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Some algebraic laws for spans

(and their connections with multirelations)

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Plan of the talk

- **A few words on motivation**

  i.e. what spans can be used for

- **An application to Petri nets with read arcs**

  i.e. an abstract domain for software architectures

- **Span(Set), (multi)relations, and some algebraic properties**

  i.e. a closer look to spans

- **CoSpan(Set), difunctionality, and some algebraic properties**

  i.e. a closer look to cospans

- **Causality and concurrency via (co)spans**

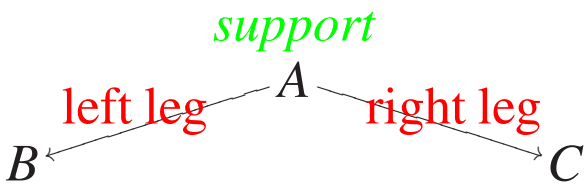
  i.e. back to Petri nets

- **Summary and conclusion**

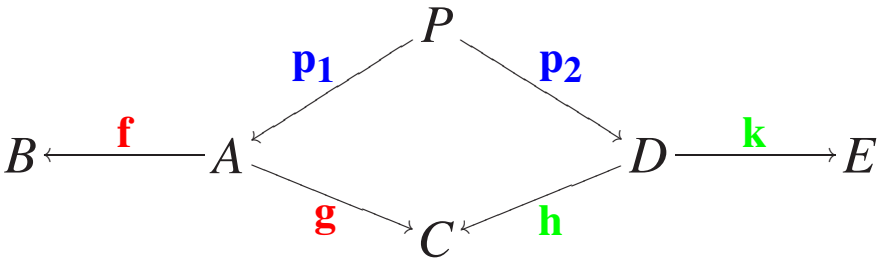
  i.e. a tableau of algebraic laws

Spans

Scenario: A category $\mathcal{C}$ with pullbacks

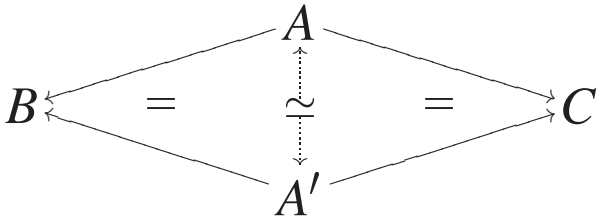


Sequential composition via pullback:



(In **Set**, $P = \{(a,d) \in A \times D \mid g(a) = h(d)\}$, with p_1 and p_2 the obvious projections)

we consider supports taken up-to-iso



Motivation

In recent years, (co)spans used for:

- **system specification**;
- **graph rewriting**;
- **predicate transformers**;
- **semantical domain for partial and multi-algebras**;
- **from the Sw Arch. point of view:**
  - abstract sw modules = programs with interfaces;
  - hidden support, plug-in composition.

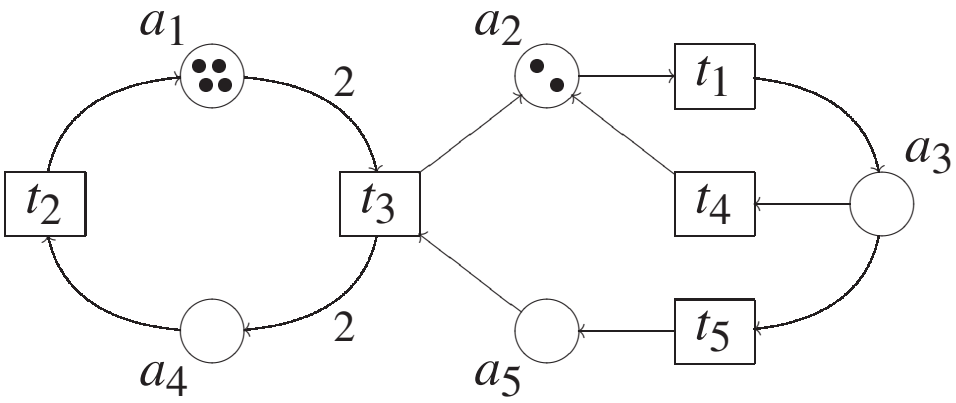
$\Rightarrow$ **(co)spans as a general purpose framework**

Well-known model for concurrent/distributed systems

Terminology and notation: places ○, transitions □, weighted arcs $\xrightarrow{n}$, tokens •, markings, firings, steps

Key concepts: multiset rewriting, (non)deterministic processes, event structures

Issues: concurrency, causality, conflict, deadlock-freedom, liveness, boundedness, reachability, coverability, place/transition invariants



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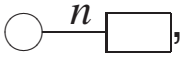
Read arcs

Multiple accesses (in reading) to the same resource

Terminology and notation: read arcs 

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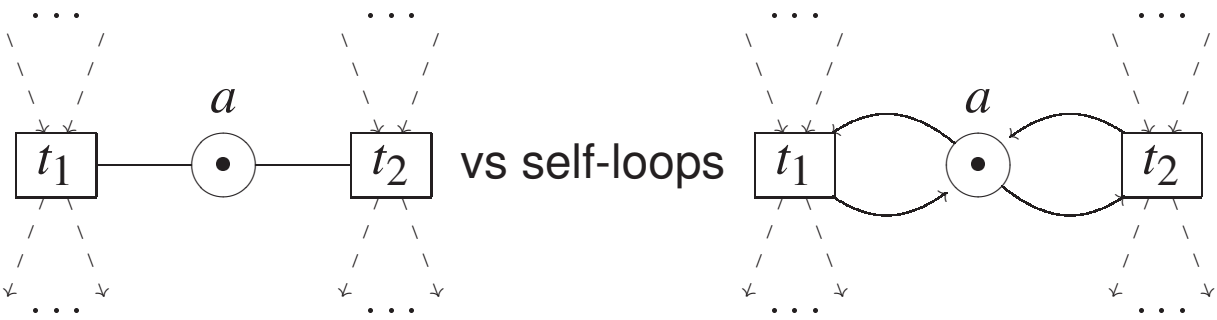
Key concepts: concurrent reading, contextual processes, asymmetric event structures



concurrent firing vs sequentialization

Algebraic models for processes

Petri nets: (symmetric) monoidal categories [MM90]

$$\langle \mathcal{C}, _;_, _ \otimes _, e, \gamma \rangle$$

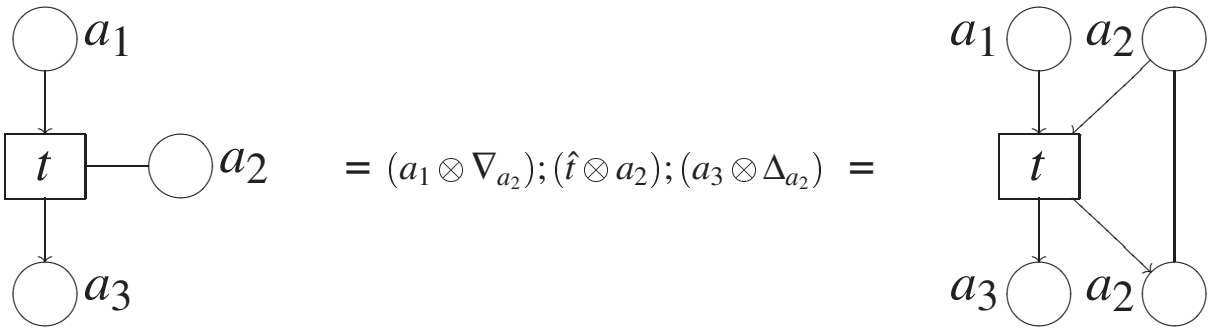
markings are objects, processes are arrows, process concatenation is $_;_$, parallel composition is $_ \otimes _$, the empty marking is the unit e, token permutations are expressed via symmetries γ

key law: $(p_1; p_2) \otimes (q_1; q_2) = (p_1 \otimes q_1); (p_2 \otimes q_2)$

(functoriality of tensor product)

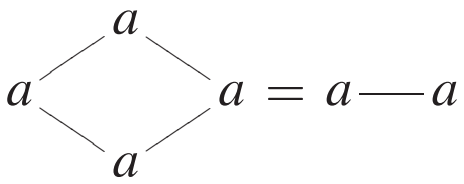
Read arcs: match-share categories [GM98]

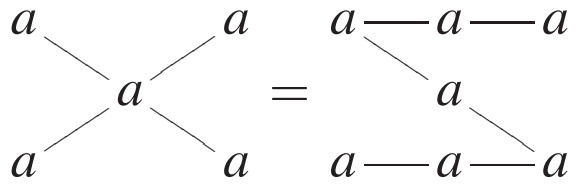
$$\langle \mathcal{C}, _;_, _ \otimes _, 0, \nabla, \gamma, \Delta \rangle$$

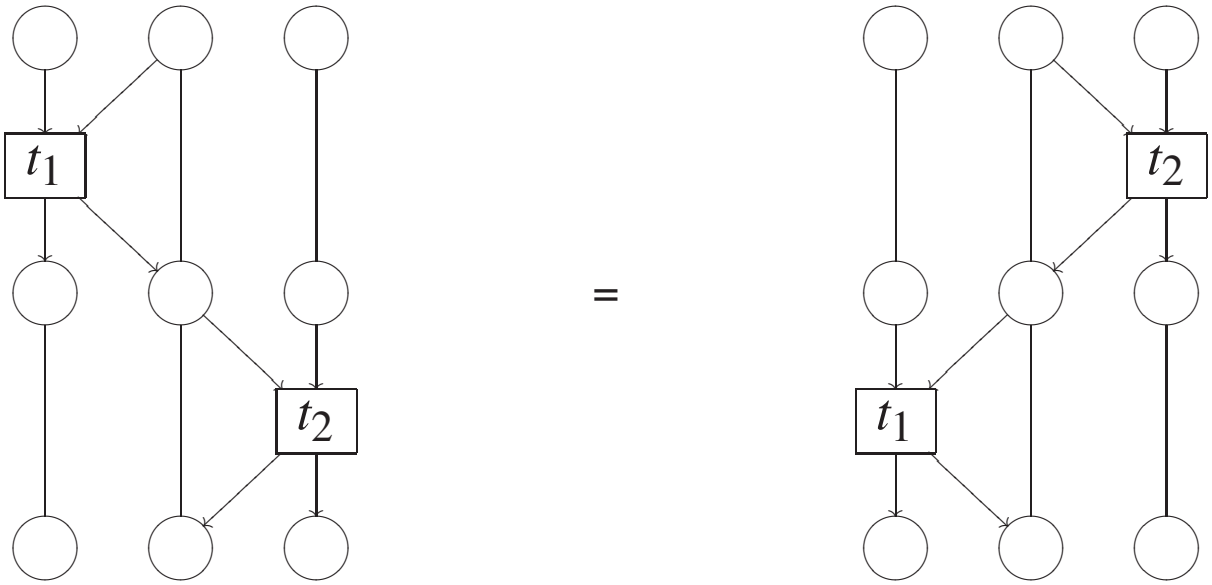


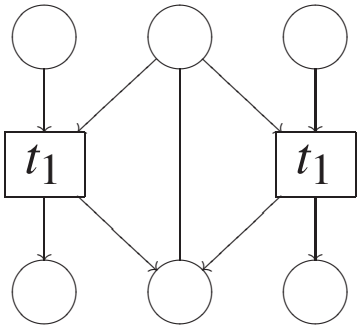
Key laws for contextual processes

$$\nabla_a;\Delta_a = a \qquad\qquad \Delta_a;\nabla_a = (\nabla_a \otimes a);(a \otimes \Delta_a)$$









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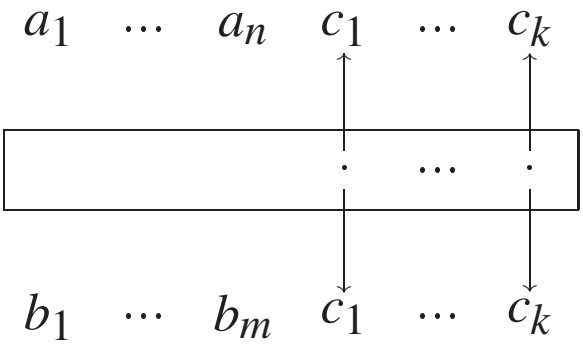
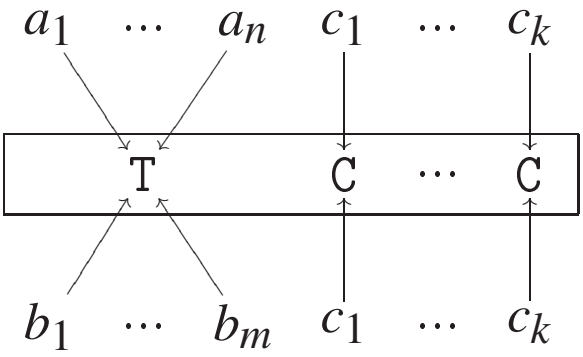
Visual representation

$pre(t) = \bigoplus_i a_i$ $\quad$ $post(t) = \bigoplus_i b_i$ $\quad$ $ctx(t) = \bigoplus_i c_i$

THE SPAN VIEW

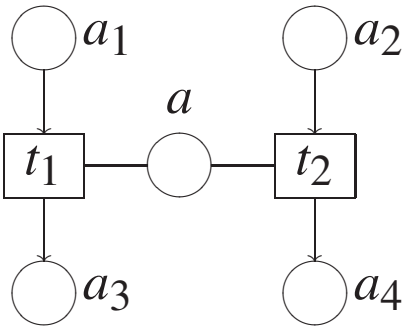


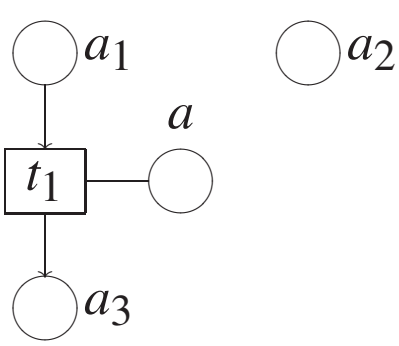
THE COSPAN VIEW



$\mathtt{C} \leq \mathtt{T}$, taking the sup when composing via pushout

A closer look at the example





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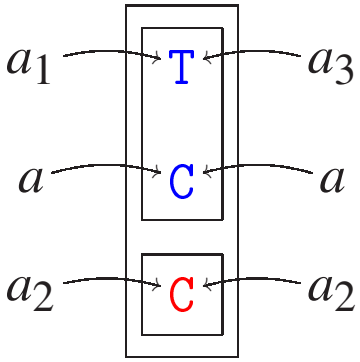

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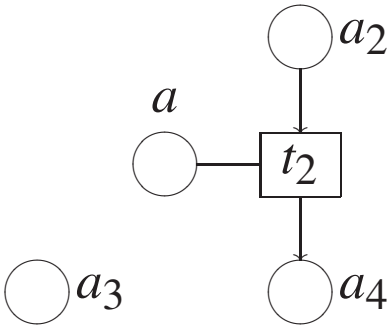
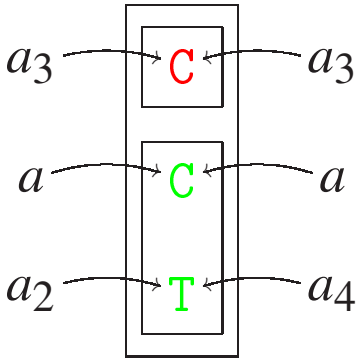
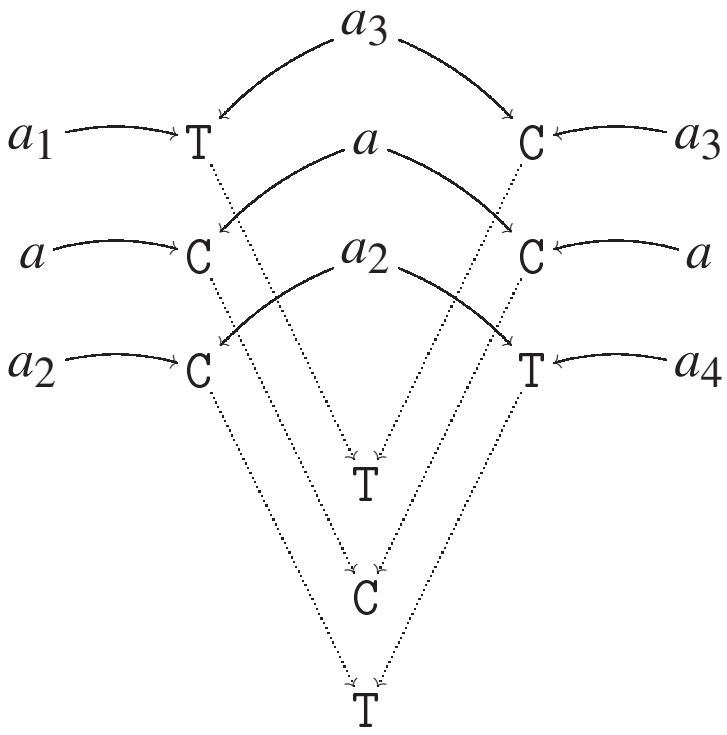




Putting things together



Different flavors of monoidal categories

INGREDIENTS

symmetries: $\gamma_{a,b}: a \otimes b \to b \otimes a$

natural transformation plus coherence axioms

$\gamma_{a\otimes b,c} = (a \otimes \gamma_{b,c}); (\gamma_{a,c} \otimes b)$ $\qquad$ $\gamma_{a,b}; \gamma_{b,a} = a \otimes b$

$(t_1 \otimes t_2); \gamma_{b_1,b_2} = \gamma_{a_1,a_2}; (t_2 \otimes t_1)$ $\qquad$ (for $t_1: a_1 \to b_1$, $t_2: a_2 \to b_2$)

duplicators, dischargers: $\nabla_a: a \to a \otimes a$, $!_a: a \to e$

coherence axioms (but not naturality!)

$\nabla_{a\otimes b} = (\nabla_a \otimes \nabla_b); (a \otimes \gamma_{a,b} \otimes b)$ $\qquad$ $!_{a\otimes b} = !_a \otimes !_b$ $\qquad$ $\nabla_e = !_e = id_e$

$\nabla_a; (\nabla_a \otimes a) = \nabla_a; (a \otimes \nabla_a)$ $\qquad$ $\nabla_a; \gamma_{a,a} = \nabla_a$ $\qquad$ $\nabla_a; (!_a \otimes a) = a$

$t; \nabla_b = \nabla_a; (t \otimes t)$ $\qquad$ $t; !_b = !_a$ $\qquad$ (for $t: a \to b$)

coduplicators, codischargers: $\Delta_a: a \otimes a \to a$, $¡_a: e \to a$

coherence axioms (but not naturality!)

The interplay between dual transformations

EIGHT LAWS FROM THE LITERATURE (e.g. [CV87,CS91]

$$\nabla_a;\Delta_a = a \quad (1)$$

$$\Delta_a;\nabla_a = (\nabla_a \otimes \nabla_a);(a \otimes \gamma_{a,a} \otimes a);(\Delta_a \otimes \Delta_a) \quad (2)$$

$$\Delta_a;\nabla_a = (\nabla_a \otimes a);(a \otimes \Delta_a) \quad (3)$$

$$\Delta_a;\nabla_a = (a \otimes \nabla_a);(\Delta_a \otimes a) \quad (4)$$

$$¡_a;!_a = e \quad (5)$$

$$!_a;¡_a = a \quad (6)$$

$$¡_a;\nabla_a = ¡_a \otimes ¡_a \quad (7)$$

$$\Delta_a;!_a = !_a \otimes !_a \quad (8)$$

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Span(Set) and disjoint union

$\langle \mathbf{Span}(\mathbf{Set}), \oplus \rangle$ $\qquad e = \varnothing$

$$A \oplus B = A \uplus B = \{(0,a) \mid a \in A\} \cup \{(1,b) \mid b \in B\}$$

$$\gamma^{\oplus}_{A,B} = A \uplus B \xleftarrow{id_{A \uplus B}} A \uplus B \xrightarrow{\chi_{A,B}} B \uplus A \qquad \chi_{A,B}(i,x) = (i+1 \bmod 2, x)$$

$$\nabla^{\oplus}_{A} = A \xleftarrow{\tau_A} A \uplus A \xrightarrow{id_{A \uplus A}} A \uplus A \qquad \tau_A(i,a) = a$$

$$!^{\oplus}_{A} = A \xleftarrow{\phi_A} \varnothing \xrightarrow{id_{\varnothing}} \varnothing \qquad \varnothing \text{ is initial in } \mathbf{Set}$$

$$\Delta^{\oplus}_{A} = A \uplus A \xleftarrow{id_{A \uplus A}} A \uplus A \xrightarrow{\tau_A} A \qquad \text{duality considerations}$$

$$¡^{\oplus}_{A} = \varnothing \xleftarrow{id_{\varnothing}} \varnothing \xrightarrow{\phi_A} A \qquad \text{duality considerations}$$

Naturality holds!

Proposition 1 $\langle \mathbf{Span}(\mathbf{Set}), \oplus \rangle$ ***is (co)cartesian.***

Valid interplay laws: **(2), (5), (7), (8)** but not **(1)!**

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Span(Set) and multirelations

SPANS ARE MORE CONCRETE THAN RELATIONS

$\mathscr{R}$: **Span(Set)** $\to$ **Rel** obvious full functor

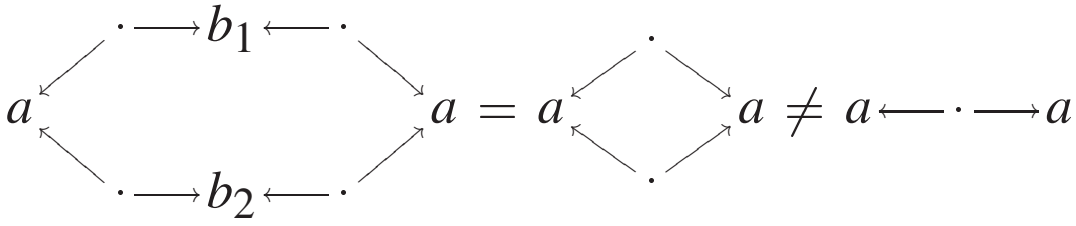
- it is identity on objects
- it maps $\langle f,g\rangle : B \leftarrow A \rightarrow C$ to

$$\mathscr{R}(\langle f,g\rangle) = \{(b,c) \in B \times C \mid \exists a \in A.\, f(a) = b \wedge g(a) = c\}$$

Proposition 2 *No lluf functor $\mathscr{S}$: **Rel** $\to$ Span(**Set**) exists such that $\mathscr{S};\mathscr{R} = 1$.*

A counteraxample based on law (1):

$$\{(a,b_1),(a,b_2)\};\{(b_1,a),(b_2,a)\} = \{(a,a)\}$$



Proposition 3 *The categories* Span(**Set**) *and* **MRel** *are equivalent.*

Span(Set) and cartesian product

$\langle \mathbf{Span(Set)}, \otimes \rangle$

$A \otimes B = A \times B = \{(a,b) \mid a \in A, b \in B\} \qquad e = 1 = \{\bullet\}$

$\gamma^{\otimes}_{A,B} = A \times B \xleftarrow{id_{A\times B}} A \times B \xrightarrow{\mathsf{X}_{A,B}} B \times A \qquad \mathsf{X}_{A,B}(a,b) = (b,a)$

$\nabla^{\otimes}_{A} = A \xleftarrow{id_A} A \xrightarrow{\nabla_A} A \times A \qquad \nabla_A(a) = (a,a)$

$!^{\otimes}_{A} = A \xleftarrow{id_A} A \xrightarrow{!_A} 1$ — 1 is final in **Set**: $!_A(a) = \bullet$

$\Delta^{\otimes}_{A} = A \times A \xleftarrow{\nabla_A} A \xrightarrow{id_A} A$ — duality considerations

$¡^{\otimes}_{A} = 1 \xleftarrow{!_A} A \xrightarrow{id_A} A$ — duality considerations

Naturality does not hold (except for $\gamma^{\otimes}$)

Valid interplay laws: **(1), (2), (3), (4)**

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CoSpan(Set) and disjoint union

$\langle \mathbf{CoSpan}(\mathbf{Set}), \oplus \rangle$

$\overline{\gamma}^{\oplus}_{A,B} = A \uplus B \xrightarrow{id_{A \uplus B}} A \uplus B \xleftarrow{\chi_{B,A}} B \uplus A$

$\overline{\nabla}^{\oplus}_{A} = A \xrightarrow{id_A} A \xleftarrow{\tau_A} A \uplus A$

$\overline{!}^{\oplus}_{A} = A \xrightarrow{id_A} A \xleftarrow{\phi_A} \varnothing$

$\overline{\Delta}^{\oplus}_{A} = A \uplus A \xrightarrow{\tau_A} A \xleftarrow{id_A} A$

$\overline{¡}^{\oplus}_{A} = \varnothing \xrightarrow{\phi_A} A \xleftarrow{id_A} A$

Naturality does not hold (except for $\overline{\gamma}^{\oplus}$)

Valid interplay laws: **(1), (2), (3), (4)**

Note that $\langle \mathbf{CoSpan}(\mathbf{Set}), \otimes \rangle$ **is just pre-monoidal**

CoSpan(Set) and difunctionality

partition $e : A \to B$ = reflexive, transitive and symmetric relation over $A \uplus B$.

redundant partition $e_p : A \to B$ = (e, n_e), for partition $e : A \to B$ and $n_e \in \mathbb{N}^{\omega}$.

$\mathscr{P}$: **CoSpan(Set)** $\to$ **RERel** obvious full functor

- it is identity on objects
- it maps $[f,g] : B \to A \leftarrow C$ to $\mathscr{P}([f,g]) = \{P_1, \ldots, P_n \mid n = |f(B) \cup g(C)|\}$ such that $d \in P_i$ iff, given a total ordering $x_1, \ldots, x_n, \ldots$ over $f(B) \cup g(C)$, then either $d \in B$ and $f(B) = x_i$, or $d \in C$ and $g(C) = x_i$.

Proposition 4 ***The categories* CoSpan(Set) *and* RERel *are equivalent.***

Proposition 5 ***Difunctionality yields a characterizable subcategory of CoSpan(Set).***

Relational properties

Connections with

- **pre-tabular allegories** [FS90]
- **Hopf algebroids** [DS87]

In fact each hom-set has a rich structure, e.g.

$$\begin{array}{rcl} s,t & : & A \leftrightarrow B \\ s \cap t & = & \nabla_A ; (s \oplus t) ; \Delta_A \\ s^{\smile} & = & (\mathsf{i}_A ; \nabla_A \oplus id_B) ; (id_A \oplus s \oplus id_B) ; (id_A \oplus \Delta_B ; !_B) \end{array}$$

We did not consider any ordering, however...

$$A \perp\!\!\!\perp B \quad = \quad A \xrightarrow{!_A} 1 \xleftarrow{!_B} B$$

but no complement!

Back to Petri nets

SPANS FOR RESOURCE PERSISTENCE

From computation σ to span $\rho_\sigma: U \xleftarrow{f} W \xrightarrow{g} V$ (mimicking parallel and sequential composition)

Claim 1 *The elements $f(a) \in U$ (images of some $a \in W$) are read, but never consumed, resources in σ.*

Moreover, for each $a \in W$, its images $f(a)$ and $g(a)$ represent the same resource in σ.

COSPANS FOR CAUSALITY THREADS

From computation σ to cospan $\eta_\sigma: U \xrightarrow{f} W \xleftarrow{g} V$

For cospans, initial model semantics can be exploited!

Claim 2 *Given $a \in U$ and $b \in V$ with $f(a) = g(b)$, then the resource a has been fetched in σ for producing b.*

Moreover, if $\tau(f(a)) = \mathtt{C}$, then a and b model the same contextual (or idle) token.

The overall picture

	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	nat. dup.	nat. dis.
Span(**Set**), $\otimes$	+	+	+	+	$\lvert A \rvert \leq 1$	$\lvert A \rvert \leq 1$	$A \simeq 1$	$A \simeq 1$	f mono	f iso
Span(**Set**), $\oplus$	$A = \varnothing$	+	$A = \varnothing$	$A = \varnothing$	+	$A = \varnothing$	+	+	+	+
CoSpan(**Set**), $\oplus$	+	+	+	+	$A = \varnothing$	$A = \varnothing$	$A = \varnothing$	$A = \varnothing$	f epi	f iso
Rel, $\otimes$	+	+	+	+	$\lvert A \rvert \leq 1$	$\lvert A \rvert \leq 1$	$A \simeq 1$	$A \simeq 1$	f mono	f iso
Rel, $\oplus$	+	+	$A = \varnothing$	$A = \varnothing$	+	$A = \varnothing$	+	+	+	+
ERel, $\oplus$	+	+	+	+	$A = \varnothing$	$A = \varnothing$	$A = \varnothing$	$A = \varnothing$	f epi	f iso

(conditions refer to generic (co)spans with support A and legs f and g)

Conclusion and future work

Similarity between $\langle \mathrm{Span}(\mathbf{Set}), \otimes \rangle$ and $\langle \mathrm{CoSpan}(\mathbf{Set}), \oplus \rangle$

Not too surprising, given the duality in their definition

Striking different behavior over the alternative structures

$\langle \mathrm{Span}(\mathbf{Set}), \oplus \rangle$ is (co)cartesian, while $\langle \mathrm{CoSpan}(\mathbf{Set}), \otimes \rangle$ is just pre-monoidal

$\langle \mathrm{CoSpan}(\mathbf{Set}), \otimes \rangle$ is one of the few "natural" examples of pre-monoidal categories

Clarification of several analogies and differences between several flavors of "categories of relations" one can easily generalize the notion of *difunctionality* to cospans: it is then preserved by composition, yielding a subcategory

Further investigate the intuitive (2-dimensional) ordering over (co)spans This is a relevant topic, both semantically (e.g. predicate transformer [GMM92]); and syntactically, (e.g. direct product in relational algebras[BHSV94])

Can our taxonomy be generalized to (complete and co-complete) categories other than Set? We are thinking in particular of **Graph**, used for rewriting systems [CG99,GHL99] and automata [KSW97]