Categorizing Binary Topological Relations Between Regions, Lines, and Points in Geographic Databases*

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Abstract

One of the fundamental concepts necessary for the analysis of spatial data in a Geographic Information System (GIS) is a formal understanding of the geometric relationships among arbitrary spatial objects. Topological relations, a particular subset of geometric relations, are preserved under topological transformations such as translation, rotation, and scaling. A comprehensive formal categorization of such binary topological relations between regions, lines, and points has been developed that is based upon the comparison of the nine intersections between the interiors, boundaries, and exteriors of the two objects. The basic criterion for the distinction of different topological relations is whether the intersections are empty or not, thus identifying 2^9 mutually exclusive topological relations. It is derived which of these 512 binary relations actually exist in $\mathbb{R}^2$ between regions, lines, and points. An equivalent model is developed that replaces the intersections with exteriors by subset conditions of the closure so that efficient implementations of topological relations are possible in geographic information systems.

1 Introduction

Queries in spatial databases, such as Geographic Information Systems (GISs) [25, 48], image databases [7, 64], or CAD/CAM systems [63], are often based on the relationships among spatial objects. For example, in geographic applications typical spatial queries are, "Retrieve all cities that are within 5 miles of the interstate highway I-95" or, "Find all highways in the states adjacent to Maine." Current commercial database query languages, such as SQL [6] and Quel [67], do not sufficiently support such queries, because they provide only tools for comparing equality or order of such simple data types as integers or strings. The incorporation of spatial relations over geometric domains into a spatial query language has been identified as an essential extension beyond the power of traditional query languages [19, 64]. Some experimental spatial query languages support queries with one

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or the other spatial relationship (Table 1); however, their diversity, semantics, completeness, and terminology vary dramatically [16, 32].

	Spatial Relationships
[30]	left of, right of, beside, above, below, near, far, touching, between, inside, outside
ATLAS [70]	area adjacency, line adjacency, boundary relationship, containment, distance, direction
MAPQUERY [25]	on, adjacent, within
KBGIS [65]	containment, subset, neighborhood, near, far, north, south, east, west
KGIS [41]	distance, overlay, adjacent, overlap
PSQL [64]	covering, coveredBy, overlapping, disjoint, nearest, furthest, within, outside, on_perimeter
SQL extension [39]	adjacent, contains, contains_point, enclosed_by, intersect, near, self_intersect
Geo-Relational Algebra [34]	equal, not equal, inside, outside, intersect
Spatial SQL [16]	disjoint, equal, meet, overlap, concur, commonBounds

Table 1: Terms proposed or used for spatial relationships in query languages.

Spatial queries can be easily solved if *all* geometric relationships between the objects of interest are explicitly stored; however, such a scenario is unrealistic, even for relatively small data collections [12], because it would need tremendous amounts of storage space—n^2 values for each kind of spatial relationships between n objects—and imply complex maintenance procedures. For instance, a GIS that explicitly recorded the geographic directions between any two objects would require extensive update operations because, with the addition of any new object, one must also determine and subsequently store the corresponding direction values from the new object to all objects already known in the database, and vice-versa (i.e., $2n$ new entries for a database with n objects). In lieu of recording all spatial relationships, it is more common to derive them, e.g., from their geometry or spatial location. This process needs a thorough understanding of *what* possible geometric relationships are and *how* they can be determined.

The development of a coherent, mathematical theory of spatial relations to overcome shortcomings in almost all geographic applications [5] is one of the goals of current GIS research [1, 56]. A formal definition, for instance, is a prerequisite for the query execution in a compiler and for reasoning about the relationships among spatial objects. Its benefits will be threefold: (1) Such a formalism may serve as a tool to identify and derive relationships. Redundant and contradicting relationships can be avoided such that a minimal set of fundamental relationships can be defined. (2) The formal methods can be applied to determine the relationship between any two spatial objects given in a formal representation. Algorithms to determine relationships can be specified exactly, and mathematically sound models will help to define the relationships formally. (3) The fundamental relationships can be used to combine more complex relationships.

The exploration of spatial relationships is a multi-disciplinary effort. Cognitive scientists, psychologists, and linguists are interested in how humans perceive the inter-relationships between spatial objects and their studies focus on the use of spatial predicates and relations in natural language [40, 49, 69]. Cartographers and geographers collected terms and prototypes of spatial relations. An early compilation of primitive spatial relations [30] lacks a formal underpinning, but is close to a list [42] that is based on a cognitive linguistics approach [47].

The scope of this paper is to use formal methods for the identification of different topological relations, a particular subset of geometric relations. Their characteristic is that they are preserved under topological transformations such as translation, rotation, and scaling. Topological information is a purely qualitative property and excludes any consideration of quantitative measures. For example, two parcels are neighbors if they share a common boundary and the neighborhood relationship is independent of the length of the boundary or the number of common boundary segments. It is important to keep in mind that topological equivalence does not preserve distances and directions, which are spatial relations that are part of other investigations [8, 26, 36, 59]; therefore, the subsequent investigations are based upon continuity, which is described in terms of *coincidence* and *neighborhood*, and no reference to the notions of distance and direction will be made. Other spatial relations, excluded from the investigations in this paper, are approximate relations, such as *close* [62] and *about five miles north-easterly of* [13], or relations that are expressions about the motion of one or several objects such as *through* and *into* [69].

We concentrate on the geometry of the objects—regions, lines, and points—irrespective of their particular meanings. While certain spatial terms may be specific to particular applications, in general all spatial relations are based upon fundamental geometric principles and models. A consistent and least redundant approach requires that the common concepts are identified at the geometry level in the form of a fundamental set of spatial relations. These generic relationships can then be applied for the definition of application-specific relationships. Linguists' observations about the use of natural language terms for the description of spatial relations support this approach [40, 69]. In the English language, spatial relations and prepositions are independently used of the size and material of the reference objects, yet context in which a specific relationship occurs is essential for the selection of the correct terms.

The remainder of this paper is organized as follows: Section 2 summarizes the spatial data model, for which the topological relations will be investigated. Section 3 introduces the 9-intersection as our model to formalize binary topological relations. Their existence for regions, lines, and points in $\mathbb{R}^2$ is investigated in Section 4. In Section 5 our model is compared with other formalisms for spatial relations and the conclusions in Section 6 describe an implementation and discuss future research activities based on these results.

2 Spatial Data Model

In order to describe the kinds of spatial objects one deals with and to determine what their particular properties are, it is necessary to introduce a spatial data model. A spatial data model is a formalization of the spatial concepts that humans employ when they organize and structure their perception of space [24, 27]. These concepts differ depending on the observers' experiences and the context in which a person views some situation. Formalizations of spatial concepts are necessary, because computer systems are essentially formal systems that manipulate symbols according to formal rules. The role of a spatial data model is similar to the conceptual schema in the 3-schema view: concepts get separated from the actual implementations, thus implementations of certain parts of the large GIS software system become more independent and may be updated without affecting the remaining software parts.

Here, the formalism will primarily serve as a means to verify that the readers' assumptions and expectations about spatial concepts concur. Without such a formal framework it would be impossible to investigate and discuss the formalization of topological relations, because it may vary considerably depending on the data model selected.

2.1 Cells and Cell Complexes

The spatial data model, upon which the definition of topological relations is based, uses *algebraic topology* [3, 66], a branch of geometry deals with the algebraic manipulation of symbols that represent geometric configurations and their relationships to one another. The application of algebraic topology has been the subject of extensive research in geographic information systems [11, 73] and led to today's most common spatial data model in GISs for modeling discrete spatial data [24, 27], e.g., in Arc/Info [53] and TIGRIS [37], and a cartographic data transfer standard [57].

The algebraic-topology spatial data model is based on primitive geometric objects, called *cells*, which are defined for different spatial dimensions[1]: A 0-cell is a node (the minimal 0-dimensional object); a 1-cell is the link between two distinct 0-cells; and a 2-cell is the area described by a closed sequences of three non-intersecting 1-cells. A *face* of an n-cell A is any $(0 \ldots n)$-cell that is contained in A.

This spatial data model differs from the simplicial data model [20, 28] primarily in one property: simplices are convex hulls, while cells may have arbitrarily shaped interiors.

The topological primitives relevant for the forthcoming investigations are the closure, interior, boundary, and exterior of a cell.

Definition 1 *The closure of an n-cell A, denoted by $\overline{A}$, is the set of all faces r-f of A, where $0 \leq r \leq n$, i.e.,*

$$\overline{A} = \bigcup_{r=0}^{n} r\text{-}f \in A$$

Definition 2 *The set-theoretic boundary of an n-cell A, denoted by ∂A, is the union of all r-faces r-f, where $0 \leq r \leq (n-1)$, that are contained in A:*

$$\partial A = \bigcup_{r=0}^{n-1} r\text{-}f \in A$$

Definition 3 *The interior of a cell A, denoted by A°, is the set difference between A's closure and A's boundary:*

$$A^\circ = \overline{A} - \partial A$$

Definition 4 *The exterior of a cell A, denoted by A^-, is the set of all cells in the universe $\mathcal{U}$ that are not elements of the closure:*

$$A^- = \mathcal{U} - \overline{A}$$

From the elementary geometric objects, more complex ones can be formed as their aggregates, called *cell complexes*. The operations on cell complexes are defined in terms of the operations on cells. Let x be the number of cells $(A_1 \ldots A_x)$ that constitute a complex C.

[1]The definition of the topological dimension of a space is based on the concept of a refinement [55]. Examples of one-dimensional spaces are a line and the border of a circle; common two-dimensional spaces are the open and the closed disks, and their topological images. An n-cell has the same dimension n as its embedding space if the cell exists in that space, but there is no homeomorphic mapping for the cell into an $(n$–$1)$-space.

Definition 5 *The boundary of C is the set of all boundaries of the x n-cells A_i that constitute C and are part of a single A in C, i.e.,*

$$\partial C = (\bigcup_{i=1}^{x} \partial A_i) - (\bigcup_{i=1}^{x} \bigcup_{j=i+1}^{x} (\partial A_i \cap \partial A_j))$$

Definition 6 *The interior of an n-complex C, denoted by C°, is the set of all $(0 \ldots n)$-cells in the closure of $A_i \in C$ that are not elements of C's boundary, i.e.,*

$$C^\circ = (\bigcup_{i=1}^{x} \overline{A}_i) - \partial C$$

Definition 7 *The exterior of a complex, denoted by C^-, is the intersection of the exteriors of all cells A_i that are part of the complex, i.e.,*

$$C^- = \bigcap_{i=1}^{x} A_i^-$$

From these definitions, it follows that (1) interior, boundary, and exterior of a cell (or a cell complex) are mutually exclusive and (2) their union coincides with the universe.

Subsequently, the term *cell* will be used as a synonym for *complexes*. For the sake of clarity, some of the interior faces will be omitted in the figures.

2.2 Integrated Topology

In order to compare cells for coincidence, it is necessary to embed all cells into the same universe. This integration allows for the solution of topological operations on a purely symbolic level, without any consideration of metric. This fundamental topological structure has to fulfill two completeness axioms [28]:

- Completeness of incidence: The intersection of two cells is either empty or a face of both cells. Hence, no two geometric objects must exist at the same location. For example, though a 1-cell may represent both a part of a state boundary and a part of the border of a nation, the geometry of the 1-cell will be recorded only once.

- Completeness of inclusion: Every n-cell is a face of a $(n + 1)$-cell. Hence, in a 2-dimensional space every 0-cell is either start- or end-node of a 1-cell, and every 1-cell is in the boundary of a 2-cell.

It is further assumed that the closure of each cell is strictly inside the universe ($A \subset R^2$), i.e., no cell is outside of or on the border of the universe.

The embedding of the cells into a universe gives rise to the definition of the codimension. The *codimension* defines the difference between the dimension of the embedding space and the dimension of a cell. For example, codimension 1 for a 2-cell describes that it is located in a 3-space. The codimension can be never less than zero and it is zero if and only if the cell and the space are of the same dimension.

2.3 Cells for Regions, Lines, and Points

Within the context of this paper, we are interested in a subset of cell complexes that are most commonly used in geographic and cartographic applications. The complexes are "homogeneously n-dimensional" and not partitioned into non-empty, disjoint parts. The commonly used geographic features of points, lines, and regions are then defined as follows:

- A *region* is a 2-complex in $\mathbb{R}^2$ with a non-empty, connected interior.
  - A *region without holes* is a region with a connected exterior and a connected boundary (thus also called a *region with connected boundaries*) (Figure 1a).
  - A *region with holes* is a region with a disconnected exterior and a disconnected boundary (Figure 1b).
- A *line* is a sequence of connected 1-complexes in $\mathbb{R}^2$ such that they neither cross each other nor form closed loops.
  - A *simple line* is a line with two disconnected boundaries (Figure 1c).
  - A *complex line* is a line with more than two disconnected boundaries (Figure 1d).
- A *point* is a single 0-cell in $\mathbb{R}^2$.

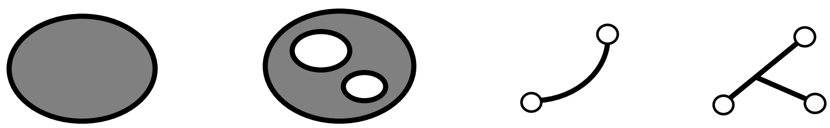

Figure 1: A region with (a) connected and (b) disconnected boundary; and a (c) simple and (d) complex line.

3 9-Intersection as a Model for Topological Relations

The binary topological relation R between two cells, A and B, is based upon the comparison of A's interior (A°), boundary (∂A), and exterior (A^-) with B's interior (B°), boundary (∂B), and exterior (B^-). These six *object parts* can be combined such that they form nine fundamental descriptions of a topological relation between two n-cells. These are:

- the intersection of A's *interior* with B's *interior* (and spelled "boundary-boundary intersection"), denoted by ($A^\circ \cap B^\circ$),
- the intersection of A's interior with B's boundary ($A^\circ \cap \partial B$),
- A's interior with B's exterior ($A^\circ \cap B^-$),
- the boundary-boundary intersection $\partial A \cap \partial B$,
- A's boundary with B's interior ($\partial A \cap B^\circ$),
- A's boundary with B's exterior ($\partial A \cap B^-$),

- the intersection of the two exteriors ($A^- \cap B^-$),
- A's exterior with B's boundary ($A^- \cap \partial B$), and
- A's exterior with B's interior ($A^- \cap B^\circ$).

Sometimes, we will also refer to more general terms like, "A's interior intersections," which encompasses the three intersections $A^\circ \cap \partial B$, $A^\circ \cap B^\circ$, and $A^\circ \cap B^-$, or "B's boundary intersections," which are $\partial A \cap \partial B$, $A^\circ \cap \partial B$, and $A^- \cap \partial B$.

The framework for the description of the topological relation between two cells, A and B, is the ordered set of these nine intersections, called the *9-intersection*, which is concisely represented as a 3×3-matrix.

$$R(A, B) = \begin{pmatrix} A^\circ \cap B^\circ & A^\circ \cap \partial B & A^\circ \cap B^- \\ \partial A \cap B^\circ & \partial A \cap \partial B & \partial A \cap B^- \\ A^- \cap B^\circ & A^- \cap \partial B & A^- \cap B^- \end{pmatrix}$$

Every different set of 9-intersections describes a different topological relation, and relations with the same specifications will be considered to be *topologically equivalent*; therefore, the 9-intersection can be employed to analyze whether or not two different configurations have the same topological relation [23]. Topological relations are characterized by the *topological invariants* of these nine intersections, i.e., properties that are preserved under topological transformations [55]. Examples of topological invariants applicable to the 9-intersection are the content (i.e., emptiness or non-emptiness) of a set, the dimension, the number of separations, and the sequence of disconnected intersections of different dimensions along the boundary [21, 38].

For the 9-intersection mode, the *content* of the nine intersections was identified as a simple and most general topological invariant [21]. It characterizes each of the nine intersections by a value *empty* ($\emptyset$) or *non-empty* ($\neg\emptyset$). For example, the 9-intersections based on empty/non-empty intersections for a configuration in which region A covers region B is:

$$R(A, B) = \begin{pmatrix} A^\circ \cap B^\circ = \neg\emptyset & A^\circ \cap \partial B = \emptyset & A^\circ \cap B^- = \emptyset \\ \partial A \cap B^\circ = \neg\emptyset & \partial A \cap \partial B = \neg\emptyset & \partial A \cap B^- = \emptyset \\ A^- \cap B^\circ = \neg\emptyset & A^- \cap \partial B = \neg\emptyset & A^- \cap B^- = \neg\emptyset \end{pmatrix}$$

or briefly:

$$R(A, B) = \begin{pmatrix} \neg\emptyset & \emptyset & \emptyset \\ \neg\emptyset & \neg\emptyset & \emptyset \\ \neg\emptyset & \neg\emptyset & \neg\emptyset \end{pmatrix}$$

Subsequently, the latter notation will be used as a shortcut. The sequence of the nine intersections, from left to right and from top to bottom, will always be (1) interior, (2) boundary, and (3) exterior.

The nine empty/non-empty intersections describe a set of relations that provides a complete coverage—any set is either empty or not empty and *tertium non datur*. Furthermore, they are mutually exclusive so that the union (OR) of all specifications is identically true, i.e., one of the specified relations holds true for any possible configuration, and the intersection (AND) of any two specified relations is identically false, i.e., only a single one exists between two cells.

For the goal of this paper—the formal identification of existing topological relations—it is extremely useful that the 9-intersection can concisely describe topological properties and constraints of both existing and non-existing relations.

3.1 Topological Properties

A variety of topological properties between two cells, A and B, can be expressed in terms of the 9-intersection [18]. Those intersections that do not matter and, therefore, can take an arbitrary value will be marked by a "wild card" (_).

Let a_i and b_j be arbitrary non-empty parts of A and B, respectively.

- If a_i is *disjoint* from b_j then the intersection between these two parts must be empty, while the other eight intersections can take any arbitrary value. For example, if A's boundary is disjoint from B's interior then the 9-intersection between A and B must match the following pattern:

$$R_{\{\emptyset,\neg\emptyset\}}(A,B) = \begin{pmatrix} \_ & \_ & \_ \\ \emptyset & \_ & \_ \\ \_ & \_ & \_ \end{pmatrix}$$

- If a_i *intersects* with b_j then the intersection between these two parts must be non-empty. For example, if A's interior intersects with B's boundary then the 9-intersection between A and B must match the following pattern:

$$R_{\{\emptyset,\neg\emptyset\}}(A,B) = \begin{pmatrix} \_ & \neg\emptyset & \_ \\ \_ & \_ & \_ \\ \_ & \_ & \_ \end{pmatrix}$$

- If a_i is a *subset* ($\subseteq$) of b_j then the intersection between these two parts must be non-empty. Furthermore, the two intersections between a_i and the other two parts of B, b_k and b_l, must be empty, because the parts are pairwise disjoint, otherwise, there would be some part of a_i outside of b_j, which would contradict the subset relation. For example, if A's boundary is a subset of B's interior (Figure 2a), then the 9-intersection between A and B must match the following pattern:

$$R_{\{\emptyset,\neg\emptyset\}}(A,B) = \begin{pmatrix} \_ & \_ & \_ \\ \neg\emptyset & \emptyset & \emptyset \\ \_ & \_ & \_ \end{pmatrix}$$

- Likewise, if a_i is a *subset* of two parts, b_j and b_k ($j \neq k$), such that $a_i \not\subseteq b_j$ and $a_i \not\subseteq b_k$, then the intersections with these two parts must be non-empty, while the intersection between a_i and the third part of B must be empty. For example, if $\partial A \subseteq (\partial B \cup B^\circ)$ such that $\partial A \not\subseteq \partial B$ and $\partial A \not\subseteq B^\circ$ (Figure 2b), then the 9-intersection between A and B must match the following pattern:

$$R_{\{\emptyset,\neg\emptyset\}}(A,B) = \begin{pmatrix} \_ & \_ & \_ \\ \neg\emptyset & \neg\emptyset & \emptyset \\ \_ & \_ & \_ \end{pmatrix}$$

- A consequence of the first subset rule is that if two object parts, a_i and b_j, *coincide*, then the intersection between a_i and b_j must be non-empty, while the other four intersections, having either a_i or b_j as an argument, must be empty. This follows from $a_i = b_j$ if $a_i \subseteq b_j$ and $b_j \subseteq a_i$. For example, if the two boundaries of A and B coincide then the 9-intersection between A and B must match the following pattern:

$$R_{\{\emptyset,\neg\emptyset\}}(A,B) = \begin{pmatrix} \_ & \emptyset & \_ \\ \emptyset & \neg\emptyset & \emptyset \\ \_ & \emptyset & \_ \end{pmatrix}$$

Figure 2: (a) A's boundary being a subset of B's interior and (b) A's boundary being a subset of B's interior and boundary.

3.2 Constraints for Non-Existing Relations

In a similar way, the 9-intersection can be used to describe "negative" topological constraints, i.e., configurations that cannot exist. Non-existing configurations may be due to particular properties of the objects (e.g., regions or lines), the embedding space (e.g., 2-D plane or surface of a 3-D object), the relation between the objects and the embedding space (i.e., the codimension), or the spatial data model (e.g., discrete or continuous). The following example is to illustrate the idea of representing non-existing relations in terms of the 9-intersection. Between two non-empty cells in $\mathbb{R}^2$, there must be at least one non-empty intersection, otherwise, no geometric interpretation can be found. In terms of the 9-intersection, it is impossible that all nine intersections are empty; therefore, the following condition holds:

$$R_{\{\emptyset,\neg\emptyset\}}(A,B) \quad \neq \quad \begin{pmatrix} \emptyset & \emptyset & \emptyset \\ \emptyset & \emptyset & \emptyset \\ \emptyset & \emptyset & \emptyset \end{pmatrix}$$

Multiple conditions for non-existing relations may be correlated such that the same non-existing relation, described by two patterns of 9-intersections, is a member of different conditions. For example, if one condition C_i is more specific than another condition C_j then all of C_i's non-existing intersections are included in the set of C_j's non-existing intersections. Using the 9-intersection, such dependencies can be easily detected by comparing the corresponding values of the two 9-intersections. For example, the condition C_1 that "all nine intersections must not be empty" is implied by condition C_2 that "the exteriors of two cells must always be non-empty," because $C_1 \subset C_2$:

$$C_1 = \begin{pmatrix} \emptyset & \emptyset & \emptyset \\ \emptyset & \emptyset & \emptyset \\ \emptyset & \emptyset & \emptyset \end{pmatrix} \quad C_2 = \begin{pmatrix} - & - & - \\ - & - & - \\ - & - & \emptyset \end{pmatrix} \quad \Rightarrow \quad C_1 \subset C_2$$

4 Existing 9-Intersections in $\mathbb{R}^2$

This section focuses on the binary relations in $\mathbb{R}^2$ between an m-cell and an n-cell, where $0 \leq m,n \leq 2$. Based upon the empty/non-empty 9-intersections, 2^9 topological relations are possible between two cells; however, only a smaller number of them can be realized in a particular space. Some of them depend on the dimensions and codimensions of the cells. The goal of this section is to identify which topological relations may be realized and which ones may not.

The approach taken is a three-step process:

- the formalization of topological conditions for *non-existing* relations in terms of the empty/non-empty 9-intersections, which are translated into specification patterns for non-existing topological relations;
- the calculation of the set of 9-intersections that exist between two cells as the set of all 512 possible relations, reduced by the union of all non-existing relations; and
- the verification of the existence of the remaining relations by realizing prototypical geometric configurations in $\mathbb{R}^2$.

Since different topological conditions apply depending on the codimensions of the objects involved, the investigations will be separated into relations between two regions in 2-D (Section 4.1); two lines in 2-D (Section 4.2); a region and a line in 2-D (Section 4.3); and the trivial relations with points in 2-D (Section 4.4). Subsequently, we present one combination of conditions that leads to the set of existing binary topological relations between any combination of regions, lines, and points. Numerous other combinations of conditions are possible. Though our set of conditions is not necessarily minimal, it is such that (1) no condition is part of another condition and (2) no condition is covered by any combination of other conditions. The first property can be easily checked by comparing the 9-intersections of all conditions (Section 3.2). To evaluate the second property a test program was used to compare those 9-intersections that fulfilled all n conditions with the 9-intersections that fulfilled only n-1 conditions. If the latter set was equal to the first set, then the condition left out was implied by the combination of the other relations and, therefore, redundant.

4.1 Relations between Two Regions with Codimension 0

4.1.1 Conditions for Regions.

The intersection between two exteriors is only empty if at least one of the two regions coincides with $\mathbb{R}^2$, or if the union of the two cells is the universe. This follows immediately from $\overline{A} \cup A^- = \mathbb{R}^2$ and $\overline{B} \cup B^- = \mathbb{R}^2$: $A^- \cap B^-$ is only empty if either $\overline{A} = \mathbb{R}^2$, or $\overline{B} = \mathbb{R}^2$, or $\overline{A} \cup \overline{B} = \mathbb{R}^2$. All three scenarios are impossible for the cell data model, because $A \subset \mathbb{R}^2$ and $B \subset \mathbb{R}^2$. Thus also $(A \cup B) \subset \mathbb{R}^2$; therefore, the following condition holds:

Condition 1 *The exteriors of two cells intersect with each other, i.e.,*

$$R_{\{\emptyset,\neg\emptyset\}}(A,B) \quad \not= \quad \begin{pmatrix} - & - & - \\ - & - & - \\ - & - & \emptyset \end{pmatrix} \tag{1}$$

The following three conditions are based upon a particular property of this spatial data model, namely the fact that if the boundaries of two regions do not coincide then there is either some interior or exterior between them. This implies that if A's interior does not intersect with B's exterior then the interiors must intersect (Condition 2), A's boundary must not intersect with B's exterior (Condition 3), and A's interior must not intersect with B's boundary (Condition 4).

Condition 2 *If both interiors are disjoint then A's interior intersects with B's exterior, and vice-versa, i.e.,*

$$R_{\{\emptyset,\neg\emptyset\}}(A,B) \quad \not= \quad \begin{pmatrix} \emptyset & - & \emptyset \\ - & - & - \\ - & - & - \end{pmatrix} \vee \begin{pmatrix} \emptyset & - & - \\ - & - & - \\ \emptyset & - & - \end{pmatrix} \tag{2}$$

Condition 3 *If A's interior is a subset of the B's closure then A's boundary must be a subset of B's closure as well, and vice-versa, i.e.,*

$$R_{\{\emptyset,\neg\emptyset\}}(A,B) \quad \neq \quad \begin{pmatrix} \_ & \_ & \emptyset \\ \_ & \_ & \neg\emptyset \\ \_ & \_ & \_ \end{pmatrix} \vee \begin{pmatrix} \_ & \_ & \_ \\ \_ & \_ & \_ \\ \emptyset & \neg\emptyset & \_ \end{pmatrix} \tag{3a \& b}$$

Condition 4 *If A's interior intersects with B's boundary then it must also intersect with B's exterior, and vice-versa, i.e.,*

$$R_{\{\emptyset,\neg\emptyset\}}(A,B) \quad \neq \quad \begin{pmatrix} \_ & \neg\emptyset & \emptyset \\ \_ & \_ & \_ \\ \_ & \_ & \_ \end{pmatrix} \vee \begin{pmatrix} \_ & \_ & \_ \\ \neg\emptyset & \_ & \_ \\ \emptyset & \_ & \_ \end{pmatrix} \tag{4}$$

A cell with a non-empty boundary cannot have all three boundary intersections empty. $\partial A = \neg\emptyset$ implies $\partial A \cap \mathbb{R}^2 = \neg\emptyset$. Since $\partial B \cup B^\circ \cup B^- = \mathbb{R}^2$ it follows that $\partial A \cap (\partial B \cup B^\circ \cup B^-) = \neg\emptyset$, which is only true if at least one part of B intersects with A's boundary.

Condition 5 *A's boundary intersects with at least one part of B, and vice-versa, i.e.,*

$$R_{\{\emptyset,\neg\emptyset\}}(A,B) \quad \neq \quad \begin{pmatrix} \_ & \_ & \_ \\ \emptyset & \emptyset & \emptyset \\ \_ & \_ & \_ \end{pmatrix} \vee \begin{pmatrix} \_ & \emptyset & \_ \\ \_ & \emptyset & \_ \\ \_ & \emptyset & \_ \end{pmatrix} \tag{5a \& b}$$

Since the boundary of a region separates its interior from the exterior, every path from the exterior to the interior crosses the boundary (Jordan-Curve-Theorem) [66]. This gives rise to the following four conditions:

Condition 6 *If both interiors are disjoint then A's boundary cannot intersect with B's interior, and vice-versa, i.e.,*

$$R_{\{\emptyset,\neg\emptyset\}}(A,B) \quad \neq \quad \begin{pmatrix} \emptyset & \neg\emptyset & \_ \\ \_ & \_ & \_ \\ \_ & \_ & \_ \end{pmatrix} \vee \begin{pmatrix} \emptyset & \_ & \_ \\ \neg\emptyset & \_ & \_ \\ \_ & \_ & \_ \end{pmatrix} \tag{6a \& b}$$

Every connected object part that intersects with both the interior and exterior of another object must also intersect with that object's boundary. For arbitrary regions, only the interior is connected.

Condition 7 *If A's interior intersects with B's interior and exterior, then it must also intersect with B's boundary, and vice-versa, i.e.,*

$$R_{\{\emptyset,\neg\emptyset\}}(A,B) \quad \neq \quad \begin{pmatrix} \neg\emptyset & \_ & \_ \\ \emptyset & \_ & \_ \\ \neg\emptyset & \_ & \_ \end{pmatrix} \vee \begin{pmatrix} \neg\emptyset & \emptyset & \neg\emptyset \\ \_ & \_ & \_ \\ \_ & \_ & \_ \end{pmatrix} \tag{7a \& b}$$

Unless the boundaries of two regions coincide, at least one boundary must intersect with the other region's exterior.

Condition 8 *If both boundaries do not intersect with each other then at least one boundary must intersect with its opposite exterior, i.e.,*

$$R_{\{\emptyset,\neg\emptyset\}}(A,B) \quad \neq \quad \begin{pmatrix} \_ & \_ & \_ \\ \_ & \emptyset & \emptyset \\ \_ & \emptyset & \_ \end{pmatrix} \tag{8}$$

Likewise, if the interiors of two regions are separated then at least one boundary must intersect with the opposite exterior.

Condition 9 *If both interiors do not intersect with each other then at least one boundary must intersect with its opposite exterior, i.e.,*

$$R_{\{\emptyset,\neg\emptyset\}}(A,B) \quad \neq \quad \begin{pmatrix} \emptyset & - & - \\ - & - & \emptyset \\ - & \emptyset & - \end{pmatrix} \tag{9}$$

4.1.2 Conditions for Regions without Holes.

Conditions (1)–(9) apply to regions—independent of whether they have holes or not. Regions without holes are a more restricted class of spatial objects than regions and, therefore, their topological relations have further constraints. The crucial property of a region without holes is that its boundary is connected. This fact, in combination with the Jordan-Curve-Theorem, gives rise to the definition of the following three conditions:

Condition 10 *If both boundaries intersect with the opposite interiors then the boundaries must also intersect with each other, i.e.,*

$$R_{\{\emptyset,\neg\emptyset\}}(A,B) \quad \neq \quad \begin{pmatrix} - & \neg\emptyset & - \\ \neg\emptyset & \emptyset & - \\ - & - & - \end{pmatrix} \tag{10}$$

Condition 11 *If A's interior intersects with B's exterior then A's boundary must also intersect with B's exterior, i.e.,*

$$R_{\{\emptyset,\neg\emptyset\}}(A,B) \quad \neq \quad \begin{pmatrix} - & - & \neg\emptyset \\ - & - & \emptyset \\ - & - & - \end{pmatrix} \vee \begin{pmatrix} - & - & - \\ - & - & - \\ \neg\emptyset & \emptyset & - \end{pmatrix} \tag{11}$$

Condition 12 *If the interiors do not intersect with each other then A's boundary must intersect with B's exterior, and vice-versa, i.e.,*

$$R_{\{\emptyset,\neg\emptyset\}}(A,B) \quad \neq \quad \begin{pmatrix} \emptyset & - & - \\ - & - & - \\ - & \emptyset & - \end{pmatrix} \vee \begin{pmatrix} \emptyset & - & - \\ - & - & \emptyset \\ - & - & - \end{pmatrix} \tag{12}$$

4.1.3 Realization of Region Relations.

The 9-intersections of the existing relations between two regions can be determined by successively applying these conditions and canceling the corresponding non-existing 9-intersections from the set of all 512 relations. Eighteen relations exist in $\mathbb{R}^2$ if the region boundaries are connected or disconnected, eight of which can be realized only for regions with connected boundaries. The existence of the topological relations corresponding to the 9-intersections has been verified by finding their geometric interpretations. Figure 3 shows prototypes of the eight relations between arbitrary regions (R_0–R_7) and the ten particular relations between regions with disconnected boundaries (R_8–R_{17}), respectively.

Some of the conditions for regions are generic so that they apply also for other cells:

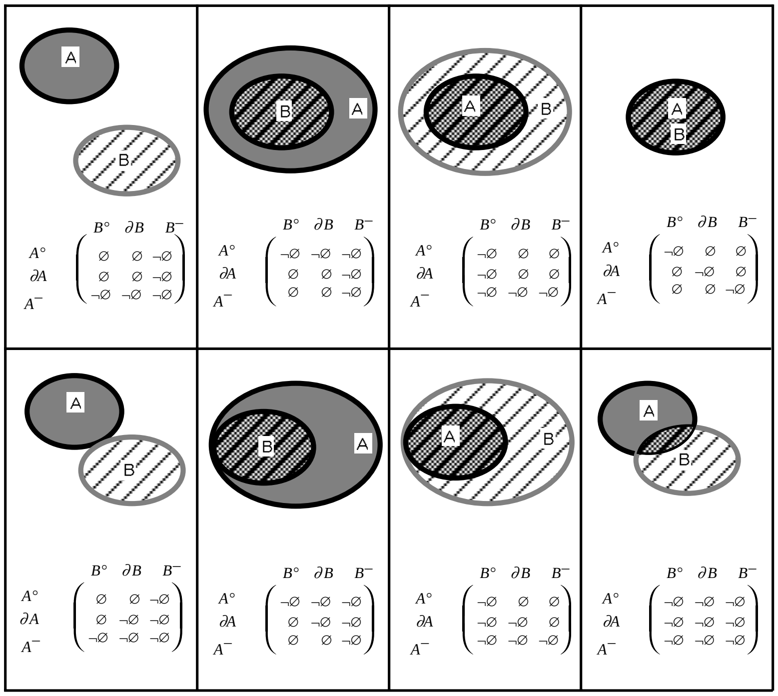

Figure 3: A geometric interpretation of the 8 relations between two regions with connected boundaries

- Condition (1) holds for any two non-empty cells.
- Conditions (2)–(4) hold for any two non-empty cells, A and B, of the same dimension. If the dimension of A is greater than the dimension of B then only the first part of each condition applies.
- Condition (5) holds for any two cells with non-empty boundaries.
- Conditions (6)–(12) apply only to regions with codimension 0.

4.2 Relations between two Lines with Codimension > 0

4.2.1 Line Conditions.

Lines are non-empty cells with non-empty boundaries, therefore, Conditions (1)–(5) apply. Additional constraints must hold for two lines due to the property of the spatial data model that another point exists between any two distinct points; therefore, if the exterior of one line intersects with the boundary of another line, the exterior must also intersect with the interior of the other line. This implies:

Condition 13 *If A's closure is a subset of B's interior then either A's exterior intersects with both B's boundary and B's interior, or not at all, and vice-versa, i.e.,*

$$R_{\{\emptyset,\neg\emptyset\}}(A,B) \;\not\simeq\; \begin{pmatrix} - & - & \emptyset \\ - & - & - \\ \emptyset & \neg\emptyset & - \end{pmatrix} \vee \begin{pmatrix} - & - & \emptyset \\ - & - & \neg\emptyset \\ \emptyset & - & - \end{pmatrix} \vee \tag{13a}$$

$$\begin{pmatrix} - & - & \emptyset \\ - & - & - \\ \neg\emptyset & \emptyset & - \end{pmatrix} \vee \begin{pmatrix} - & - & \neg\emptyset \\ - & - & \emptyset \\ \emptyset & - & - \end{pmatrix} \tag{13b}$$

4.2.2 Simple Line Conditions.

If the two lines are simple then both boundaries consist of two points, each of which has no extend and, therefore, can only intersect with one part of another object. This particular property of simple lines leads to the following condition:

Condition 14 *Each boundary can intersect with at most two opposite parts, i.e.,*

$$R_{\{\emptyset,\neg\emptyset\}}(A,B) \;\not\simeq\; \begin{pmatrix} - & \neg\emptyset & - \\ - & \neg\emptyset & - \\ - & \neg\emptyset & - \end{pmatrix} \vee \begin{pmatrix} - & - & - \\ \neg\emptyset & \neg\emptyset & \neg\emptyset \\ - & - & - \end{pmatrix} \tag{14a \& b}$$

Likewise, the fact that the boundary of a simple line A is a subset of the boundary of another simple line B implies that no part of the boundary can be outside of A's boundary. If there were some part of B's boundary outside of A's boundary, this would mean that B's boundary has more than two disconnected boundaries, and then the line would not be simple anymore.

Condition 15 *If A's boundary is a subset of B's boundary, then the two boundaries coincide, and vice-versa, i.e.,*

$$R_{\{\emptyset,\neg\emptyset\}}(A,B) \;\not\simeq\; \begin{pmatrix} - & \neg\emptyset & - \\ \emptyset & \neg\emptyset & \emptyset \\ - & - & - \end{pmatrix} \vee \begin{pmatrix} - & - & - \\ \emptyset & \neg\emptyset & \emptyset \\ - & \neg\emptyset & - \end{pmatrix} \vee \tag{15a}$$

$$\begin{pmatrix} - & \emptyset & - \\ \neg\emptyset & \neg\emptyset & - \\ - & \emptyset & - \end{pmatrix} \vee \begin{pmatrix} - & \emptyset & - \\ - & \neg\emptyset & \neg\emptyset \\ - & \emptyset & - \end{pmatrix} \tag{15b}$$

4.2.3 Realization of Line Relations.

There are 57 relations between two lines, 33 of them can be also realized between simple lines. Figures 4 and 5 show the 9-intersections and corresponding geometric interpretations of the 33 relations between two simple lines and of the 24 relations that exist only for complex lines, respectively.

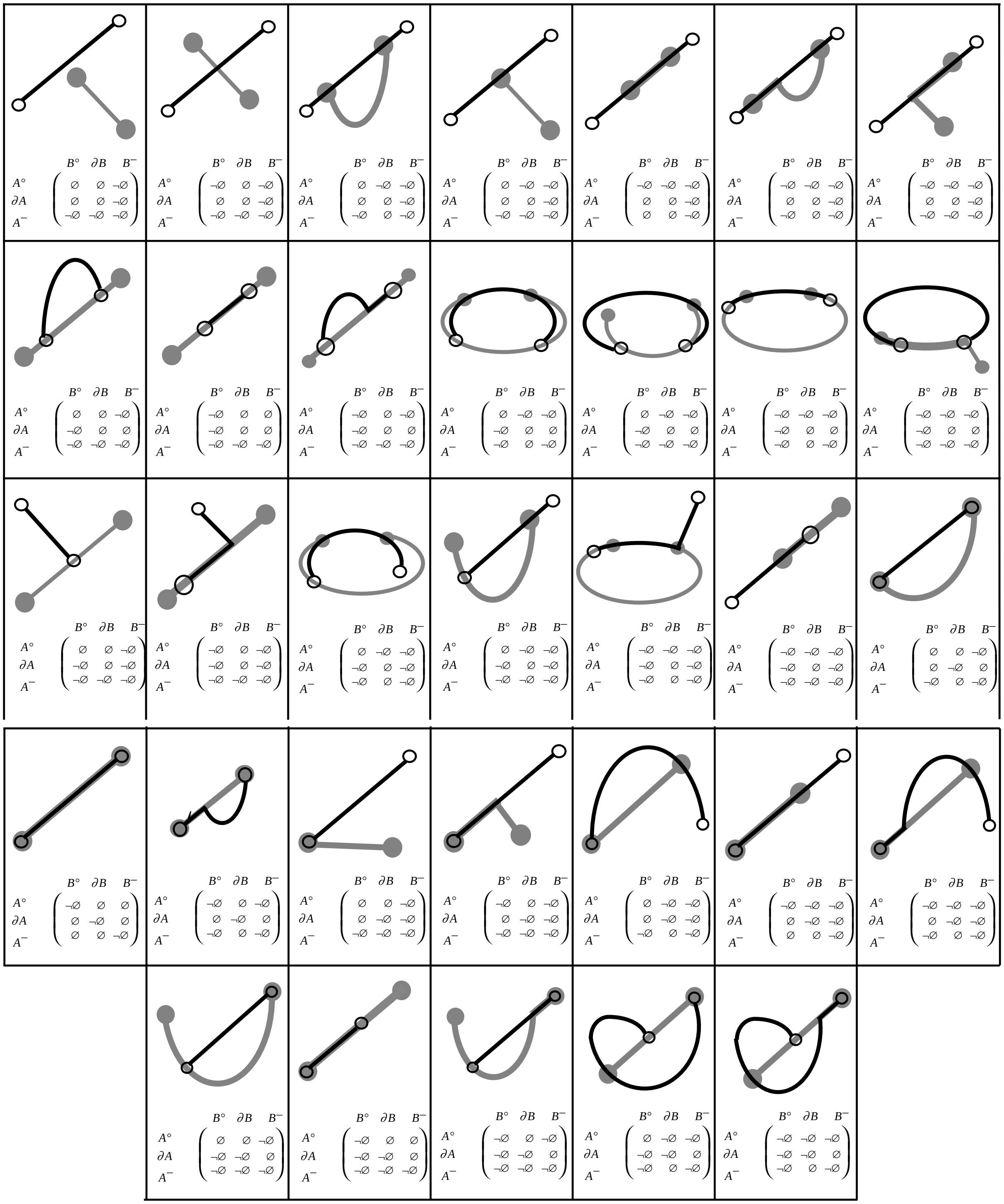

Figure 4: A geometric interpretation of the 33 relations that can be realized between two simple lines.

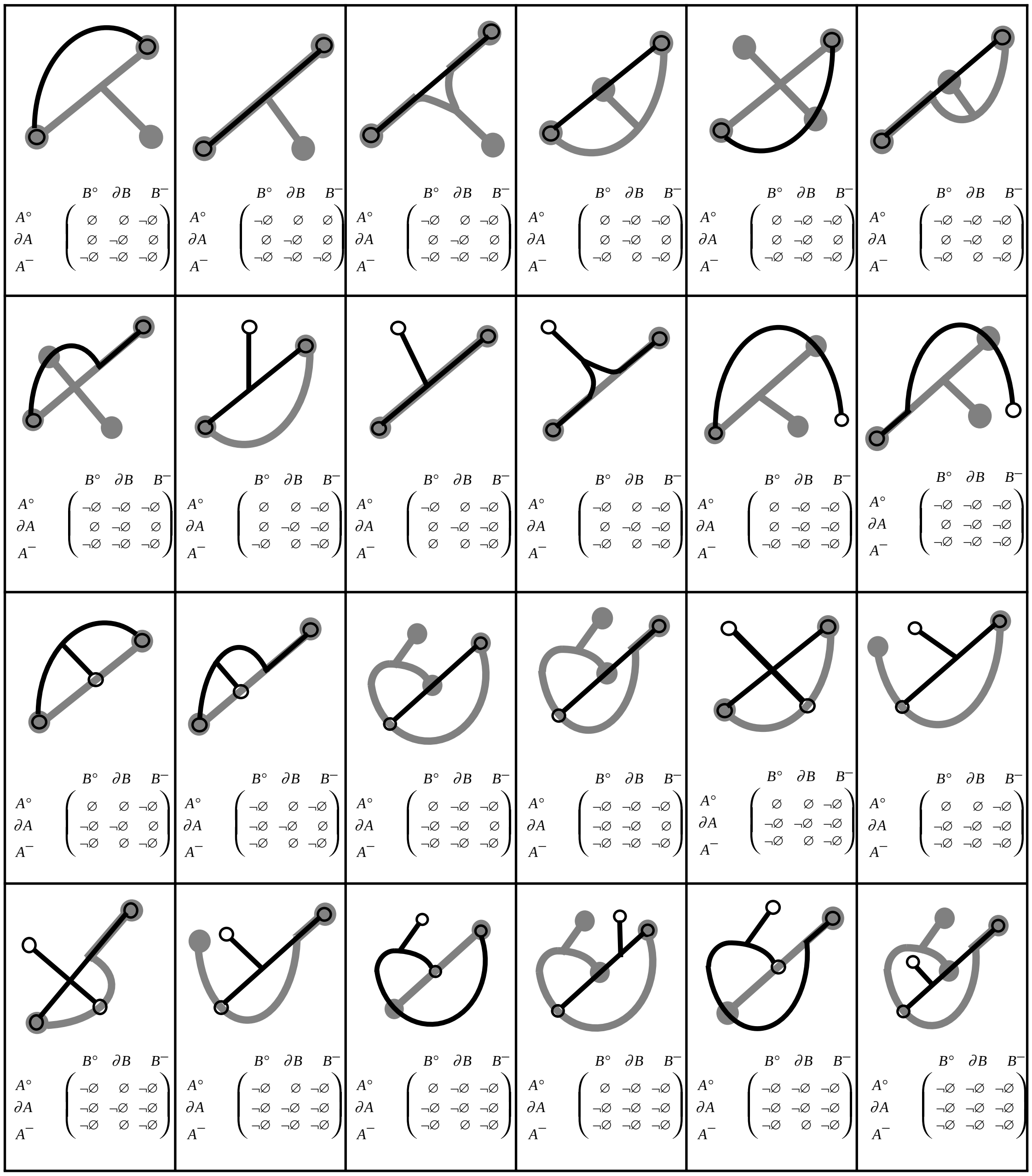

Figure 5: A geometric interpretation of the 24 additional relations between two non-simple lines.

4.3 Relations between a Region and a Line

The relations between a region and a line involve two objects of different dimensions, therefore, conditions that hold between a region and a line do not necessarily hold between a line and a region.

From the previous definitions for regions, the symmetric Condition (1), and the asymmetric parts of Conditions (3a), (5a), (6a), and (7a) apply also for the relations between a region and a line. Further constraints are due to the fact that the regions and lines have different dimensions. The dimension of the interior of a region A is always greater than the dimension of the closure of a line B, therefore, $A^\circ \supset \overline{B}$:

Condition 16 *The interior of a region A always intersects with the exterior of a line B, i.e.,*

$$R_{\{\emptyset,\neg\emptyset\}}(A,B) \quad \not= \quad \begin{pmatrix} - & - & \emptyset \\ - & - & - \\ - & - & - \end{pmatrix} \tag{16}$$

By definition, a line has a non-empty boundary and contains no loops. A region's boundary, on the other hand, is a closed 1-cell. This implies that the closure of a line is at most a true subset ($\subset$) of the region's boundary:

Condition 17 *The boundary of a region A always intersects with the exterior of a line B, i.e.,*

$$R_{\{\emptyset,\neg\emptyset\}}(A,B) \quad \not= \quad \begin{pmatrix} - & - & - \\ - & - & \emptyset \\ - & - & - \end{pmatrix} \tag{17}$$

The interior of a line is always non-empty, which implies the following condition:

Condition 18 *The interior of a line B must intersect with at least one of the three parts of a region A, i.e.,*

$$R_{\{\emptyset,\neg\emptyset\}}(A,B) \quad \not= \quad \begin{pmatrix} \emptyset & - & - \\ \emptyset & - & - \\ \emptyset & - & - \end{pmatrix} \tag{18}$$

Twenty 9-intersections fulfill these conditions for the topological relations between two lines. One of them can be realized only if the line consists of more than one segment, i.e., if it is a non-simple line. If the line has only a single segment then the intersections must also fulfill the conditions for simple lines, (14a) and (15a). The 9-intersections and their geometric representations for a region and a line are shown in Figure 6.

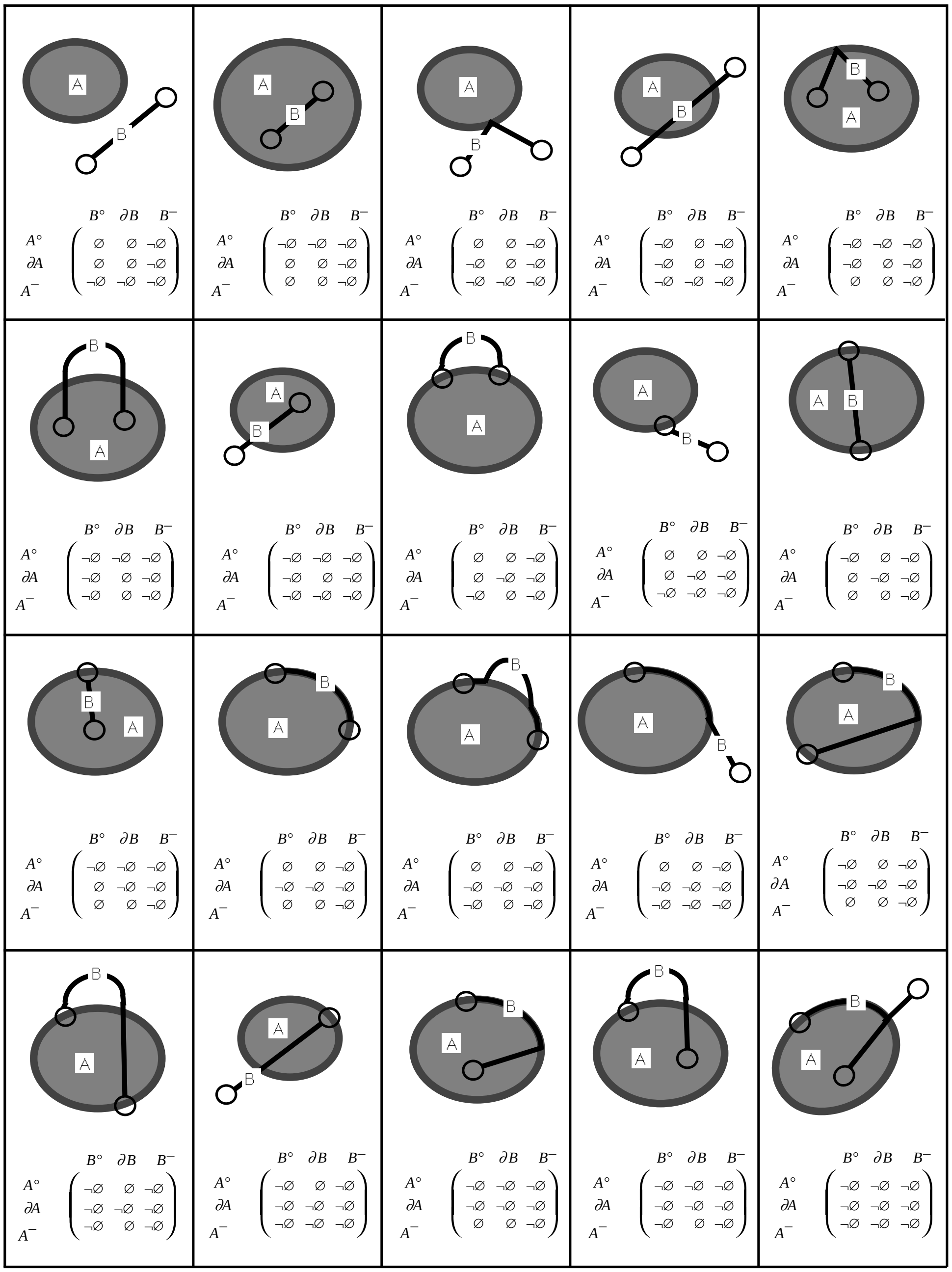

Figure 6: A geometric interpretation of the 20 relations between a region and a line (one of them can be only realized if the line is non-simple).

4.4 Relations with Points

Since the boundary of a point is empty, it is irrelevant to analyze its three boundary intersections. This leaves six significant intersections for describing the topological relations between a non-point (region or line) and a point and gives rise to 2^6 possible relations. The conditions for non-existing intersections are based on the fact that a point is always a true subset ($\subset$) of one of the three parts—interior, boundary, and exterior—of a non-point object.

Condition 19 *Interior, boundary, and exterior of any non-point object A intersect with the exterior of a point B, i.e.,*

$$R_{\{\emptyset,\neg\emptyset\}}(A,B) \quad \neq \quad \begin{pmatrix} - & - \\ - & \emptyset \\ - & - \end{pmatrix} \vee \begin{pmatrix} - & \emptyset \\ - & - \\ - & - \end{pmatrix} \vee \begin{pmatrix} - & - \\ - & - \\ - & \emptyset \end{pmatrix} \tag{19}$$

Condition 20 *The interior of a point can only intersect with a single part of another object, i.e.,*

$$R_{\{\emptyset,\neg\emptyset\}}(A,B) \quad \neq \quad \begin{pmatrix} \neg\emptyset & - \\ \neg\emptyset & - \\ - & - \end{pmatrix} \vee \begin{pmatrix} - & - \\ \neg\emptyset & - \\ \neg\emptyset & - \end{pmatrix} \vee \begin{pmatrix} \neg\emptyset & - \\ - & - \\ \neg\emptyset & - \end{pmatrix} \tag{20}$$

Condition 21 *The interior of a point must be a subset of one of the three parts of another object, i.e.,*

$$R_{\{\emptyset,\neg\emptyset\}}(A,B) \quad \neq \quad \begin{pmatrix} \emptyset & - \\ \emptyset & - \\ \emptyset & - \end{pmatrix} \tag{21}$$

This leaves three combinations of intersections between a point and a non-point that can be realized for point-region and point-line configurations).

Finally, for the sake of completeness, the trivial case between two points. Since both boundaries are empty, there are only four relevant intersections, that is, the intersections between interiors and exteriors. Condition (1) for the 9-intersection can be immediately applied to these four intersections:

Condition 22 *Both exteriors must intersect, i.e.,*

$$R_{\{\emptyset,\neg\emptyset\}}(A,B) \quad \neq \quad \begin{pmatrix} - & - \\ - & \emptyset \end{pmatrix} \tag{22}$$

Since a point is "atomic," it cannot intersect with more than one part of another cell. On the other hand, points are non-empty and therefore, they must intersect with at least one part of another cell.

Condition 23 *The interior of a point intersects with exactly one opposite object part, i.e.,*

$$R_{\{\emptyset,\neg\emptyset\}}(A,B) \quad \neq \quad \begin{pmatrix} \emptyset & \emptyset \\ - & - \end{pmatrix} \vee \begin{pmatrix} \emptyset & - \\ \emptyset & - \end{pmatrix} \vee \begin{pmatrix} \neg\emptyset & \neg\emptyset \\ - & - \end{pmatrix} \vee \begin{pmatrix} \neg\emptyset & - \\ \neg\emptyset & - \end{pmatrix} \tag{23}$$

This leaves two combinations of intersections for which the corresponding topological relations, disjoint and equal, can be realized between two points.

5 Related Work

A common thread in most spatial reasoning systems is the attempt to formalize spatial reasoning tasks by translating the problem into Cartesian coordinate space and to use common Euclidean geometry to find the solution [14, 51, 54, 72]. The field of *geometric reasoning* is based on this premise [46]. By using a propositional representation, such as predicates for the relations between objects, it is possible to describe qualitative spatial concepts without the need to bring them into a quantitative environment [9].

Computational approaches focusing on mathematical models to formalize relations among symbolic representations of conceptually modeled objects have been mainly investigated in artificial intelligence and engineering. Various models for *cardinal directions*, such as north, east, and northeast, have been discussed [59] and formalized for point objects [26], and their properties have been analyzed and compared with desirable properties of models for cardinal directions. It has also been proposed to derive topology from metric by using the primitives of *distance* and *direction* in combination with the logical connectors *AND*, *OR*, and *NOT* [58], which is only described for precise metric positions and leads to serious implementation problems in computers [31, 52] due to the finiteness of the underlying number system [20, 29].

5.1 Symbolic Projections

The most extensively investigated formalism for spatial relations is based on a segmentation of the plane, called *symbolic projections* [9]. Symbolic projections translate exact metric information into a qualitative form and allow for reasoning about the spatial relations among objects in a 2-D plane [8]. The order in which objects appear, projected vertically and horizontally, is encoded into two strings, called *2D string*, upon which spatial queries are executed as fast substring searches [9]. Initially, this approach has been proposed only for non-overlapping objects (using the two operators "less" and "equal"). An extension of this algebra with the operator "edge-to-edge" [43] allows for overlapping objects. By including the "empty space" into the 2D strings ambiguities that may exist for certain configurations can be resolved [44].

It was shown that symbolic projections and the 9-intersection are both suitable for powerful spatial reasoning [8, 18]. The major differences are:

- Symbolic projections and their derivatives subdivide the space, while the 9-intersection considers the objects and how they are embedded into space.
- Symbolic projections are primarily based on the relation "less" along to perpendicular axes, therefore, modeling directions such as north, south, east, and west, from which topological relations are derived. The 9-intersection, on the other hand, is only concerned with topological relations.
- Unlike the 9-intersection, which is invariant under topological transformations, symbolic projections depend on the orientation of the objects and, therefore, they are not invariant under rotation.
- The shapes of the objects (convex/concave) matter for the relations modeled by the symbolic projections, while the 9-intersection is independent of the shape of the objects.

5.2 Derivatives of Allen's Interval Relations

Another popular framework are the relations between one-dimensional intervals, initially proposed for modeling time [4]. They have been frequently extended to describe spatial relations in 2- and

3-dimensional space [33, 36, 61]. Some of the extension from 1-dimensional intervals, initially designed to model time, carry over the ordering (start/end) of the interval boundaries to the higher dimension. Most of these approaches assume that spatial objects are described by their bounding rectangles, to which Allen's approach can be easily generalized; however, rectangles are sometimes only crude approximations of the actual shapes of the objects and, therefore, they represent only a simplified model of spatial data. Variations for imprecise boundaries, using fuzzy logic [74], have been also studied [15].

The 9-intersection can be also considered a derivative of Allen's approach. Initially it was proposed to use only the four intersections of the two interiors and boundaries [17, 23], which was shown to be sufficient for codimension 0 [21]. Pigot's extension for triangles in $\mathbb{R}^3$ uses the five intersections of A's boundary with B's interior, boundary, "exterior," "above," and "below" [60]. Actually, this "exterior" is the exterior of B projected into $\mathbb{R}^{n-1}$, and "above" and "below" are then the two sets in $\mathbb{R}^n$ that are separated by the union of B's interior, boundary, and "exterior." Based on this classification schema, a total of fourteen topological relations are distinguished between two triangles in $\mathbb{R}^3$.

5.3 4-Intersection

The initial model for binary topological relations was developed for regions embedded in $\mathbb{R}^2$ [21]. This model, called the *4-intersection*, considers the two objects' interiors and boundaries and analyzes the intersections of these four object parts for their content (i.e., emptiness and non-emptiness). Several researchers have tried to model line-region and line-line relations in $\mathbb{R}^2$ just with the 4-intersection [10, 35, 68]. It is obvious that the 4-intersection is a subset of the 9-intersection, so that the 9-intersection would be able to distinguish more details than the 4-intersection. For region-region configurations in $\mathbb{R}^2$, the 4-intersection and the 9-intersection provide the same eight relations; however, for line-line and region-line relations, the 4-intersection distinguishes only 16 and 11 relations, respectively. The major difference for line-line relations is that the 4-intersection does not suffice to establish an equivalence relation [22], because several different line-line configurations have the same empty/non-empty 4-intersection. Similarly for region-line relations, the 4-intersection does not distniguish between certain topologically distinct configurations that may be critical for defining natural-language spatial predicates to be used in spatial query languages [50]. With the 9-intersection, these problems are overcome.

6 Conclusions

6.1 Summary

A formalism for the definition of binary topological relations has been presented that is based upon purely topological properties and, therefore, independent from the existence of such non-topological concepts as distance or direction. Binary topological relations are described by putting the three topologically distinct parts of one object—its interior, boundary, and exterior—into relation with the parts of the other object. Formally, this has been described as the 9-intersection, i.e., all possible set intersections of the parts. The criterion for distinguishing different topological relations is the content of the 9-intersections, i.e., whether the intersections are empty or non-empty.

The search for a method that provides also an efficient implementation led to the separation of the 9-intersection into the primary criterion or 4-intersection—empty or non-empty intersections between interiors and boundaries—and the secondary criterion, whether or not boundaries and interiors are subsets of the other objects closure. The 4-intersection representation proved to be sufficient for modeling the topological relations between two n-cells if their boundaries are connected and their codimensions are zero; however, the 4-intersection is insufficient if the objects are embedded into a higher dimensional space and the secondary criterion has also to be examined to resolve ambiguities; however, the 4-intersection is insufficient if the objects are embedded into a higher dimensional space and the secondary criterion has also to be examined to resolve ambiguities.

6.2 Implementation

A variation of the 9-intersection has been implemented in MGE-Dynamo [38]. Since each part of a cell is an aggregate of primitives with a unique identifier, the relevant operations, such as interior and boundary, can be implemented as symbolic, rather than arithmetic, operations. The implementation needs four fundamental operations:

- testing whether an intersection of two parts is empty;
- testing whether an intersection of two parts is non-empty;
- testing whether a part is included in another part; and
- testing whether a part is not included in another part.

These are standard operations, for which most efficient implementations have been proposed, for instance in language compilers [2].

The particular benefit of this approach for the implementation of a GIS is that it provides a complete coverage of binary topological relations. Users can build from them customized topological relations, accessible in their spatial query language [39]. For example, some applications may disregard the topological difference between *inside* and *coveredBy* and integrate the two into a single relation, say *within*, such that *within* has a non-empty interior intersection, an empty and a non-empty boundary-intersection, while the value of the boundary intersection does not matter.

This framework may also serve as an internal representation for a graphical spatial query language in which users sketch the spatial constraints graphically. In order to process such queries in a geographic database, the topological constraints contained in the sketch must be parsed and translated into a symbolic representation such a the 9-intersection.

6.3 Discussion and Future Work

The results of this paper represent a significant advancement in the investigations of formalisms for topological relations. Compared to our previous results [21], the novel findings are:

- The application of the framework of empty and non-empty intersections to objects with codimension greater than zero. This was achieved by introducing the 9-intersection.
- The inclusion of objects with connected *or* disconnected boundaries, giving rise to treat lines and n-dimensional objects ($n > 1$) with holes.
- With the 9-intersection we have found a model within which topological constraints can be formalized and compared.

Issues still to be investigated include:

- Topological relations between complex objects, i.e., objects that are made up of simpler ones—either of the same dimension or mixed dimensions, such as a line ending at a region and both together form a single object.

- Optimization strategies of queries with multiple topological constraints are necessary to improve the processing of complex spatial queries. For a small subset—the eight relations between two regions without holes—we have derived the composition table [18] upon which a *relation algebra* [71] can be based.

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