Algorithms for matrix functions and equations

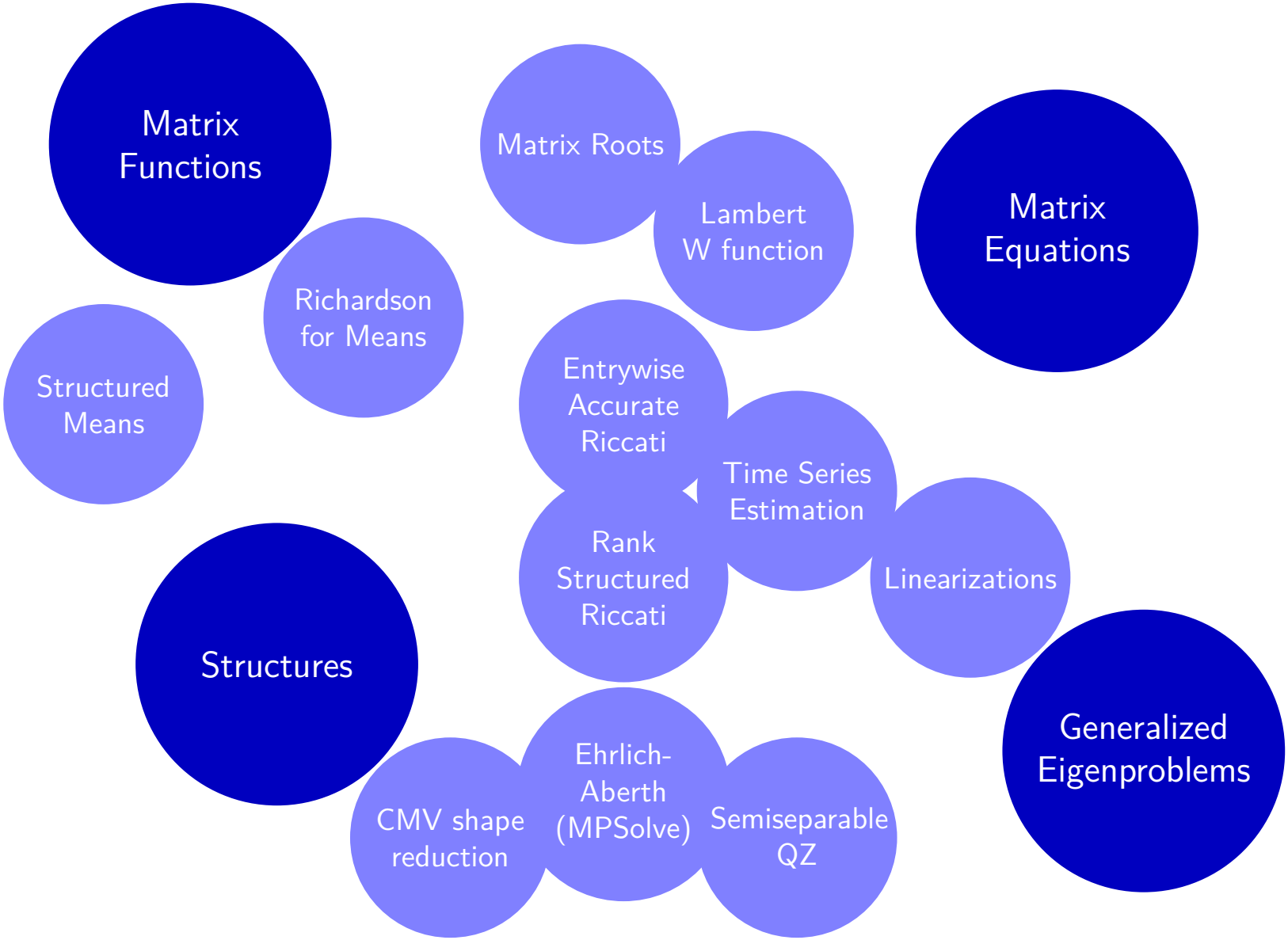
Consuntivo scientifico progetto di ricerca 2013

Federico Poloni[1]

Project participants: D.A. Bini[1], G. Del Corso[1], L. Gemignani[1], B. Iannazzo[2], B. Meini[1], V. Noferini[3], L. Robol[4]

[1]U Pisa [2]U Perugia [3]U Manchester [4]Scuola Normale Superiore, Pisa

Convegno GNCS 2014
Montecatini, 19-20th February 2014


Matrix Functions
Matrix Roots
Lambert W function
Matrix Equations
Richardson for Means
Structured Means
Entrywise Accurate Riccati
Time Series Estimation
Rank Structured Riccati
Linearizations
Structures
Generalized Eigenproblems
CMV shape reduction
Ehrlich-Aberth (MPSolve)
Semiseparable QZ

Why all these together?

Most of the "real-life" matrix equations are (generalized) eigenproblems in disguise:

$$XBX + XA - DX - C = 0 \iff \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} I \\ X \end{bmatrix} = \begin{bmatrix} I \\ X \end{bmatrix} Y$$

$$AX^2 + BX + C = 0 \iff \begin{bmatrix} 0 & I \\ -C & -B \end{bmatrix} \begin{bmatrix} I \\ X \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & A \end{bmatrix} \begin{bmatrix} I \\ X \end{bmatrix} Y$$

- Solution of interest $\iff$ spectrum in a certain region
- How do you ensure correct number of eigenvalues there? Structure
- How do you solve them efficiently? Structure
- How do you "move around" eigenvalues by region and preserve eigenvectors? Matrix functions

Where do they appear?

Riccati-type equations Control theory, modelling (probabilistic+engineering), queuing theory, time series...

Matrix functions Scientific imaging, radar, probability/statistics, delay differential equations

Structured eigenproblems matrix equations, everything that needs to compute roots of polynomials

Entrywise accurate solution of Riccati equations from fluid queues

- Riccati equations appearing in a probability application (fluid queues: model buffers with different input/output rates modeled by Markov chain)
- Improved accuracy of existing algorithms
- Now fully entrywise accurate algorithm: $\frac{|\tilde{X}_{ij} - X_{ij}|}{X_{ij}} \leq \varepsilon$
- Entrywise accuracy important in probability ("failure rates", entries span several orders of magnitude)
- Error analysis (long and boring)
- New ideas: do the same for quadratic problems/cyclic reduction
- [Nguyen, Poloni]

Estimation of MA time series models

- "Fitting" coefficients of a model from observations
- Looking for something faster than Maximum Likelihood
- Results in a matrix equation problem, or more generally matrix polynomial factorization $A\lambda + B + A^T\lambda^{-1} = (I - \lambda X)Y(I - \lambda^{-1}X^T)$
- Applying standard theory to solve it
- How to make the equation solvable when it isn't (observation errors in coefficients?) Structured eigenproblem perturbation
- New ideas: reduce to many scalar problems, reduce # of simultaneous variables
- [Poloni, Sbrana], [Brüll, Poloni, Sbrana, Schröder]

Rank-structured Riccati

- Algebraic Riccati equation $XBX + XA - DX - C = 0$ where A is block-diagonal, other coefficients have 1 small dimension
- Idea: a sort of quadratic block Jacobi/Gauss-Seidel
- Every iteration is a small-scale ARE.
- Asymptotic convergence theory, applicability
- New ideas: use this for eigensolvers
- [Bini, Meini], [Meini]

Matrix roots and Lambert W function

- Compute matrix versions of inverses of functions such as $x \mapsto x^p$, $x \mapsto x \exp(x)$
- Main issue: several branches to choose from
- Matrix roots: Schur form + back-substitution
- Strategies to perform back-substitution in # steps logarithmic in p
- Faster than competing algorithms on practically relevant parameter ranges
- Lambert W function: Newton's method
- Starting point to ensure correct branches
- [Iannazzo, Manasse], [Iannazzo]

Efficient matrix means

- Matrix geometric mean: minimizer of a certain Riemannian distance on positive definite matrices
- Equivalently, solves nonlinear equation $\sum_{i=1}^{k} \log(A_i^{-1} X) = 0$
- Algorithm: "simple" Richardson iteration with special step-size, motivated by local convergence theory
- Good convergence properties, faster in literature
- Other problem: Toeplitz means of Toeplitz matrices
- How to define them? Minimizer of another Riemannian distance
- Richardson-based algorithms; differential geometry motivates step size/preconditioner
- [Bini, Iannazzo], [Bini, Iannazzo, Jeuris, Vandebril]

Linearizations

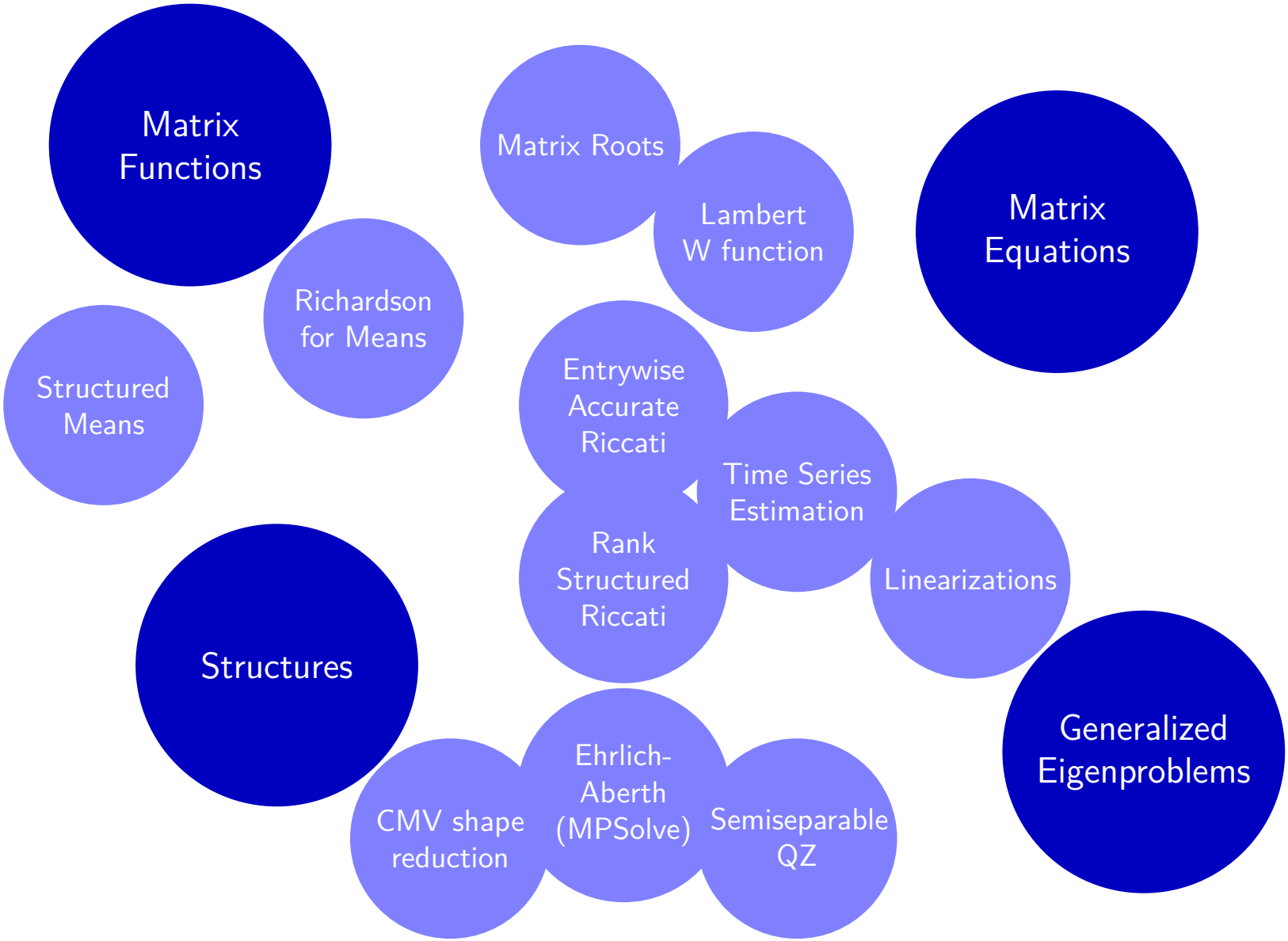
- Methods to turn a polynomial eigenproblem into a linear one
- Simplest one: (block) companion form. Many variants, often motivated by structure preservation
- Brought other linear algebra topics into the picture to simplify proofs and theory: duality, Wong chains, Bezoutians
- New ideas: apply this to multiparameter eigenproblems / relation to polynomial algebra algos; use semiseparable technology and notation
- [Noferini, Poloni] [Townsend, Noferini, Nakatsukasa], [Del Corso, Poloni]

Semiseparable QZ

- QR algorithms known in quadratic time for special structures (defined by low ranks of special submatrices)
- Key technique: updating "generators" in a linear number of parameters
- QZ more challenging — two rank structures to keep track of
- Developed Semisep-QZ versions for two different structures appearing in (generalized) companion forms, including unitary-plus-rank-1
- Almost-normal matrices: a more general structure: $(A^H + uv^T)A = A(A^H + uv^T)$
- Developed method to reduce almost-normal A to QAQ^H in a special block tridiagonal form (CMV shape, preserved by QR)
- [Boito,Eidelman,Gemignani]$\times 2$, [Bevilacqua, Del Corso, Gemignani]

Ehrlich-Aberth variants

- EA: simultaneous Newton-like iteration for finding polynomial roots
- Good for badly-scaled problems
- structured version for palindromic polynomials: work on "structured pairs of roots" at the same time
- Use it for eigenvalue computations (without forming determinants explicitly, need only $\frac{p'(x)}{p(x)}$)
- Use it for structured eigenvalue computations
- Choice of polynomial basis can give benefits, direction still to explore
- Add all this to state-of-the-art rootfinder MPSolve
- [Bini, Noferini], [Gemignani, Noferini], [Bini, Robol]


Matrix Functions
Matrix Roots
Lambert W function
Matrix Equations
Richardson for Means
Structured Means
Entrywise Accurate Riccati
Time Series Estimation
Rank Structured Riccati
Linearizations
Structures
Generalized Eigenproblems
CMV shape reduction
Ehrlich-Aberth (MPSolve)
Semiseparable QZ

Results

Caveat:

- Research doesn't start or stop at grant boundaries
- 1 year short compared to lifetime of a research project, from idea to publication

Anyway, in the project lifetime:

- 8 relevant journal publications (1×SIMAX, 1×NLAA, 3×LAA, 1×BIT, 2×application journals)
- 5 relevant preprints/submitted
- 6 more ongoing ideas that will (hopefully!) lead to one

Expenses

What we used the grant for

- Conferences: Providence (USA), Trieste, Lausanne (Suisse), Cagliari
- Research visits: @C. Schröder, @G. Nguyen

What we didn't use it for

- Summer school for L. Robol (completely financed by organizers)
- Incoming visits (bureaucracy...) :(
  (mostly on other funds, as well as other visits)

Thanks

- My colleagues in the project: Bini, Del Corso, Gemignani, Iannazzo, Meini, Noferini, Robol
- Your attention till this late in the evening
- INDAM/GNCS for this grant!