a \otimes a) &= a \end{aligned}$$

$$t; \nabla_b = \nabla_a; (t \otimes t) \qquad t; !_b = !_a \qquad \text{(for } t: a \rightarrow b)$$

coduplicators, codischargers: $\Delta_a: a \otimes a \rightarrow a, !_a: e \rightarrow a$

coherence axioms (but not naturality!)

The interplay between dual transformations

EIGHT LAWS FROM THE LITERATURE (e.g. [CV87,CS91])

$$\nabla_a; \Delta_a = a \quad (1)$$

$$\Delta_a; \nabla_a = (\nabla_a \otimes \nabla_a); (a \otimes \gamma_{a,a} \otimes a); (\Delta_a \otimes \Delta_a) \quad (2)$$

$$\Delta_a; \nabla_a = (\nabla_a \otimes a); (a \otimes \Delta_a) \quad (3)$$

$$\Delta_a; \nabla_a = (a \otimes \nabla_a); (\Delta_a \otimes a) \quad (4)$$

$$i_a; !a = e \quad (5)$$

$$!a; i_a = a \quad (6)$$

$$i_a; \nabla_a = i_a \otimes i_a \quad (7)$$

$$\Delta_a; !a = !a \otimes !a \quad (8)$$

Span(Set) and disjoint union

$\langle \mathbf{Span}(\mathbf{Set}), \oplus \rangle$

$e = \emptyset$

$$A \oplus B = A \uplus B = \{(0, a) \mid a \in A\} \cup \{(1, b) \mid b \in B\}$$

$$\gamma_{A,B}^{\oplus} = A \uplus B \xleftarrow{id_{A \uplus B}} A \uplus B \xrightarrow{\chi_{A,B}} B \uplus A \quad \chi_{A,B}(i, x) = (i+1 \bmod 2, x)$$

$$\nabla_A^{\oplus} = A \xleftarrow{\tau_A} A \uplus A \xrightarrow{id_{A \uplus A}} A \uplus A \quad \tau_A(i, a) = a$$

$$!_A^{\oplus} = A \xleftarrow{\phi_A} \emptyset \xrightarrow{id_{\emptyset}} \emptyset \quad \emptyset \text{ is initial in } \mathbf{Set}$$

$$\Delta_A^{\oplus} = A \uplus A \xleftarrow{id_{A \uplus A}} A \uplus A \xrightarrow{\tau_A} A \quad \text{duality considerations}$$

$$i_A^{\oplus} = \emptyset \xleftarrow{id_{\emptyset}} \emptyset \xrightarrow{\phi_A} A \quad \text{duality considerations}$$

Naturality holds!

Proposition 1 $\langle \mathbf{Span}(\mathbf{Set}), \oplus \rangle$ *is (co)cartesian.*

Valid interplay laws: **(2), (5), (7), (8)** but not **(1)!**

Span(Set) and multirelations

SPANS ARE MORE CONCRETE THAN RELATIONS

$\mathcal{R} : \mathbf{Span}(\mathbf{Set}) \rightarrow \mathbf{Rel}$ obvious full functor

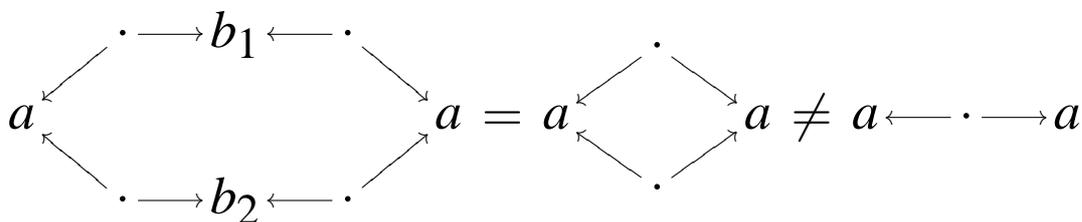
- it is identity on objects
- it maps $\langle f, g \rangle : B \leftarrow A \rightarrow C$ to

$$\mathcal{R}(\langle f, g \rangle) = \{(b, c) \in B \times C \mid \exists a \in A. f(a) = b \wedge g(a) = c\}$$

Proposition 2 *No lluf functor $\mathcal{S} : \mathbf{Rel} \rightarrow \mathbf{Span}(\mathbf{Set})$ exists such that $\mathcal{S};\mathcal{R} = 1$.*

A counterexample based on law (1):

$$\{(a, b_1), (a, b_2)\}; \{(b_1, a), (b_2, a)\} = \{(a, a)\}$$



Proposition 3 *The categories $\mathbf{Span}(\mathbf{Set})$ and \mathbf{MRel} are equivalent.*

Span(Set) and cartesian product

$\langle \text{Span}(\text{Set}), \otimes \rangle$

$$A \otimes B = A \times B = \{(a, b) \mid a \in A, b \in B\} \quad e = 1 = \{\bullet\}$$

$$\gamma_{A,B}^{\otimes} = A \times B \xleftarrow{id_{A \times B}} A \times B \xrightarrow{X_{A,B}} B \times A \quad X_{A,B}(a, b) = (b, a)$$

$$\nabla_A^{\otimes} = A \xleftarrow{id_A} A \xrightarrow{\nabla_A} A \times A \quad \nabla_A(a) = (a, a)$$

$$!_A^{\otimes} = A \xleftarrow{id_A} A \xrightarrow{!_A} 1 \quad 1 \text{ is final in } \mathbf{Set}: !_A(a) = \bullet$$

$$\Delta_A^{\otimes} = A \times A \xleftarrow{\nabla_A} A \xrightarrow{id_A} A \quad \text{duality considerations}$$

$$i_A^{\otimes} = 1 \xleftarrow{!_A} A \xrightarrow{id_A} A \quad \text{duality considerations}$$

Naturality does not hold (except for γ^{\otimes})

Valid interplay laws: **(1), (2), (3), (4)**

CoSpan(Set) and disjoint union

$\langle \mathbf{CoSpan}(\mathbf{Set}), \oplus \rangle$

$$\bar{\gamma}_{A,B}^{\oplus} = A \uplus B \xrightarrow{id_{A \uplus B}} A \uplus B \xleftarrow{\chi_{B,A}} B \uplus A$$

$$\bar{\nabla}_A^{\oplus} = A \xrightarrow{id_A} A \xleftarrow{\tau_A} A \uplus A$$

$$\bar{!}_A^{\oplus} = A \xrightarrow{id_A} A \xleftarrow{\phi_A} \emptyset$$

$$\bar{\Delta}_A^{\oplus} = A \uplus A \xrightarrow{\tau_A} A \xleftarrow{id_A} A$$

$$\bar{i}_A^{\oplus} = \emptyset \xrightarrow{\phi_A} A \xleftarrow{id_A} A$$

Naturality does not hold (except for $\bar{\gamma}^{\oplus}$)

Valid interplay laws: **(1), (2), (3), (4)**

Note that $\langle \mathbf{CoSpan}(\mathbf{Set}), \otimes \rangle$ is just pre-monoidal

CoSpan(Set) and difunctionality

partition $e: A \rightarrow B$ = reflexive, transitive and symmetric relation over $A \uplus B$.

redundant partition $e_p: A \rightarrow B = (e, n_e)$, for partition $e: A \rightarrow B$ and $n_e \in \mathbb{N}^\omega$.

$\mathcal{P}: \mathbf{CoSpan}(\mathbf{Set}) \rightarrow \mathbf{RERel}$ obvious full functor

- it is identity on objects
- it maps $[f, g]: B \rightarrow A \leftarrow C$ to $\mathcal{P}([f, g]) = \{P_1, \dots, P_n \mid n = |f(B) \cup g(C)|\}$ such that $d \in P_i$ iff, given a total ordering x_1, \dots, x_n, \dots over $f(B) \cup g(C)$, then either $d \in B$ and $f(B) = x_i$, or $d \in C$ and $g(C) = x_i$.

Proposition 4 *The categories $\mathbf{CoSpan}(\mathbf{Set})$ and \mathbf{RERel} are equivalent.*

Proposition 5 *Difunctionality yields a characterizable subcategory of $\mathbf{CoSpan}(\mathbf{Set})$.*

Relational properties

Connections with

- **pre-tabular allegories** [FS90]
- **Hopf algebroids** [DS87]

In fact each hom-set has a rich structure, e.g.

$$\begin{aligned} s, t & : A \leftrightarrow B \\ s \cap t & = \nabla_A; (s \oplus t); \Delta_A \\ s \smile & = (i_A; \nabla_A \oplus id_B); (id_A \oplus s \oplus id_B); (id_A \oplus \Delta_B; !_B) \end{aligned}$$

We did not consider any ordering, however...

$$A \perp\!\!\!\perp B = A \xrightarrow{!_A} 1 \xleftarrow{!_B} B$$

but no complement!

Back to Petri nets

SPANS FOR RESOURCE PERSISTENCE

From computation σ to span $\rho_\sigma: U \xleftarrow{f} W \xrightarrow{g} V$ (mimicking parallel and sequential composition)

Claim 1 *The elements $f(a) \in U$ (images of some $a \in W$) are read, but never consumed, resources in σ .*

Moreover, for each $a \in W$, its images $f(a)$ and $g(a)$ represent the same resource in σ .

COSPANS FOR CAUSALITY THREADS

From computation σ to cospan $\eta_\sigma: U \xrightarrow{f} W \xleftarrow{g} V$

For cospans, initial model semantics can be exploited!

Claim 2 *Given $a \in U$ and $b \in V$ with $f(a) = g(b)$, then the resource a has been fetched in σ for producing b .*

Moreover, if $\tau(f(a)) = C$, then a and b model the same contextual (or idle) token.

The overall picture

	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	nat. dup.	nat. dis.
Span (Set), \otimes	+	+	+	+	$ A \leq 1$	$ A \leq 1$	$A \simeq 1$	$A \simeq 1$	f mono	f iso
Span (Set), \oplus	$A = \emptyset$	+	$A = \emptyset$	$A = \emptyset$	+	$A = \emptyset$	+	+	+	+
CoSpan (Set), \oplus	+	+	+	+	$A = \emptyset$	$A = \emptyset$	$A = \emptyset$	$A = \emptyset$	f epi	f iso
Rel , \otimes	+	+	+	+	$ A \leq 1$	$ A \leq 1$	$A \simeq 1$	$A \simeq 1$	f mono	f iso
Rel , \oplus	+	+	$A = \emptyset$	$A = \emptyset$	+	$A = \emptyset$	+	+	+	+
ERel , \oplus	+	+	+	+	$A = \emptyset$	$A = \emptyset$	$A = \emptyset$	$A = \emptyset$	f epi	f iso

(conditions refer to generic (co)spans with support A and legs f and g)

Conclusion and future work

Similarity between $\langle \text{Span}(\mathbf{Set}), \otimes \rangle$ and $\langle \text{CoSpan}(\mathbf{Set}), \oplus \rangle$

Not too surprising, given the duality in their definition

Striking different behavior over the alternative structures

$\langle \text{Span}(\mathbf{Set}), \oplus \rangle$ is (co)cartesian, while $\langle \text{CoSpan}(\mathbf{Set}), \otimes \rangle$ is just pre-monoidal

$\langle \text{CoSpan}(\mathbf{Set}), \otimes \rangle$ is one of the few “natural” examples of pre-monoidal categories

Clarification of several analogies and differences between several flavors of “categories of relations” one can easily generalize the notion of *difunctionality* to cospans: it is then preserved by composition, yielding a subcategory

Further investigate the intuitive (2-dimensional) ordering over (co)spans This is a relevant topic, both semantically (e.g. predicate transformer [GMM92]); and syntactically, (e.g. direct product in relational algebras[BHSV94])

Can our taxonomy be generalized to (complete and co-complete) categories other than **Set**? We are thinking in particular of **Graph**, used for rewriting systems [CG99,GHL99] and automata [KSW97]