Counting Fiedler pencils with repetitions

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Abstract

We introduce a new notation based on diagrams to deal with Fiedler pencils with repetitions (FPR), and use it to solve several counting problems. In particular, we give explicit recurrences to count the number of FPRs of a given degree d, the number of symmetric, palindromic and antipalindromic ones (where the latter two structures are intended in the sense of [4]). We relate these structures to the presence of symmetries in the associated diagrams.

Keyword: Matrix polynomials, linearizations, Fiedler pencils with repetitions
MSC 2010: 65F15, 15A18, 15A22

1 Introduction

In this paper, we deal with several combinatorial problems and properties of a family of pencils called *Fiedler pencils with repetition* (FPR). These objects, introduced by M. Fiedler [6] for scalar polynomials and later extended to the matrix case [1, 14], are generalizations of the companion pencil constructed as products of elementary matrices that satisfy certain commutation relations. The main interest in Fiedler pencils comes from the study of linearizations of a matrix polynomial $A(x) = \sum_{i=0}^{d} A_i x^i \in \mathbb{C}^{n\times n}[x]$. A *linearization* is a pencil that can be constructed easily from the entries of a matrix polynomial $A(x)$, and has the same eigenvalues and partial multiplicities as the matrix polynomial [9]. Fiedler pencils are a large family of linearizations, and inside this family it is possible to identify some members with specific structures, for instance symmetric or palindromic pencils. There has been considerable research interest in the past years in finding new methods to produce linearizations (and, in particular, structure-preserving linearizations; see for instance [2, 11]) and studying their numerical properties.

In this paper, we introduce new notation and terminology to deal with Fiedler pencils and Fiedler pencils with repetitions; most notably, we use diagrams that represent the action of Fiedler pencils on vectors, like the computation graphs used in error analysis or the butterfly diagrams used in the study of FFT. These diagrams are not only useful for visualization, but allow simplifications to some proofs through the use of concepts such as 'the lowest horizontal segment in a path'.

With this notation at hand, we can give explicit canonical forms, characterizations and constructions for members of this family that have special structures. Structured FPRs have already been studied in [3, 4, 5], but the characterizations given there do not allow one to construct explicitly all the Fiedler pencils that satisfy certain conditions, or find out easily how many there are. Here we expand on their work: our new formalism and constructions allow us, for instance, to solve combinatorial problems such as 'how many different FPRs are there with a given degree'. In particular, we give recurrences or explicit expressions for the number of Fiedler pencils possessing specific structures (symmetric, palindromic and antipalindromic).

2 Structured matrix polynomials

In the following, $d \geq 1$ is a fixed integer and $A(x) = \sum_{i=0}^{d} A_i x^i \in \mathbb{C}^{n\times n}[x]$ is a matrix polynomial. We do not assume that $A_d \neq 0$, but allow for the possibility that d (called *grade* of $A(x)$) is higher than the true degree of the matrix polynomial.

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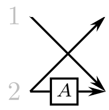


Figure 1: The diagram corresponding to $G(A)$.

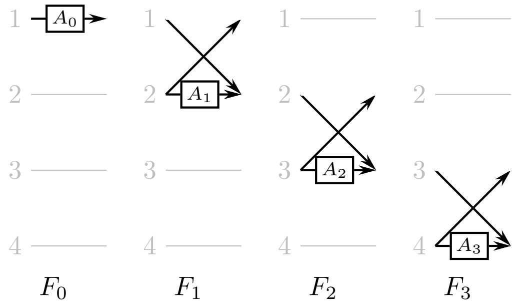


Figure 2: The diagrams corresponding to F_i for varying i and $d = 4$.

With the symbol $M^\star$, $\star \in \{*, T\}$ we denote either the transpose conjugate M^* or the complex transpose M^T of a matrix $M \in \mathbb{C}^{n\times n}$; all the results that we present in this paper hold in both cases.

A matrix polynomial is called *symmetric* if $A_i = A_i^\star$ for each i. It is called *palindromic* if $A_i = A_{d-i}^\star$ for each i, and *antipalindromic* if $A_i = -A_{d-i}^\star$ for each i. When d is odd, $A(x)$ is palindromic if and only if $A(-x)$ is antipalindromic.

3 Diagram notation for Fiedler matrices

In this section, we define Fiedler matrices associated to a given matrix polynomial $A(x) = \sum_{i=0}^d A_i x^i \in \mathbb{C}^{n\times n}[x]$. In the following, all matrices have $n \times n$ blocks. For each $k \in \mathbb{N}$ (zero included), the symbol I_k denotes an identity matrix of size k; when $k = n$, we can omit the subscript, i.e., $I := I_n$. Our basic building blocks are matrices of the form

$$G(A) := \begin{bmatrix} 0 & I \\ I & A \end{bmatrix}, \tag{1}$$

where $A \in \mathbb{C}^{n\times n}$. We represent them using diagrams like the one in Figure 1. This drawing represents visually the action of $G(A)$ on (block) row vectors; indeed, the construction of the vector

$$\begin{bmatrix} w_1 & w_2 \end{bmatrix} = \begin{bmatrix} v_1 & v_2 \end{bmatrix} G = \begin{bmatrix} v_2 & v_1 + v_2 A \end{bmatrix}$$

corresponds to following the paths marked with arrows from the left to the right, multiplying by A when the box is encountered and summing where the two arrow tips meet. We now consider block matrices with d blocks, with $d > 2$, and define *elementary Fiedler matrices*

$$\begin{aligned} F_0 &:= \begin{bmatrix} A_0 & \\ & I_{n(d-1)} \end{bmatrix}, \\ F_i &:= \begin{bmatrix} I_{n(i-1)} & & \\ & G(A_i) & \\ & & I_{n(d-i-1)} \end{bmatrix}, \qquad \text{for } i = 1, 2, \dots, d-1, \end{aligned}$$

where the matrices A_i, $i = 0, 1, \dots, d-1$, are arbitrary $n \times n$ matrices. These matrices act nontrivially each on a pair of adjacent blocks; we visualize them using similar diagrams with more rows, such as the one in Figure 2. Notice the grey horizontal lines that serve as a reminder that nothing happens on the block components that are unaffected by the transformation.

A *Fiedler product* is the formal product of a sequence of elementary Fiedler matrices

$$F = F_{i_1} F_{i_2} \cdots F_{i_\ell}, \quad 0 \le i_k < d \text{ for each } k. \tag{2}$$

Each elementary Fiedler matrix F_j may appear multiple times (or none at all) in it. We denote by $\operatorname{rev}(F)$ the Fiedler product where the elementary Fiedler matrices are taken in the reverse order, i.e. $\operatorname{rev}(F_{i_1} F_{i_2} \cdots F_{i_\ell}) = F_{i_\ell} F_{i_{\ell-1}} \cdots F_{i_1}$. We speak of 'formal product' because we are interested in the sequence of factors. When we need to refer to the actual matrix obtained by carrying out the multiplications, we call it $\mathcal{M}(F)$.

In the following, we are mostly interested in determining whenever Fiedler matrices are equal for every choice of the matrix polynomial $A(x)$; hence, when we write $\mathcal{M}(F) = \mathcal{M}(G)$, we mean that they are equal for *all* admissible values of the polynomial $A(x)$; that is, if they are equal when the coefficients A_i are regarded as (non-commuting) indeterminates. Equality for specific values of the A_i (for instance, if they are all equal to 0) is merely a coincidence and does not concern us.

We say that two products F and G are *equivalent*, and we write $F \sim G$, if one can be obtained from the other by repeatedly swapping pairs adjacent factors $F_i F_j$ with $|i - j| > 1$: for instance,

$$F_3 F_0 F_2 F_1 F_3 F_0 \sim F_0 F_3 F_2 F_3 F_1 F_0 \sim F_3 F_2 F_0 F_1 F_0 F_3. \tag{3}$$

Two elementary matrices F_i and F_j commute whenever $|i - j| > 1$, because they act on different rows. Hence, if $F \sim G$ then $\mathcal{M}(F) = \mathcal{M}(G)$, which is the motivation to consider this specific equivalence relation. As we shall see in Section 6, this is an 'if and only if' for a large subset of Fiedler products (*operation-free products*).

We represent a product (2) as a diagram in which the diagrams of the factors are concatenated horizontally in the same order as in the product, as in Figure 3. Diagrams corresponding to equivalent products can be

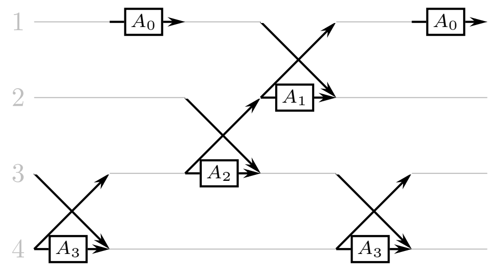


Figure 3: The diagram of $F = F_3 F_0 F_2 F_1 F_3 F_0$

obtained one from another by sliding horizontally the diagram elements without altering the interconnection of the diagram, because when $|i - j| > 1$ the elements acts on different rows. Moreover, in this case we can draw a tighter version of the diagram by putting commuting blocks in the same column, one on top of the other, as shown in Figure 4. Indeed, Figures 3 and 4 differ only by the horizontal spacing, but they have the same interconnections.

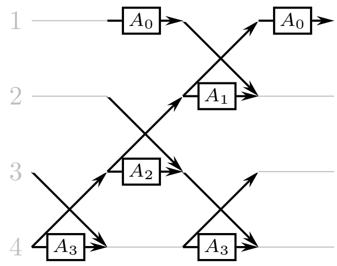


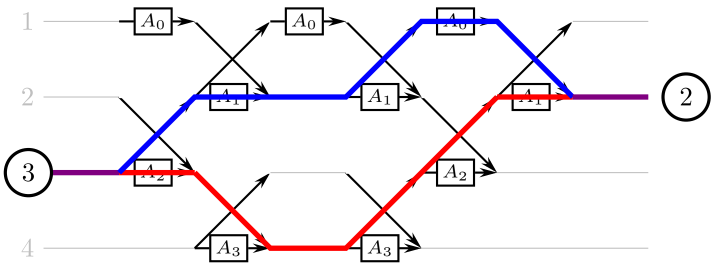
Figure 4: The diagram of the same matrix F as in Figure 3, compressed.

The main reason for considering these diagrams is that they represent visually the action of a Fiedler matrix. We call *path* a polygonal line obtained by following the lines on the diagram, going always left to right following the direction of the arrows, and possibly going through a certain number of the boxes labelled with a matrix.

The following result holds.

Theorem 1 (Block-path correspondence)**.** *Let F be a Fiedler product. The block (i, j) of $\mathcal{M}(F)$ is a (possibly empty) sum of terms of the form $A_{k_1}A_{k_2}\cdots A_{k_\ell}$; each term $A_{k_1}A_{k_2}\cdots A_{k_\ell}$ appears in the sum if and only if the diagram associated to F contains a continuous polygonal path going left to right in the direction of the arrows which starts in row i, ends in row j, and crosses ℓ boxes containing respectively $A_{k_1}, A_{k_2}, \dots, A_{k_\ell}$ (in this order).*

A formal proof can be obtained easily by induction on the number of elementary Fiedler factors in F. An example, which hopefully makes this description clearer, is in Figure 5.



$$\mathcal{M}(F_0F_2F_1F_3F_0F_1F_3F_0F_2F_1) = \begin{bmatrix} 0 & A_0^2 & A_0A_1 & 0 \\ I & A_1 & A_2 & A_3 \\ A_2 & {\color{blue}A_1A_0} + {\color{red}A_2A_1} & A_1^2 + A_2^2 + A_0 & A_2A_3 \\ A_3 & A_3A_1 & A_3A_2 & A_3^2 + I \end{bmatrix}$$

Figure 5: The summands in the $(3, 2)$ block of $\mathcal{M}(F)$ correspond to all possible paths that start in row 2 on the left of the diagram and end in row 3 on its right. In this case there are two such paths, one (drawn in red) going through two boxes, first the one labeled A_1 and then A_0, and one (drawn in blue) going through A_2 and then A_1.

A second tool that can be used to work with Fiedler products is the following.

Definition 1. For each $k = 0, \dots, d$, the *layer* $\mathcal{L}_{k-1:k}(F)$ of a Fiedler product (2) is the sequence formed by the elementary Fiedler factors of F of the kinds F_{k-1} and F_k, taken in the order in which they appear in F.

Example 1. Let $F = F_2F_3F_4F_3F_0F_1F_2F_0$. Then, $\mathcal{L}_{1:2}(F) = (F_2, F_1, F_2)$, and $\mathcal{L}_{2:3} = (F_2, F_3, F_3, F_2)$.

Note that we have defined the layers $\mathcal{L}_{-1:0}(F)$ and $\mathcal{L}_{d-1:d}(F)$ as well, even if there exist no F_{-1} or F_d factors. So, for the product in Example 1, $\mathcal{L}_{-1:0}(F) = (F_0, F_0)$ and $\mathcal{L}_{4:5}(F) = (F_4)$.

If $F \sim G$, then $\mathcal{L}_{k-1:k}(F) = \mathcal{L}_{k-1:k}(G)$ for each k: indeed, since in the definition of our equivalence relation we cannot swap F_{k-1} and F_k, the order in which they appear is uniquely determined.

The layer $\mathcal{L}_{k-1:k}(F)$ encodes the sequence of boxes and gaps appearing on the horizontal row labelled k on the diagram.

4 Operation-free products

A Fiedler product $F = F_{i_1}F_{i_2}\dots F_{i_\ell}$, with $0 \le i_j < d$ for each j, is called *operation-free* if each $n \times n$ block of $\mathcal{M}(F)$ contains either 0, I or one of the matrices A_k (for some k). For instance,

$$\mathcal{M}(F_3F_2F_0F_1F_0) = \begin{bmatrix} 0 & A_0 & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \\ A_0 & A_1 & A_2 & A_3 \end{bmatrix}$$

is operation-free, while the example of Figure 3,

$$\mathcal{M}(F_3F_0F_2F_1F_3F_0) = \begin{bmatrix} 0 & A_0 & 0 & 0 \\ 0 & 0 & 0 & I \\ 0 & 0 & I & A_3 \\ A_0 & A_1 & A_3 & A_2 + A_3^2 \end{bmatrix},$$

is not, because the block $(4,4)$ contains the expression $A_2 + A_3^2$.

We give here a characterization of operation-free products using diagrams.

Lemma 2. *For a Fiedler product F, the following are equivalent.*

1. *F is operation-free;*
2. *in the diagram associated to F, no path contains more than one box;*
3. *No layer $\mathcal{L}_{k-1:k}$ contains two consecutive F_{k-1} elements.*

Proof. We prove a circular chain of implications between the converses of each statement.

not 1 $\implies$ not 2 Assume that F is not operation-free; this means that in one of the blocks $\mathcal{M}(F)_{(i,j)}$ we need to perform (at least) an addition or a multiplication. By the block-path correspondence, a multiplication means that the same path contains two successive boxes, and we are done. So we focus on the case of an addition: by the block-path correspondence, this means that there are two different paths which go from row i to row j. An example is in Figure 5. In this case, a path starting from row i passes through an element F_h, where it separates into two different sections which join again at an element F_k (possibly all these things happening at different heights in the diagram), and then terminate at the right end of the diagram in row j. When this happens, the lower section of the path starts and ends by passing through the boxes belonging to F_h and F_k.

not 2 $\implies$ not 3 Our hypothesis is now that there is a section of a path which starts and ends by going through two different boxes (as, for instance, the red section in Figure 5). When this happens, consider the horizontal segment belonging to this section of the path which is at the lowest height k. At its end, there must be two F_{k-1} elements, and since the segment is uninterrupted there are no F_k elements in between.

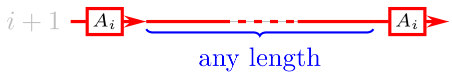


Figure 6: Path section forbidden in an operation-free product. The (uninterrupted) horizontal segment can have an arbitrary length, including zero.

not 3 $\implies$ not 1 We have to prove that if the configuration in Figure 6 appears in the diagram, then the product is not operation-free. This statement is obvious: in view of the block-path correspondence, the path (i,j) containing this configuration goes through two boxes one after the other, and thus computing the element in block $\mathcal{M}(F)_{(i,j)}$ requires a product.

□

An result equivalent to $1 \iff 3$ already appeared in literature in the form of the following statement.

Corollary 3 (Successor-infix property, SIP [14])**.** *A Fiedler product F is operation-free if the following property holds for each $k = 1, 2, \dots, d$: for each pair of identical elementary Fiedler factors F_{k-1} appearing in the product F, there must be a factor of type F_k in an intermediate position between them.*

Lemma 2 has the following consequence.

Corollary 4. *Let F be an operation-free Fiedler product. Then, no path across its diagram contains an arrow pointing towards south-west $\searrow$ followed (at some point) by an arrow pointing towards north-west $\nearrow$. In particular, once a path goes below a horizontal line, it stays below it.*

Proof. Suppose, instead, that the path passes through the south-west arrow of an element F_h and then through the north-west arrow of an element F_k. Then, there is another path sharing the same central section, but going through the boxes of elements F_h and F_k instead of following the arrows. In particular, this path goes through two boxes, so F is not operation-free. □

Corollary 4 is not an "if and only if": a counterexample is $F = F_0^2$. The only case when it fails, though, is when the shape depicted in Figure 6 appears in the uppermost row, because the elements F_0 do not have diagonal arrows.

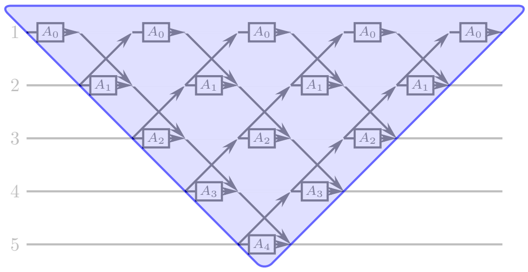


Figure 7: A diagram of the "densest possible" Fiedler product with $d = 5$. Its elements can be drawn inside the triangular region shaded in blue.

Corollary 5. *Suppose that h factors of the kind F_k appear in an operation-free Fiedler product. Then, there can be at most $h+1$ factors F_{k-1}: one between each consecutive pair of F_k, plus one to the left and one to the right of all of them. Hence, any operation-free product contains the factor F_{d-1} at most once, the factor F_{d-2} at most twice, ..., the factor F_0 at most d times.*

The operation-free Fiedler matrix with the most factors has a diagram with the shape of an upside-down triangle. An example is shown in Figure 7. All other operation-free Fiedler matrices can be obtained by taking diagrams which have only a subset of its elements.

5 The middle standard form

In this section, we introduce a standard form for operation-free products and for their diagrams, which we call *middle standard form* (MSF). It is analogous to the *column standard form* (CSF) and *row standard form* (RSF) introduced in [14] (which we briefly discuss in Remark 8), but designed to better highlight the properties of structured matrices and pencils.

Theorem 6. *For each operation-free Fiedler product F,*

$$F \sim SMT,$$

where

- $M = F_{k_1} F_{k_2} \cdots F_{k_m}$ *for a sequence* $0 \leq k_1 < k_2 < \cdots < k_m < d$ *with* $k_{j+1} - k_j \geq 2$*;*
- $S = S_m S_{m-1} \cdots S_1$*, and each factor S_j is either the identity matrix or the product* $F_{s_j} F_{s_j - 1} \cdots F_{k_j - 2} F_{k_j - 1}$ *for some $s_j < k_j$;*
- $T = T_1 T_2 \cdots T_m$*, and each T_i is either the identity matrix or the product* $F_{k_i - 1} F_{k_i - 2} \cdots F_{t_i + 1} F_{t_i}$ *for some $t_i < k_i$.*

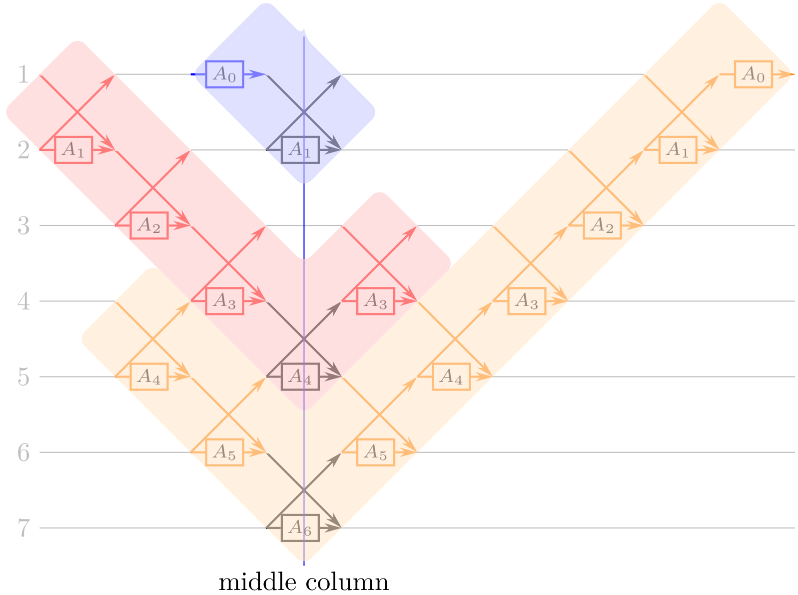
Moreover, one can draw a diagram associated to F in which for each $h = 1, 2, \ldots, m$, the three factors $S_h F_{k_h} T_h$ form a V shape (with the two arms possibly of different lengths), and all the factors belonging to M are stacked one on top of the other in the central column.

For an example of the diagram, see Figure 8.

Proof. We shall first describe how to construct the diagram as described, and then we can read off the factorization from it.

Let $F = F_{i_1} F_{i_2} \cdots F_{i_\ell}$ by an operation-free Fiedler product with $0 \leq i_j < d$. The construction that we make is inductive, starting from the factor F_j with largest j, and building the diagram upwards one row at a time. Let $i_{\max} := \max_{j=1,2,\ldots,\ell} i_j$. There must be a single $F_{i_{\max}}$ factor for F to be operation-free; we start by drawing it in the middle of our diagram, along the central column. Then, we suppose that we have laid out on the diagram all the factors of the form F_i for $i > \bar{\imath}$, and we show how to position the factors $F_{\bar{\imath}}$ in the row directly above them.

Each factor $F_{\bar{\imath}}$ present in the product has to appear in a specific position relative to the row immediately below it: between two given elements, or left of all of them, or right of all of them. Within these constraints,



$$F = \underbrace{F_4F_5}_{S_3}\underbrace{F_1F_2F_3}_{S_2}\underbrace{F_0}_{S_1}\underbrace{F_1F_4F_6}_{M}\underbrace{I}_{T_1}\underbrace{F_3}_{T_2}\underbrace{F_5F_4F_3F_2F_1F_0}_{T_3}$$

Figure 8: A Fiedler product in middle standard form and the corresponding diagram, with the 'V shapes' drawn in different colors.

they can be moved freely left or right (which corresponds to altering the order of commuting factors). To choose where to put them, we follow this rule: *we draw each element as close as possible to the central column.* For instance, see the two examples in Figure 9, in which we have drawn only the rows in $\mathcal{L}_{\bar{\imath}:\bar{\imath}+1}$. If a factor $F_{\bar{\imath}}$ comes between two $F_{\bar{\imath}+1}$ that are both to the right of the middle column, then we draw it close to the left one; if they are both to the left of the middle column, then we draw it close to the right one; and if they are on two different sides then we draw it in the middle column.

If one follows these rules, the diagram elements are arranged naturally in sequences that are first increasing and then decreasing in the shape of a V, e.g., $F_{s_h}F_{s_h+1}\cdots F_{k_h-1}F_{k_h}F_{k_h-1}\cdots F_{t_h+1}F_{t_h}$. We call $k_1, k_2, \dots, k_m$ the indices of the factors appearing in the middle column of the diagram; there cannot be two with consecutive indices F_i and F_{i+1} in this column, since we can only stack in the same column commuting matrices, by our rules to construct diagrams. We order them by index from the smallest to the largest to form $M = F_{k_1}F_{k_2}\cdots F_{k_m}$.

This constructive procedure builds a diagram equivalent to the one for F by only swapping pairs of commuting matrices, so it implies that $F \sim SMT$. $\square$

Some results that relate the indices of the MSF with the content of the blocks of $\mathcal{M}(F)$ can be obtained visually from the diagrams; for instance the following one.

Lemma 7. *Let F be an operation-free Fiedler product with middle standard form SMT, with the indices k_i, s_i, t_i as above. Then,*

$$\mathcal{M}(F) = \begin{bmatrix} \mathcal{B} & 0 \\ 0 & I_{n(d-1-k_m)} \end{bmatrix} \quad \textit{for some } \mathcal{B} \in \mathbb{C}^{n(k_m+1)\times n(k_m+1)}. \tag{4}$$

Moreover, the last block row of $\mathcal{B}$ contains

- $\begin{bmatrix} 0 & 0 & \cdots & 0 & I & A_{t_m} & A_{t_m+1} & \cdots & A_{k_m} \end{bmatrix}$ *if $t_m > 0$, or*
- $\begin{bmatrix} A_0 & A_1 & \cdots & A_{k_m} \end{bmatrix}$ *if $t_m = 0$.*

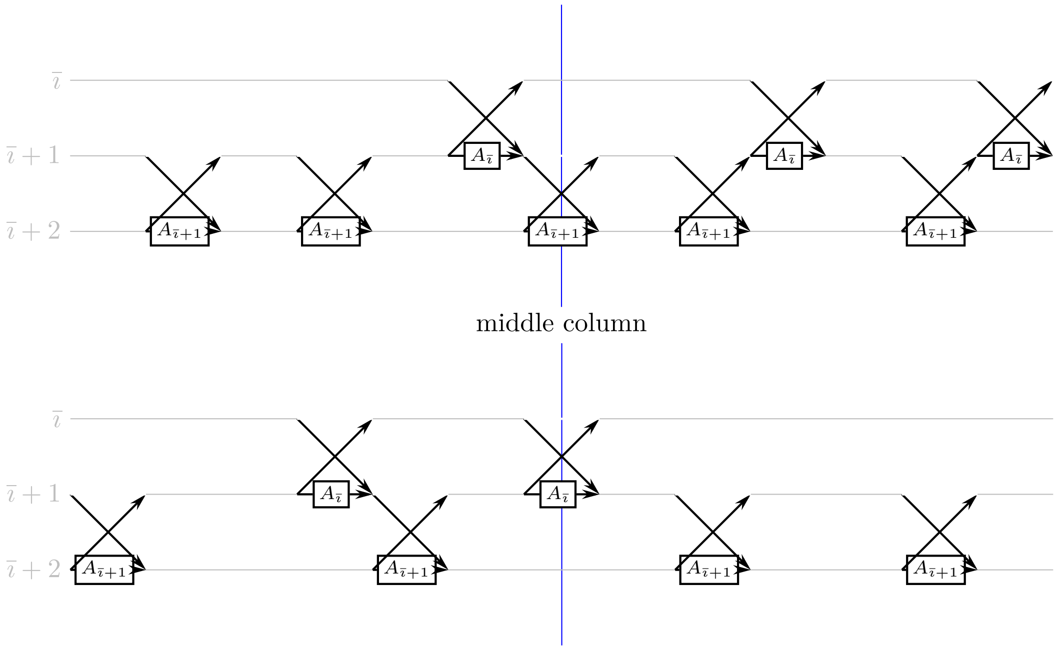


Figure 9: An example of how to lay down the factors $F_{\bar\imath}$ (with respect to the row below) when constructing the MSF diagram of an operation-free Fiedler product.

Similarly, the last block column of $\mathcal{B}$ contains

$$\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ I \\ A_{s_m} \\ A_{s_m+1} \\ \vdots \\ A_{k_m} \end{bmatrix} \textit{ if } s_m > 0, \textit{ or } \begin{bmatrix} A_0 \\ A_1 \\ \vdots \\ A_{k_m} \end{bmatrix} \textit{ if } s_m = 0.$$

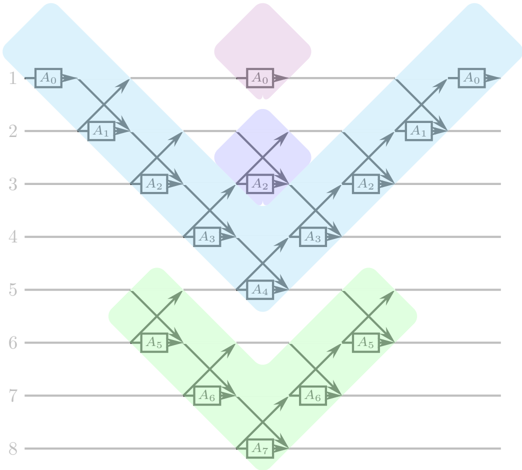
Proof. Note that there are no elementary factors with index larger than k_m; hence the last $d-1-k_m$ horizontal rows of the diagram contain uninterrupted straight lines, and the shape (4) follows. In view of the block-path correspondence, to determine the content of the last row of $\mathcal{B}$ it is sufficient to consider the paths that start from the row labeled k_m+1 of the MSF diagram for F. These are:

- a straight horizontal path that goes through a block A_{k_m}; this corresponds to A_{k_m} in block (k_m+1, k_m+1)
- for each elementary factor F_i present in T_m, a path that bends up just before reaching A_{k_m}, proceeds diagonally for a while, and then goes through the block A_i corresponding to said factor;
- if $t_m \neq 0$, an additional path that proceeds the same way but then does not go through a block.

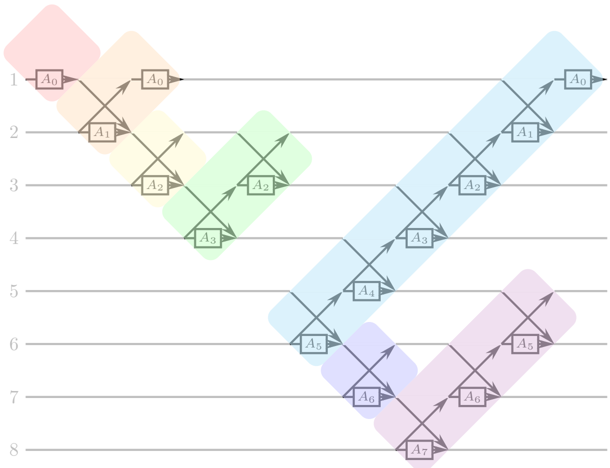
The content of the last column of $\mathcal{B}$ can be determined analogously, by checking the paths that end in the row labeled k_m+1 on the diagram. □

Remark 8. *The column standard form and row standard form introduced in [14] also correspond to special ways of choosing how to draw the diagrams, as following:*

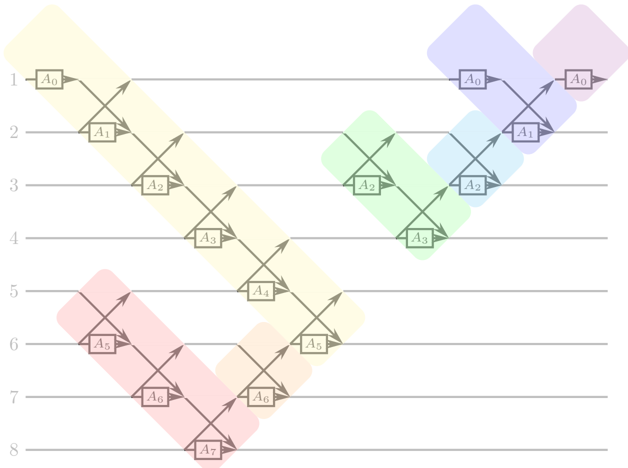
- *The row standard form is obtained by drawing each diagram element inside the region shaded in blue in Figure 7 as far left as possible, and taking the product $R_1R_2\cdots R_d$, where $R_i = F_{i-1}F_{i-2}\ldots F_{r_i}$ are the products that correspond to the diagonal sequences of diagonal elements in direction $\nearrow$ obtained with this procedure.*



$F_5F_6F_0F_1F_2F_3F_0F_2F_4F_7F_3F_2F_1F_0F_6F_5$
Middle standard form



$F_0F_1F_0F_2F_3F_2IF_5F_4F_3F_2F_1F_0F_6F_7F_6F_5$
Row standard form



$F_5F_6F_7F_6F_0F_1F_2F_3F_4F_5IF_2F_3F_2F_0F_1F_0$
Column standard form

Figure 10: The three different standard forms of the same (symmetric) Fiedler product F.

- *The column standard form is obtained by drawing each diagram element inside the region shaded in blue in Figure 7 as far right as possible, and taking the product $C_dC_{d-1}\cdots C_1$, where $C_i = F_{c_i}F_{c_{i+1}}\dots F_{i-1}$ are the products that correspond to the diagonal sequences of diagonal elements in direction $\nwarrow$ obtained with this procedure.*

An example with the three standard forms of the same F is displayed in Figure 10.

6 The role of equivalence

The following result pins down the exact meaning of equivalence.

Theorem 9. *Let F and G be two operation-free Fiedler products. Then, the following are equivalent.*

1. $F \sim G$;
2. $\mathcal{M}(F) = \mathcal{M}(G)$ *for each matrix polynomial* $A(x)$;
3. $\mathcal{M}(F) = \mathcal{M}(G)$ *for a matrix polynomial* $A(x)$ *such that* $A_i \neq I$ *for each* $i = 0, 1, \dots, d$;
4. $\mathcal{L}_{k-1:k}(F) = \mathcal{L}_{k-1:k}(G)$ *for each* k;

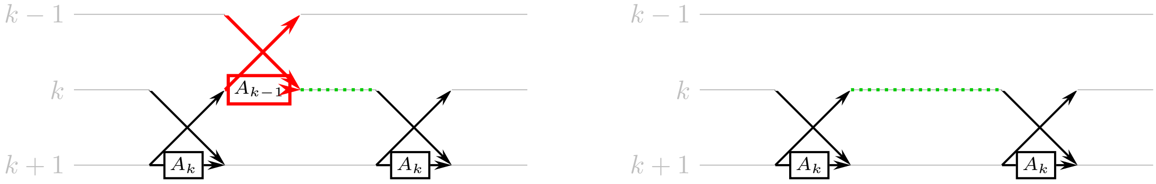


Figure 11: Parts of diagrams that are different if two layers do not coincide. These pictures are used in the proof of Theorem 9, the first corresponds to product F and the second to G.

5. *The middle standard forms of F and G are equal.*

Proof. $1 \implies 2$ This part is clear since F_i and F_j commute whenever $|i-j|>1$.

$2 \implies 3$ This is obvious.

$3 \implies 4$ We prove that different layers imply different matrices. Let k be the largest index for which $\mathcal{L}_{k-1:k}(F) \neq \mathcal{L}_{k-1:k}(G)$. The only way in which the two layers can be different is when F contains an elementary factor F_{k-1} between two successive F_k (or the beginning/end of the product) while G does not (or vice versa). The parts of the two diagrams that differ are pictured in Figure 11. Note that, by our choice of k, the parts of the two diagram that are below row k are identical. Consider the paths in the two diagram that go through the horizontal segments drawn in green (dotted) on the picture. By Corollary 4, the remaining part of those paths lies entirely below row k of the diagram. By our choice of k, the parts of the two diagrams that lie below row k are identical. Hence the paths in the two diagram both go from the same row i to the same row j, and the only difference between the two is that one goes through the box in red containing A_{k-1}, and the other doesn't. By Lemma 2, there are no other boxes on this path, because F is operation-free. This means that the block (i,j) of $\mathcal{M}(F)$ contains A_{k-1}, while the same block of $\mathcal{M}(G)$ contains I, hence the two matrices are different (if $A_{k-1} \neq I$).

$4 \implies 5$ This holds because our construction of the MSF only uses the layers of a Fiedler product.

$5 \implies 1$ The relation $\sim$ is an equivalence relation, so $F \sim SMT \sim G$ implies $F \sim G$.

$\square$

In the previous theorem, the implications $1 \implies 2 \implies 3$ are easy and well-known. In Proposizion 2.12 in [3] the implication $1 \iff 4$ is proved also when F and G are not operation free. The part $5 \implies 1$ (although with the column and row standard form instead of the MSF introduced here) is in [14], and the results obtained there relating the content of rows and columns can be used to derive a result similar to $3 \implies 5$, although this is never mentioned explicitly.

Remark 10. *We conjecture that the equivalences $1 \iff 2 \iff 3 \iff 4$ in Theorem 9 hold even if one does not assume that F and G are operation-free.*

7 Transposition and symmetric matrix polynomials

Suppose that the matrix polynomial $A(x) = \sum_{i=0}^{d} A_i x^i$ on which our construction is based is symmetric. Then, $F_i = F_i^\star$ for each i. Moreover, by the properties of transposition,

$$\left(\mathcal{M}(F_{i_1} F_{i_2} \cdots F_{i_\ell})\right)^\star = \mathcal{M}(F_{i_\ell} F_{i_{\ell-1}} \cdots F_{i_2} F_{i_1}),$$

that is, to transpose a Fiedler product it is sufficient to reverse the order of its factors. This can be reformulated with a simple interpretation in terms of the associated diagrams.

Lemma 11. *Let F be a Fiedler product. A diagram representing $F^\star$ can be obtained by flipping horizontally a diagram representing F.*

By *flipping horizontally* here we mean reflecting it along a vertical axis. For instance, the two bottom diagrams in Figure 10 can be obtained one from the other via a horizontal flip.

With a slight abuse of notation, we say that a Fiedler product F is *symmetric* if the matrix $\mathcal{M}(F)$ satisfies $\mathcal{M}(F) = \left(\mathcal{M}(F)\right)^\star$ for every symmetric matrix polynomial $A(x)$. In view of Theorem 9, this is equivalent to asking that $F \sim \operatorname{rev} F$. Using Lemma 11, it is easy to show that some Fiedler products F are symmetric.

Lemma 12. *Let F a Fiedler product. If the diagram has a vertical symmetry axis, then F is symmetric.*

The converse does not hold: since there is some freedom in the horizontal positioning of the blocks, different diagrams can be associated to the same F; some may be symmetric, some may be not. For instance, the Fiedler product appearing in Figure 10 is symmetric; this is apparent from the MSF diagram, which is symmetric along its middle column, but not at all from the CSF or the RSF one.

Indeed, the MSF always displays when an operation-free Fiedler product is symmetric.

Lemma 13. *Let F be an operation-free Fiedler product with middle standard form $F \sim SMT$. Then, the following are equivalent.*

1. *F is symmetric;*
2. *$s_i = t_i$ for each $i = 1, 2, \dots, m$, that is, $T = \operatorname{rev} S$;*
3. *The MSF diagram of F has reflection symmetry along its central column.*

Proof. $1 \implies 2$ Let $F \sim SMT$ be the MSF of F. We have $F \sim \operatorname{rev} F \sim (\operatorname{rev} T)M(\operatorname{rev} S)$, and the three factors $\operatorname{rev} T, M, \operatorname{rev} S$ satisfy the definition of MSF; hence, by the uniqueness of the MSF, they must coincide with S, M, T, which means that $s_i = t_i$.

$2 \implies 3$ This is clear from the shape of the diagram.

$3 \implies 1$ This follows by Lemma 12.

□

With the middle standard form at our disposal, it is easy to derive the number of symmetric Fiedler products of a given degree.

Theorem 14. *Let Σ_d be the number of different (non-equivalent) operation-free symmetric Fiedler products of degree d. Then Σ_d satisfies the recursion*

$$\Sigma_d = \Sigma_{d-1} + d\,\Sigma_{d-2}, \quad \Sigma_1 = 2,\ \Sigma_2 = 4. \tag{5}$$

Proof. For $d = 1$, $\Sigma_1 = 2$ since the only possible Fiedler products are F_0 and I. For $d = 2$, we have the previous ones plus two more products, F_1 and $F_0F_1F_0$. Let now $d > 2$. We have two cases, which are also depicted in Figure 12:

- F does not contain the factor F_{d-1}. Then F is also a symmetric Fiedler product of degree $d-1$; there are Σ_{d-1} possibilities for this case.
- F contains the factor F_{d-1}. Then its MSF can be obtained by adding a V-shape to the MSF of a Fiedler product $F^{(d-2)}$ of degree $d-2$. There are Σ_{d-2} ways to choose $F^{(d-2)}$ and d possible choices for the length of the two arms of the V.

The Fiedler products obtained in this way all have different MSF, hence they are non-equivalent.

□

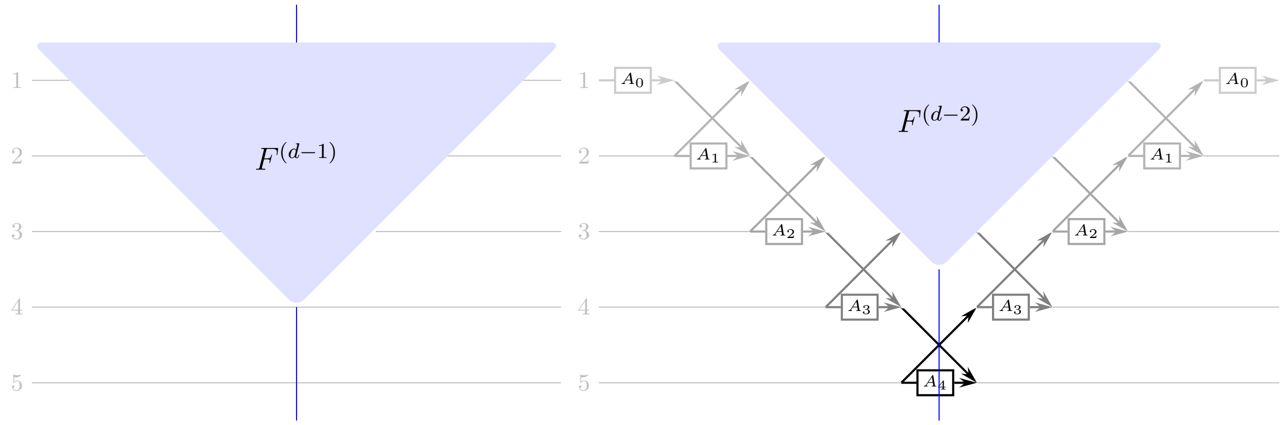
The recurrence (5) is widely studied in combinatorics and algebraic geometry: it is the number of Young tableaux of $d+1$ cells [8]. See also [13, Sequence A000085] for more properties and references.

8 Infix pairs

In this section, we study infix pairs, which are the building block to construct and understand Fiedler pencils with repetitions.

Definition 2. We call *infix pair of height g* a pair of matrices $(\mathcal{F}, \mathcal{G})$, with $\mathcal{F}, \mathcal{G} \in \mathbb{C}^{nd \times nd}$ such that $\mathcal{F} = \mathcal{M}(LQR)$ and $\mathcal{G} = \mathcal{M}(LR)$, where

- Q is a Fiedler product which contains each of the factors $F_0, F_1, \dots, F_{g-1}$ exactly once (in some order);
- L and R are Fiedler products which may only contain the factors $F_0, F_1, \dots, F_{g-2}$;



(a) (b)

Figure 12: The two possibilities to construct a symmetric Fiedler product of degree $d = 5$ starting from a lower-degree one.

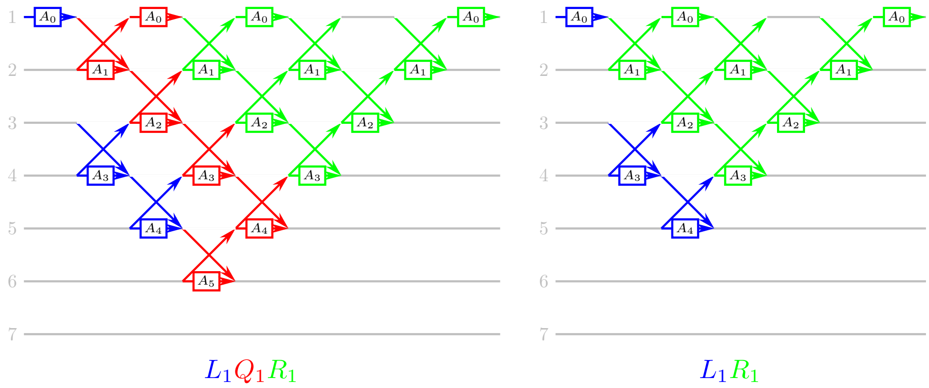


Figure 13: The diagrams of the infix pair (with $d = 7$ and $g = 6$) generated by $L_1 = F_0F_3F_4$, $Q_1 = F_1F_0F_2F_3F_5F_4$, $R_1 = F_1F_0F_2F_1F_3F_2F_1F_0$, with the three factor depicted in different colors.

- LQR is operation-free.

A triple of Fiedler products L, Q, R satisfying Definition 2 is called a set of *generators* for $(\mathcal{F}, \mathcal{G})$.

All our matrices are $nd \times nd$, but the factors that can appear in a height-g infix pair are only $F_0, F_1, \dots, F_{g-1}$. Hence the matrix $\mathcal{F}$ acts like an identity matrix in the last $d-g$ blocks, while the matrix $\mathcal{G}$ (which only contains factors up to F_{g-2}) acts like an identity matrix in the last $d-g+1$ blocks, i.e., they can be partitioned as

$$\mathcal{F} = \begin{bmatrix} \widehat{\mathcal{F}} & 0 \\ 0 & I_{n(d-g)} \end{bmatrix}, \quad \mathcal{G} = \begin{bmatrix} \widehat{\mathcal{G}} & 0 \\ 0 & I_{n(d-g+1)}, \end{bmatrix} \quad \widehat{\mathcal{F}} \in \mathbb{C}^{ng \times ng}, \ \widehat{\mathcal{G}} \in \mathbb{C}^{n(g-1) \times n(g-1)}. \tag{6}$$

An example is shown using diagrams in Figure 13.

Note that the same pair of matrices can be obtained with different choices of the generators: for instance, Figures 13 and 14 show two non-equivalent choices L_1, Q_1, R_1 and L_2, Q_2, R_2 that produce the same pair $(\mathcal{F}, \mathcal{G})$, as they have the same diagrams. It is also possible that $L_1Q_1R_1 \sim L_3Q_3R_3$, but $L_1R_1 \not\sim L_3R_3$; an example is in Figure 15.

The following lemma shows that no matter how we reorder the factors in LQR, the three factors stay separated in each layer, and we can still identify which of the individual elementary factors 'belong' to L, Q or R.

Lemma 15. *Let $(\mathcal{M}(LQR), \mathcal{M}(LR))$ be an infix pair with generators L, Q, R. Let F' be any Fiedler product equivalent to $F = LQR$, obtained by swapping pairs of commuting factors. Suppose further that the elementary factors are labeled individually so that we can recognize them after swapping and reordering them. Then, in every layer $\mathcal{L}_{k-1:k}(F')$, $k = 1, 2, \dots, g-1$,*

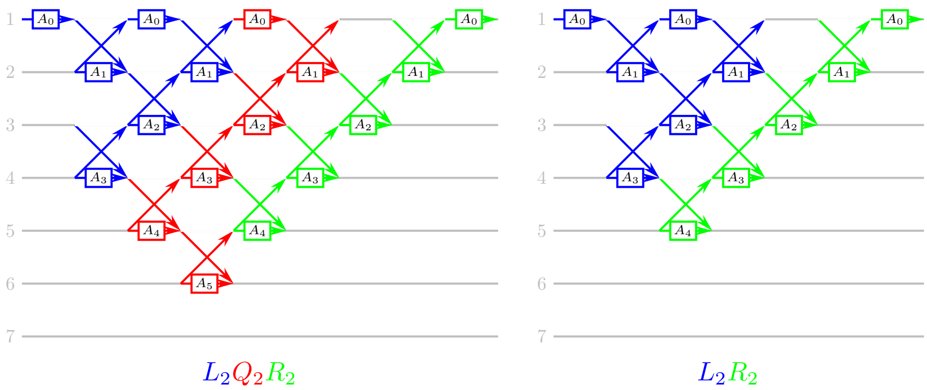


Figure 14: The diagrams of the infix pair (with $d = 7$ and $g = 6$) generated by $L_2 = F_0F_1F_3F_0F_2F_1$, $Q_2 = F_4F_3F_5F_0F_2F_1$, $R_2 = F_4F_3F_2F_1F_0$, for which $L_1Q_1R_1 \sim L_2Q_2R_2$ and $L_1R_1 \sim L_2R_2$.

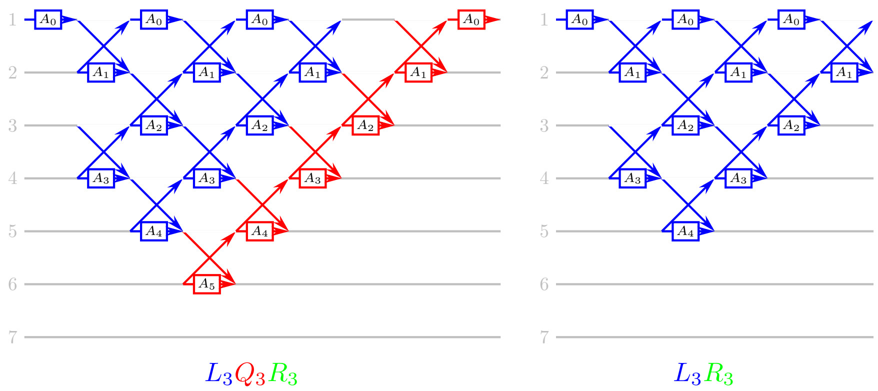


Figure 15: The diagrams of the infix pair (with $d = 7$ and $g = 6$) generated by $L_3 = F_0F_1F_3F_0F_2F_4F_1F_3F_0F_2F_1$, $Q_3 = F_5F_4F_3F_2F_1F_0$, $R_3 = I$, for which $L_1Q_1R_1 \sim L_3Q_3R_3$, but $L_1R_1 \not\sim L_3R_3$.

$$\mathcal{L}_{k-1:k}(G) = (\underbrace{F_?, \dots, F_{k-1}, F_{k-1} \dots, F_?}_{\mathcal{L}_{k-1:k}(L)}, \underbrace{F_?, \dots, F_?}_{\mathcal{L}_{k-1:k}(R)}) \quad \mathcal{L}_{k-1:k}(F) = (\underbrace{F_?, \dots, F_{k-1}, F_{k-1} \dots, F_?}_{\mathcal{L}_{k-1:k}(L)}, \underbrace{F_?, F_?}_{\mathcal{L}_{k-1:k}(Q)}, \underbrace{F_?, \dots, F_?}_{\mathcal{L}_{k-1:k}(R)})$$

$$\mathcal{L}_{k-1:k}(G) = (\underbrace{F_?, \dots, F_?, F_{k-1}}_{\mathcal{L}_{k-1:k}(L)}, \underbrace{F_{k-1}, F_?, \dots, F_?}_{\mathcal{L}_{k-1:k}(R)}) \qquad \mathcal{L}_{k-1:k}(F) = (\underbrace{F_?, \dots, F_?, F_{k-1}}_{\mathcal{L}_{k-1:k}(L)}, \underbrace{F_{k-1}, F_k}_{\mathcal{L}_{k-1:k}(Q)}, \underbrace{F_{k-1}, F_?, \dots, F_?}_{\mathcal{L}_{k-1:k}(R)})$$

Figure 16: Examples of cases 1 and 3 in the proof of Lemma 16.

- *the elementary factors which appeared in Q before the reordering are still consecutive.*
- *the elementary factors which appeared in L before the reordering are still the ones to the left of those in Q, and those which belonged to R are still to the right of Q.*

Proof. Swapping commuting factors does not alter the order in which the elements come in each layer. □

We now prove a first result about infix pairs: not only is LQR operation-free, as mandated by the definition, but LR is too. This result appears in [14, Theorem 4], with a proof which is essentially the same as ours, only without the layer formalism.

Lemma 16 ([14, Theorem 4])**.** *Let $(\mathcal{M}(LQR), \mathcal{M}(LR))$ be an infix pair with generators L, Q, R. Then, LR is operation-free.*

Proof. We work by contradiction: suppose that $G = LR$ is not operation-free. Then, by Lemma 2, for some $k \in \{1, 2, \dots, g-1\}$ the layer $\mathcal{L}_{k-1:k}(G)$ (which is the concatenation of $\mathcal{L}_{k-1:k}(L)$ and $\mathcal{L}_{k-1:k}(R)$) contains two consecutive elements equal to F_{k-1}. As shown in Figure 16we have three cases:

1. The two consecutive element of type F_{k-1} both belong to $\mathcal{L}_{k-1:k}(L)$. Then, $\mathcal{L}_{k-1:k}(F)$ (which is the concatenation of $\mathcal{L}_{k-1:k}(L)$, $\mathcal{L}_{k-1:k}(Q)$, and $\mathcal{L}_{k-1:k}(R)$) contains two consecutive F_{k-1}, too. This means that F is not operation-free, which is a contradiction.
2. The two consecutive elements of type F_{k-1} both belong to $\mathcal{L}_{k-1:k}(R)$. This case is analogous to the one above.
3. The two elements of type F_{k-1} are the last elements of $\mathcal{L}_{k-1:k}(L)$ and the first element of $\mathcal{L}_{k-1:k}(R)$. Note that $\mathcal{L}_{k-1:k}(Q)$ equals either (F_{k-1}, F_k) or (F_k, F_{k-1}); hence in either case when we concatenate the three lists to form $\mathcal{L}_{k-1:k}(F)$ we find two adjacent elements of type F_{k-1}.

□

The following lemma gives some insight on the structure of different sets of generators that define the same infix pair.

Lemma 17. *Let L_1, Q_1, R_1 and L_2, Q_2, R_2 be two different sets of generators for the same infix pair (i.e., $L_1Q_1R_1 \sim L_2Q_2R_2$ and $L_1R_1 \sim L_2R_2$). Then,*

1. *For each $k = 0, 1, \dots, g-2$, the elements of $\mathcal{L}_{k-1:k}(F)$ which lie between those belonging to Q_1 and those belonging to Q_2 are alternatingly F_{k-1} and F_k.*
2. *Let L_3, Q_3, R_3 be the set of generators of some infix pair $(\mathcal{F}', \mathcal{G}')$ satisfying $\mathcal{F} = \mathcal{F}'$ (i.e., $F = L_1Q_1R_1 \sim L_3Q_3R_3$). Suppose that in every layer $\mathcal{L}_{k-1:k}(F)$ the elements belonging to Q_3 lie between those belonging to Q_1 and those belonging to Q_2 (possibly coinciding). Then, $\mathcal{G}' = \mathcal{G}$; that is, L_3, Q_3, R_3 are a different set of generators for the same infix pair.*

Proof. We prove Part 1 by induction on g, by adding a new row at the top of an existing infix pair. Suppose that the result holds up to height $g-1$, and consider an infix pair of height g. Let us define a height-$g-1$ infix pair by removing the factors F_0 from L_1, Q_1, R_1 and shifting down by 1 each other index: that is, if $L_1 = F_{i_1}F_{i_2}\dots F_{i_\ell}$, then we define $\hat{L}_1 = F_{i_1-1}F_{i_2-1}\dots F_{i_\ell-1}$, with the convention that $F_{-1} = I_{nd}$ (and analogously $\hat{Q}_1, \hat{R}_1$). Clearly, $\hat{L}_1, \hat{Q}_1, \hat{R}_1$ are the generators of a height-$(g-1)$ infix pair $(\hat{\mathcal{F}}, \hat{\mathcal{G}})$ whose diagrams are obtained by removing the first row from those of $(\mathcal{F}, \mathcal{G})$ and relabeling.

By inductive hypothesis, Part 1 holds for the layers of $(\hat{\mathcal{F}}, \hat{\mathcal{G}})$, hence also for the layers of $(\mathcal{F}, \mathcal{G})$ apart from (possibly) the first one. Let us focus on the first layer $\mathcal{L}_{0:1}(L_1Q_1R_1)$, then. Two consecutive elements F_1, F_0 (or F_0, F_1) of this sequence belong to Q_1; we color them in red. Similarly, two consecutive elements F_1, F_0 (or F_0, F_1) belong to Q_2; we color them in purple. We present the proof in detail for the case in which the red elements are F_0, F_1 (in this order), and come before the purple elements F_1, F_0 (in this order). The other cases are analogous. Hence our setting is

$$\mathcal{L}_{0:1}(L_1Q_1R_1) = (\underbrace{?,\dots,?}_{A}, {\color{red}F_0}, {\color{red}F_1}, \underbrace{?,\dots,?}_{B}, {\color{purple}F_1}, {\color{purple}F_0}, \underbrace{?,\dots,?}_{C}). \tag{7}$$

We call A, B, C the subsequences formed by the elements to the left of the red elements, in between the red and purple, and to the right of the purple elements, respectively, as shown in (7).

Since $L_1R_1 \sim L_2R_2$, the two subsequences

$$AF_0F_1BC \text{ and } ABF_1F_0C \tag{8}$$

obtained by removing either the red or purple factors are equal. In particular, both must end with F_0C: hence B ends with F_0. If $B = (F_0)$, then we are done. Otherwise, since both subsequences in (8) end with F_1F_0C, B must end with F_1F_0, and we have now

$$\mathcal{L}_{0:1}(L_1Q_1R_1) = (\underbrace{?,\dots,?}_{A}, {\color{red}F_0}, {\color{red}F_1}, \underbrace{?,\dots,?, F_1, F_0}_{B}, {\color{purple}F_1}, {\color{purple}F_0}, \underbrace{?,\dots,?}_{C}).$$

If B contains further elements, we can prove by the same argument that it ends with F_0, F_1, F_0, and so on, until we reach the red factors.

Part 2 is also proved by induction on g. Using the same notation $\hat{}$ introduced in the previous point, we know by inductive hypothesis that $\hat{L}_1\hat{R}_1 = \hat{L}_2\hat{R}_2 = \hat{L}_3\hat{R}_3$, that is, the layers of L_1R_1, L_2R_2, L_3R_3 coincide apart from possibly the topmost one $\mathcal{L}_{0:1}(L_3R_3)$. We focus on this first layer then. By the previous part, this layer is of the form

$$\mathcal{L}_{0:1}(L_1Q_1R_1) = (\underbrace{?,\dots,?}_{A}, {\color{red}F_0}, {\color{red}F_1}, \underbrace{F_0, F_1, F_0, \dots, F_1, F_0}_{B}, {\color{purple}F_1}, {\color{purple}F_0}, \underbrace{?,\dots,?}_{C}).$$

(again, the red and purple subsequences can be either F_0, F_1 or F_1, F_0; the other cases are analogous). The hypothesis of Part 2 ensures that the two elements belonging to Q_3 lie in the region between the elements colored in red and purple (possibly overlapping with them), that is, the one in which F_0 and F_1 alternate. Now it is clear that removing any two consecutive elements F_0, F_1 or F_1, F_0 from this region leaves the same subsequence $\mathcal{L}_{0:1}(L_3R_3) = \mathcal{L}_{0:1}(L_1R_1) = \mathcal{L}_{0:1}(L_2R_2)$. $\square$

In terms of diagrams, Lemma 17 means that if we build an infix pair by choosing two different 'paths' Q_1, Q_2 inside a diagram for $\mathcal{F}$, then the space between Q_1 and Q_2 is filled with as many elements as possible, and choosing any other 'path' Q_3 inside this region results in the same infix pair; see for instance the example in Figure 17.

We can give a recurrence for the number of infix pairs.

Theorem 18. *Let Π_g be the number of distinct (non-equivalent) infix pairs of height g. Then, $\Pi_1 = 1, \Pi_2 = 3$, and for each $g \geq 3$,*

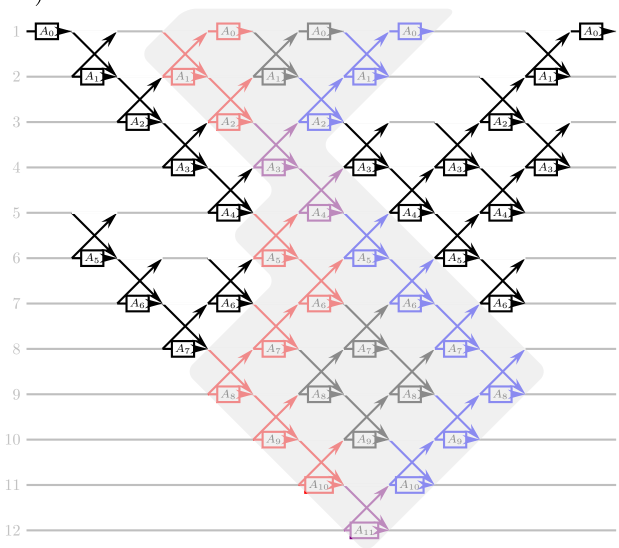
$$\Pi_g = 2g\Pi_{g-1} - (g-1)^2\Pi_{g-2}. \tag{9}$$

Proof. We present two ways to construct a height-g infix pair with generators L, Q, R starting from a height-$(g-1)$ infix pair with generators L', Q', R'.

Left extension $L = WL'$, $Q = F_{g-1}Q'$, $R = R'$, where W is either the identity or the product $F_wF_{w+1}\dots F_{g-2}$, for some $w \in \{0, 1, \dots, g-2\}$. This construction amounts to inserting an additional factor C_{g-1} into the column standard form of $L'Q'R'$.

Right extension $L = L'$, $Q = Q'F_{g-1}$, $R = R'H$, where H is either the identity or the product $F_{g-2}F_{g-3}\dots F_h$, for some $h \in \{0, 1, \dots, g-2\}$. This construction amounts to inserting an additional factor R_{g-1} into the row standard form of $L'Q'R'$.

Figure 17: An example of Lemma 17. Choosing either the red elements (Q_1) or the blue elements (Q_2) as Q gives equivalent infix pairs; hence, the shaded portion of the diagram contains as many elements as possible (Part 1 of the lemma), and choosing as Q any continuous 'path' Q_3 contained inside the shaded region gives an equivalent infix pair (Part 2).



Moreover, we present a way to construct a height-g infix pair with generators L, Q, R starting from a height-$(g-2)$ infix pair L'', Q'', R''.

Double extension $L = WF_{g-2}L''$, $Q = F_{g-1}Q''F_{g-2}$, $R = R''H$, where W is either the identity or the product $F_wF_{w+1}\dots F_{g-3}$, for some $w \in \{0,1,\dots,g-3\}$ and H is either the identity or the product $F_{g-2}F_{g-3}\dots F_h$, for some $h \in \{0,1,\dots,g-2\}$. Equivalently, one can choose $L = WL''$, $Q = F_{g-2}Q''F_{g-1}$, $R = R''F_{g-2}H$, because F_{g-2} commutes with both L'' and R'' and F_{g-1} commutes with Q''. This construction amounts to inserting an additional V shape into the middle standard form of $L''Q''R''$.

These constructions are represented in Figure 18.

We prove the following statements.

1. *Each height-g infix pair can be obtained by either left or right extension.* Indeed, consider a height-g infix pair with generators L, Q, R, and take the middle standard form $LQR \sim SMT$, with S, M, T as in Theorem 6. The unique factor F_{g-1} belongs to M, by the construction of the MSF, and $k_m = g-1$. Suppose now that the factor F_{g-2} in Q belongs to T; then, we claim that the elementary factors of S_m all belong to L. Otherwise, take the maximum possible $i < g-1$ such that the same elementary factor F_i belongs to both S_m and Q. Clearly, $i \neq g-2$, because the factor F_{g-2} in Q is after its factor F_{g-1}. By Lemma 15, the factor F_{i+1} in Q is adjacent to its factor F_i in $\mathcal{L}_{i:i+1}(LQR)$, hence it must be the factor F_{i+1} that appears in S_m. This is a contradiction, by maximality of i. Hence all factors of S_m belong to L. In particular, L, Q, R are obtained from the left extension of the infix pair obtained by deleting the factor F_{g-1} and those belonging to S_m. Analogously, if the factor F_{g-2} in Q belongs to S, we can obtain L, Q, R as right extension of the infix pair obtained by deleting F_{g-1} and T_m.

2. *Each height-g infix pair that can be obtained by double extension can also be obtained by left extension* and *by right extension.* This is clear by considering the two equivalent forms of double extension pictured in Figure 18: from the first one, one sees that it is the right extension of the infix pair obtained by deleting the rightmost diagonal; from the second one, one sees that it is the left extension of the infix pair obtained by deleting the leftmost diagonal. More formally, let $L = WF_{g-2}L''$, $Q = F_{g-1}Q''F_{g-2}$, $R = R''H$ be a double extension of the height-$(g-2)$ infix pair $(L''Q''R'', L''R'')$; then (LQR, LR) is a left extension of $L' = L'', Q' = Q''F_{g-2}, R' = R''H$, and a right extension of $L' = WL'', Q' = F_{g-2}Q'', R' = R''$.

3. *Each height-g infix pair that can be obtained by both left and right extension can also be obtained by double extension.* This part requires more attention because of a subtle point: it could be the case that

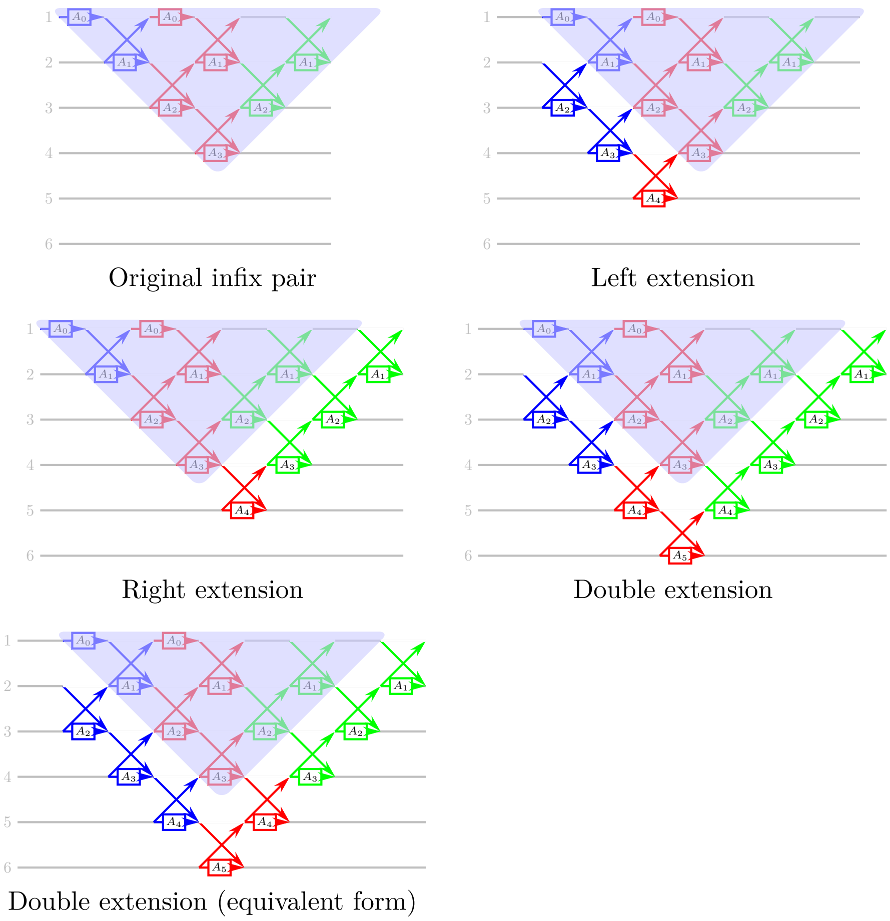
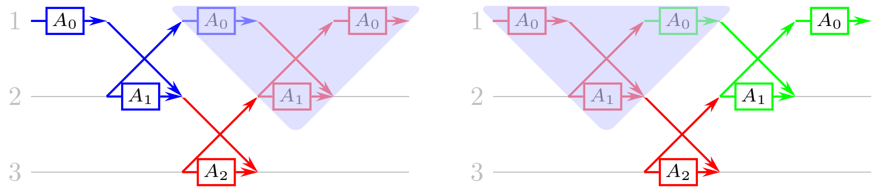


Figure 18: The different kind of extensions described in Theorem 18. The diagonal sequences of factors added can have arbitrary length.



$L_\ell = F_0F_1F_0$, $Q_\ell = F_2F_1F_0$, $R_\ell = I$ $\quad$ $L_r = I$, $Q_r = F_0F_1F_2$, $R_r = F_0F_1F_0$

Figure 19: An example of the problematic configuration in Part 3 of Theorem 18. The set of generators L_ℓ, Q_ℓ, R_ℓ on the left can be obtained by left extension but not by right extension, while the one on the right L_r, Q_r, R_r can be obtained by right extension only.

there is a choice of the generators L_ℓ, Q_ℓ, R_ℓ that we can obtain as left extension, and a different choice L_r, Q_r, R_r that we can obtain as right extension. An example is in Figure 19. We choose again a MSF $SMT \sim L_\ell Q_\ell R_\ell \sim L_r Q_r R_r$. If the factor F_{g-3} in Q_ℓ belongs to M, then it is easy to see that the infix pair is a double extension of the infix pair obtained by deleting the factors S_m, F_{g-1} and T_m. The same holds if the factor F_{g-3} in Q_r belongs to M.

Hence the only remaining case is the one in which Q_ℓ contains the factor $F_{g-1}F_{g-2}F_{g-3}$ in this order, while Q_r contains the factor $F_{g-3}F_{g-2}F_{g-1}$ in this order, exactly like in Figure 19. In this case, we rely on Lemma 17: Part 1 tells us that there must be a third factor F_{g-3} between those two, and Part 2 tells us that we can find a third set of generators $LQR \sim L_\ell Q_\ell R_\ell \sim L_r Q_r R_r$, $LR \sim L_\ell R_\ell \sim L_r R_r$ for which the factor F_{g-3} is in Q.

4. *Left extension produces $g\Pi_{g-1}$ different (non-equivalent) pairs of height g, right extension produces $g\Pi_{g-1}$ different (non-equivalent) pairs, and double extension produces $(g-1)^2\Pi_{g-2}$ pairs.* When we make a left extension of an infix pair $(\mathcal{F}', \mathcal{G}')$, we have g different ways to choose the length of the diagonal sequence of factors W to add; the only constraint is that F_{g-1} must be present. For each choice and for each $(\mathcal{F}', \mathcal{G}')$, we get a different infix pair. Indeed, Lemma 7 shows that different choices of W (and hence of S_m) produce different matrices. The proof for the right extension is analogous. For double extension, we can choose independently the length of the two added factors W and H, so we have $(g-1)^2$ choices. Again, by Lemma 7 these lengths determine the blocks of the last nontrivial row and column of $\mathcal{F}$, so all the infix pairs produced by these choices are distinct.

Once we have established all these facts, the result follows by inclusion-exclusion:

$$\Pi_g = \underbrace{g\Pi_{g-1}}_{\text{produced by left extension}} + \underbrace{g\Pi_{g-1}}_{\text{produced by right extension}} - \underbrace{(g-1)^2\Pi_{g-2}}_{\text{produced by both}}.$$

□

Surprisingly, the sequence Π_g has already been studied in a completely different area: it counts the number of so-called *two-sided generalized Fibonacci sequences* [13, Sequence A005189]. The paper [7] contains a definition of these sequences, a derivation of the recurrence (9), and some asymptotic properties such as the rate of growth of the sequence.

9 Symmetric infix pairs

One can give a simple canonical form for *symmetric* infix pairs, i.e., those for which LQR and LR are symmetric.

Lemma 19. *Let $(\mathcal{F}, \mathcal{G})$ be a symmetric infix pair of height g. Then,*

1. *In the MSF diagram of F, the three central columns are completely filled, i.e., the central column 0 contains the product $P = F_{g-1}F_{g-3}F_{g-5}\cdots F_{(1 \text{ or } 0)}$, ending with either F_1 or F_0 depending on the parity of g, and the two adjacent columns 1 and -1 contain $U = F_{g-2}F_{g-4}F_{g-6}\cdots F_{(0 \text{ or } 1)}$, where the final index depends again on the parity of g.*

2. *One can choose generators for $(\mathcal{F}, \mathcal{G})$ as follows:*

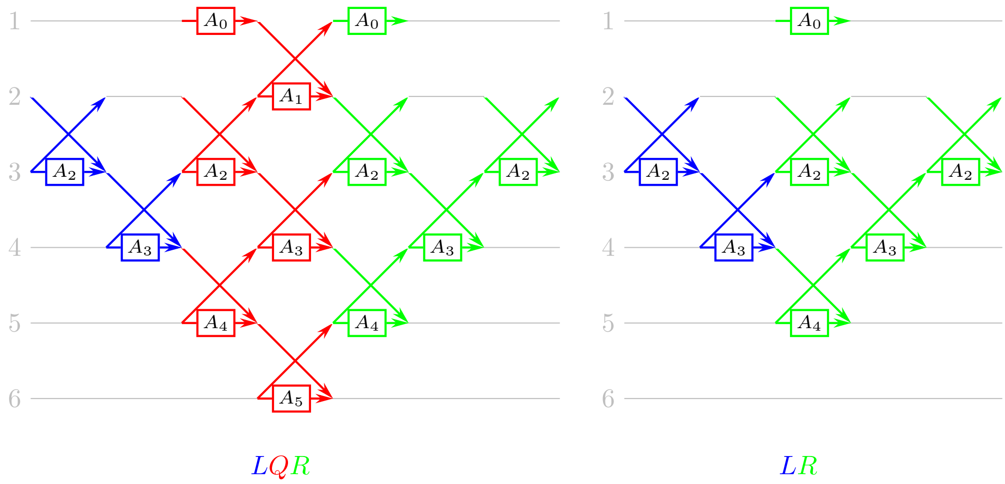


Figure 20: An example of the canonical set of generators described in Lemma 19.

- $L = C_{g-3}C_{g-5}\cdots C_{(1 \text{ or } 0)}$, *where each* C_i *is either the identity or the product* $F_{c_i}F_{c_i+1}\cdots F_i$, *for some* $0 \leq c_i \leq i$;
- $Q = UP$;
- $R = U \operatorname{rev} L$.

An example of this configuration is in Figure 20: the three central columns are filled, and two of them form Q.

Proof. Assume that $A(x)$ is a symmetric matrix polynomial, and let L_1, Q_1, R_1 be any set of generators for $(\mathcal{F}, \mathcal{G})$. Then, one can check that $\operatorname{rev} R_1, \operatorname{rev} Q_1, \operatorname{rev} L_1$ are a set of generators for $(\mathcal{F}^*, \mathcal{G}^*) = (\mathcal{F}, \mathcal{G})$. In particular, the MSF diagrams of $L_1Q_1R_1$ and $\operatorname{rev} R_1 \operatorname{rev} Q_1 \operatorname{rev} L_1^*$ are the same, and the elements corresponding to Q_1 and those corresponding to $\operatorname{rev} Q_1$ are mirror symmetric along the middle column. By Part 1 of Lemma 17, the region between Q_1 and $\operatorname{rev} Q_1$ contains as many elements as possible. In particular, when we follow the algorithm to construct the MSF inductively, there is always one element in the column 0 for type $F_{g-1}, F_{g-3}, F_{g-5}, \cdots, F_{(1 \text{ or } 0)}$ (because column 0 is always between any pair of mirror-symmetric positions), and always one element in columns -1 and 1 for types $F_{g-2}, F_{g-4}, F_{g-6}, \cdots, F_{(0 \text{ or } 1)}$. By Part 2 of the same lemma, we can choose a new set of factors L, Q, R for which Q contains exactly the elements in columns -1 and 0. With this choice, L is the set of elements in the MSF diagram of $L_1Q_1R_1$ that is left of column -1, hence it has the structure of a product $L = C_{g-3}C_{g-5}\cdots C_{(1 \text{ or } 0)}$ as required. The expression of R follows by symmetry. □

Using this canonical form, one can count explicitly symmetric infix pairs. Essentially, we need to count the possible lengths of the 'arms' of the MSF of LQR.

Corollary 20. *There are* $(g-1)!!$ *distinct (non-equivalent) symmetric infix pairs of height* g.

Here, the notation $k!!$ denotes the double factorial (or semi-factorial) of an integer k, i.e., the product of all the integers between 1 and k that have the same parity as k.

Proof. We count the number of distinct canonical forms generated according to Part 2 of Lemma 19. Such a canonical form is essentially the MSF of LQR, with the only restriction that the three central columns are full. This implies that $k_1 = 1, k_2 = 3, \ldots, k_m = g-1$ or $k_1 = 0, k_2 = 2, \ldots, k_m = g-1$ (according to the parity of g; hence $m = \lfloor \frac{g+1}{2} \rfloor$), and that none of the factors S_h or T_h is the identity matrix. Hence we have $0 \leq s_h = t_h \leq k_h - 1$, and there are $g-1$ choices for s_{g-1}, $g-3$ for s_{g-3}, and so on down to s_1 or s_0. □

10 Inverse Fiedler matrices

The inverse of $G(A)$ in (1) is the matrix

$$G(A)^{-1} = \begin{bmatrix} -A & I \\ I & 0 \end{bmatrix},$$

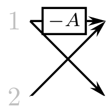


Figure 21: The diagram corresponding to $G(A)^{-1}$

to which we can associate, analogously, the diagram in Figure 21. Similarly to the construction above, we can define the following elementary inverse Fiedler matrices

$$F_{-i} := F_i^{-1} = \begin{bmatrix} I_{n(i-1)} & & \\ & G(A_i)^{-1} & \\ & & I_{n(d-i-1)} \end{bmatrix}, \qquad \text{for } i = 1, 2, \dots, d-1,$$
$$F_{-d} := \begin{bmatrix} I_{n(d-1)} & \\ & -A_d \end{bmatrix},$$

which are associated to the elementary diagrams in Figure 22. Notice that F_0 and F_{-d} do not admit an

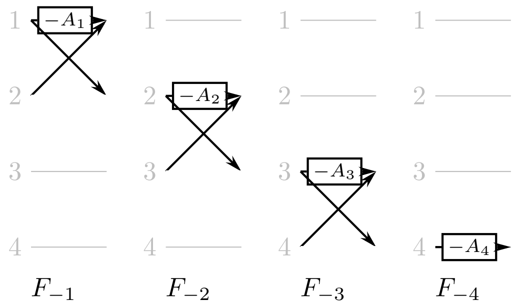


Figure 22: The elementary diagrams associated to F_{-i}.

operation-free inverse; indeed, their invertibility is not guaranteed and depends on that of A_0 and A_d.

We call *inverse Fiedler matrix* a product of elementary Fiedler matrices with negative indices

$$F = F_{i_1} F_{i_2} \cdots F_{i_\ell}, \quad -d \le i_k < 0 \text{ for each } k.$$

All the constructions described earlier can be replicated for inverse Fiedler matrices. The resulting diagrams look like a vertically mirrored version of the diagrams associated to regular Fiedler matrices. Every statement that we have proved regarding diagrams associated Fiedler matrices holds true for inverse Fiedler matrices, provided we switch the "up" and "down" direction. For instance, the analogue of Figure 7 is Figure 23, and the analogue of Corollary 4 states that in an operation-free inverse Fiedler matrix we cannot have an arrow in direction $\nearrow$ followed by one in direction $\searrow$. The MSF of an operation-free inverse Fiedler product can be constructed analogously, by building the diagram downwards starting from $i_{\min}$, and creates sequences of factors in the shape of an upside-down V. We do not repeat here all the theorems and their proofs.

Similarly, we define an *inverse infix pair of height g* as a pair $(\mathcal{M}(LQR), \mathcal{M}(LR))$ in which LQR is operation-free, Q contains the factors $F_{-d}, F_{-d+1}, \dots, F_{-d+g-1}$ exactly once, and L, R may only contain factors $F_{-d}, F_{-d+1}, \dots, F_{-d+g-2}$. An example is in Figure 24.

11 Generalized Fiedler pencils and Fiedler pencils with repetitions

A *generalized Fiedler pencil* [1] for the matrix polynomial $A(x)$ is a matrix pencil $\mathcal{M}(Q^+) - \mathcal{M}(Q^-)x \in \mathbb{C}^{nd \times nd}[x]$ such that Q^+ is a Fiedler product which contains only the blocks F_i, $i = 0, 1, \dots, g-1$ each one exactly once, and Q^- is an inverse Fiedler matrix containing only the blocks F_{-i}, $i = g, g+1, \dots, d$, each one exactly once, for some g such that $0 < g \le d$. In particular, Q^+ always contains at least the factor F_0 and Q^- always contains at least the factor F_{-d}.

To these pencils we associate two diagrams, one for each matrix. An example is in Figure 25. Counting

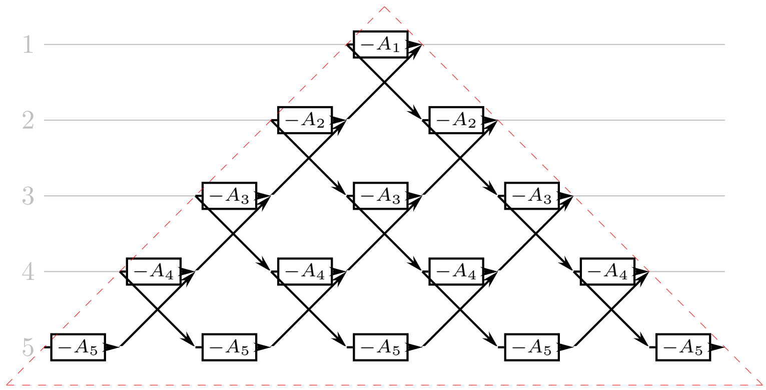


Figure 23: The “densest possible” inverse Fiedler product with $d = 5$.

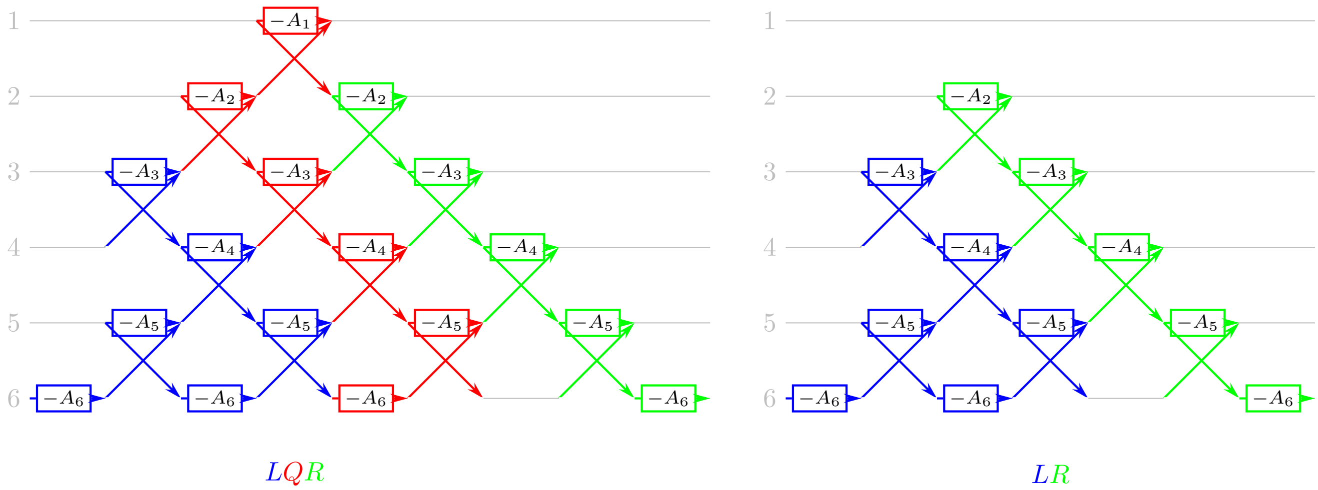


Figure 24: A height-6 inverse infix pair.

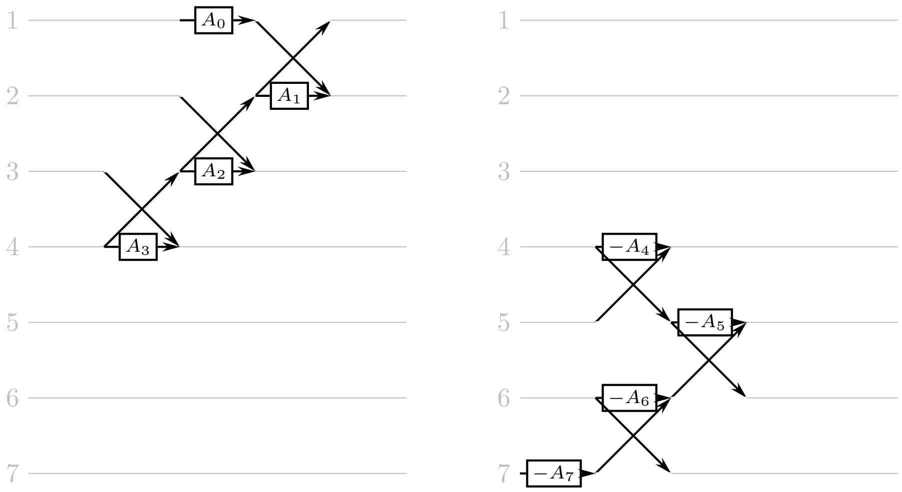


Figure 25: The two diagrams associated to the generalized Fiedler pencil $F_3F_2F_0F_1 - F_{-7}F_{-6}F_{-4}F_{-5}x$.

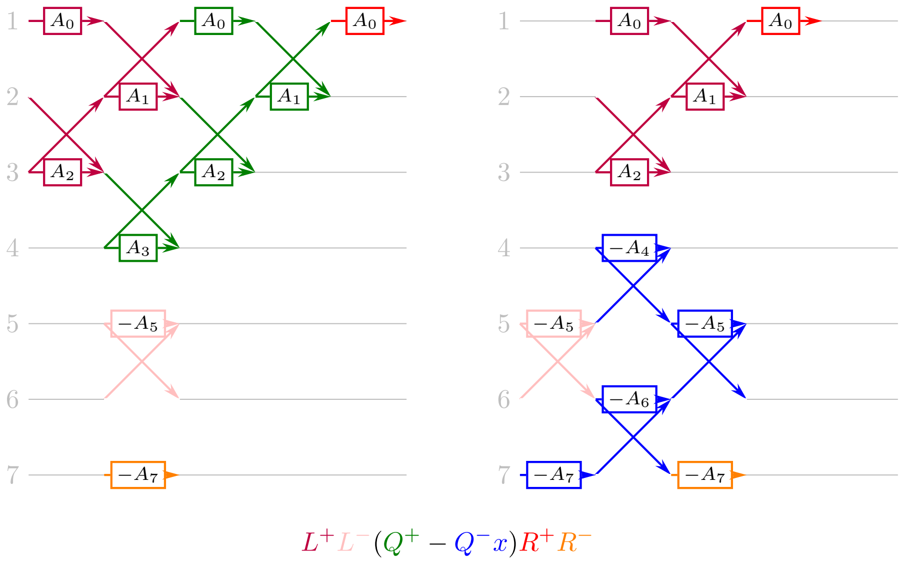


Figure 26: Diagrams associated to the Fiedler pencil with repetitions with $d = 7$, $g = 4$, $Q^+ = F_3 F_2 F_0 F_1$, $Q^- = F_{-7} F_{-6} F_{-4} F_{-5}$, $L^+ = F_0 F_2 F_1$, $R^+ = F_0$, $R^- = F_{-7}$, $L^- = F_{-5}$.

the number of distinct generalized Fiedler pencils is simple. The first mention of this result that we could find in the literature is in [10], but it was likely known earlier.

Lemma 21. *For a matrix polynomial $A(x)$ of degree d, there are $d \cdot 2^{d-1}$ distinct generalized Fiedler pencils.*

Proof. For a fixed value of g, there are 2^{g-1} possible choices for Q^+: we start by a diagram containing only F_{g-1}; then we can insert F_{g-2} either to its left or its right, F_{g-3} either to the left or the right of F_{g-2}, and so on, up to F_0; every time we have 2 choices for the position of the next elementary factor. Different choices give non-equivalent pencils, because their layers are different. Analogously, there are 2^{d-g} choices for Q^-. Summing over the possible values of g gives the result. □

A more general family of linearizations is the following. Let L^+, Q^+, R^+ be the generators of a height-g infix pair $(\mathcal{F}^+, \mathcal{G}^+)$, and L^-, Q^-, R^- be the generators of a height-$(d+1-g)$ inverse infix pair $(\mathcal{F}^-, \mathcal{G}^-)$. The matrix pencil $\mathcal{A}^+ - \mathcal{A}^- x$, with $\mathcal{A}^+ = \mathcal{M}(L^+ L^- Q^+ R^+ R^-)$ and $\mathcal{A}^- = \mathcal{M}(L^+ L^- Q^- R^+ R^-)$, is called a *Fiedler pencil with repetitions*. Note that this reduces to a generalized Fiedler pencil in the case $L^+ = L^- = R^+ = R^- = I$, and indeed it corresponds to pre- and post-multiplying a generalized Fiedler pencil with $L^+ L^-$ and $R^+ R^-$, respectively.

To each FPR we can associate two diagrams, one representing the degree-0 coefficient and one representing the degree-1 coefficient. An example is in Figure 26. The only difference between these two diagrams is that one contains the upper part Q^+ of a generalized Fiedler pencil, and the other contains its lower part Q^- instead.

In the matrix $\mathcal{A}^+$, the products L^+, Q^+, R^+ contain only elementary matrices in the set $\{F_0, F_1, \dots, F_{g-1}\}$, while the factors L^-, R^- contain only inverse elementary matrices in the set $\{F_{-(g+1)}, F_{-(g+2)}, \dots, F_{-d}\}$. These two sets act on separate rows of the diagram, and hence they commute. Correspondingly, $\mathcal{A}^+$ can be divided into two diagonal blocks $\mathcal{A}^+_U \in \mathbb{C}^{ng \times ng}$ and $\mathcal{A}^+_L \in \mathbb{C}^{n(d-g) \times n(d-g)}$, and we have

$$\mathcal{F}^+ = \begin{bmatrix} \mathcal{A}^+_U & 0 \\ 0 & I_{n(d-g)} \end{bmatrix} = \mathcal{M}(L^+ Q^+ R^+), \quad \mathcal{G}^- = \begin{bmatrix} I_{ng} & 0 \\ 0 & \mathcal{A}^+_L \end{bmatrix} = \mathcal{M}(L^- R^-), \quad \mathcal{A}^+ = \mathcal{F}^+ \mathcal{G}^- = \begin{bmatrix} \mathcal{A}^+_U & 0 \\ 0 & \mathcal{A}^+_L \end{bmatrix}. \tag{10}$$

Analogously, in $\mathcal{A}^-$ the products L^+, R^+ only contains factors in $\{F_0, F_1, \dots, F_{g-2}\}$, while L^-, Q^-, R^- only contain factors in $\{F_{-g}, F_{-(g+1)}, \dots, F_{-d}\}$; the two sets commute and determine a block diagonal decomposition of $\mathcal{A}^-$, into $\mathcal{A}^-_U \in \mathbb{C}^{n(g-1) \times n(g-1)}$ and $\mathcal{A}^-_L \in \mathbb{C}^{n(d-g+1) \times n(d-g+1)}$. Note that the sizes of these blocks are different from those of $\mathcal{A}^+$. The analogue of (10) is

$$\mathcal{G}^+ = \begin{bmatrix} \mathcal{A}^-_U & 0 \\ 0 & I_{n(d-g+1)} \end{bmatrix} = \mathcal{M}(L^+ R^+), \quad \mathcal{F}^- = \begin{bmatrix} I_{n(g-1)} & 0 \\ 0 & \mathcal{A}^-_L \end{bmatrix} = \mathcal{M}(L^- Q^- R^-), \quad \mathcal{A}^- = \mathcal{G}^+ \mathcal{F}^- = \begin{bmatrix} \mathcal{A}^-_U & 0 \\ 0 & \mathcal{A}^-_L \end{bmatrix}. \tag{11}$$

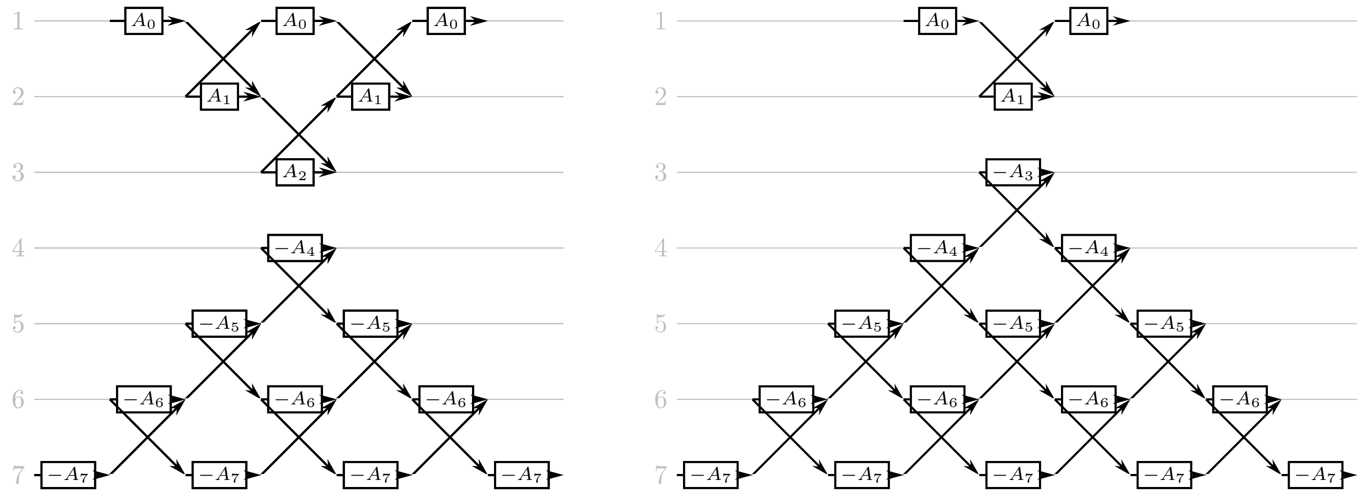


Figure 27: The "hourglass shapes" corresponding to the FPR with the most possible factors among those with $d = 7$ and $g = 3$.

For instance, in the example in Figure 26 we have $g = 4$, and $d = 7$, and the matrices of the pencil are

$$\mathcal{A}^+ = \begin{bmatrix} 0 & 0 & A_0 & 0 & & & \\ 0 & 0 & 0 & I & & & \\ 0 & A_0 & A_1 & A_2 & & & \\ A_0 & A_1 & A_2 & A_3 & & & \\ & & & & -A_5 & I & 0 \\ & & & & I & 0 & 0 \\ & & & & 0 & 0 & -A_7 \end{bmatrix} \quad \mathcal{A}^- = \begin{bmatrix} 0 & A_0 & 0 & & & & \\ 0 & 0 & I & & & & \\ A_0 & A_1 & A_2 & & & & \\ & & & -A_4 & -A_5 & I & 0 \\ & & & -A_5 & -A_6 & 0 & -A_7 \\ & & & I & 0 & 0 & 0 \\ & & & 0 & -A_7 & 0 & 0 \end{bmatrix}$$

For a fixed value of g, the FPR with the most elementary factors corresponds to two diagrams with a hourglass shape such as the ones in Figure 27.

A FPR is uniquely determined by the choices of g and of the two infix pairs $(\mathcal{F}^+, \mathcal{G}^+)$ and $(\mathcal{F}^-, \mathcal{G}^-)$, hence the following corollary holds.

Corollary 22. *There are $\Xi_d = \sum_{g=1}^{d} \Pi_g \Pi_{d+1-g}$ different Fiedler pencils with repetitions of degree d.*

The first values of this sequence are $\Xi_1 = 1$, $\Xi_2 = 6$, $\Xi_3 = 37$, $\Xi_4 = 254$, $\Xi_5 = 1958$, $\Xi_6 = 16910$. This sequence does not appear in the comprehensive reference database [13].

12 Symmetric FPRs

We call a FPR $\mathcal{A}^+ - \mathcal{A}^- x$ *symmetric* if both $\mathcal{A}^+$ and $\mathcal{A}^-$ are symmetric for each symmetric matrix polynomial $A(x)$. Then, thanks to the block diagonal decompositions, the following result holds.

Corollary 23. *A Fiedler pencil with repetitions is symmetric if and only if both infix pairs $(\mathcal{F}^+, \mathcal{G}^+)$ and $(\mathcal{F}^-, \mathcal{G}^-)$ are symmetric.*

Hence most of the work that we need to characterize symmetric FPRs has already been done in Section 9. In particular, the following result is now clear in view of Corollaries 20 and 22.

Corollary 24. *For a symmetric matrix polynomial $A(x)$ of degree d, there are $\Omega_d = \sum_{g=1}^{d} (g-1)!!(d-g)!!$ different symmetric FPRs.*

The first values of this sequence are $\Omega_1 = 1$, $\Omega_2 = 2$, $\Omega_3 = 5$, $\Omega_4 = 10$, $\Omega_5 = 26$, $\Omega_6 = 58$. This sequence does not appear in [13] either.

Moreover, replacing $(L^+Q^+R^+, L^+R^+)$ and $(L^-Q^-R^-, L^-R^-)$ with their canonical form described in Part 2 of Lemma 19, one obtains a visually simple canonical diagram in which the only difference between the two matrices in the pencil is in columns -1 and 0, which we display in an example in Figure 28. A similar

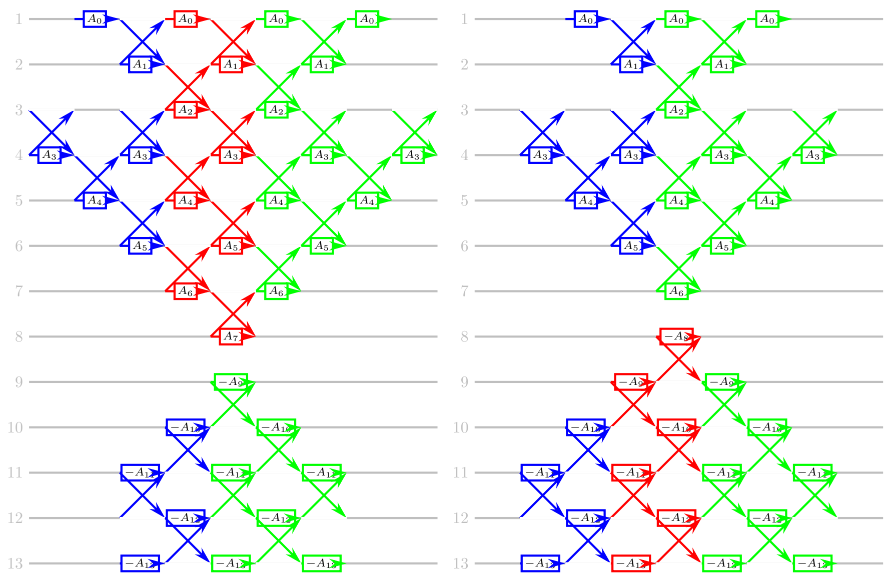


Figure 28: An example of the standard form for symmetric FPRs. Both the top and bottom parts are in the standard form for symmetric infix pairs; the two pencils differ only by the red part, and the only freedom of choice that we have is the length of the blue diagonal 'arms' extending out of the central columns (the green part is fixed by symmetry).

standard form for symmetric FPRs is given in [5], in algebraic terms and without a corresponding visualization. The form described there is slightly different, though: translating to our notation, in the standard form for infix pairs suggested in [5], Q may contain at its top a sequence of factors that leaves the columns 0 and -1 in a straight diagonal line. Correspondingly, the factor U such that $R = U \operatorname{rev} L$ (called *symmetric complement* there) has a more complex structure, and the characterization of the possible left factors L is more involved. An example in which the two forms differ is given in Figure 29.

13 Palindromic and antipalindromic FPRs

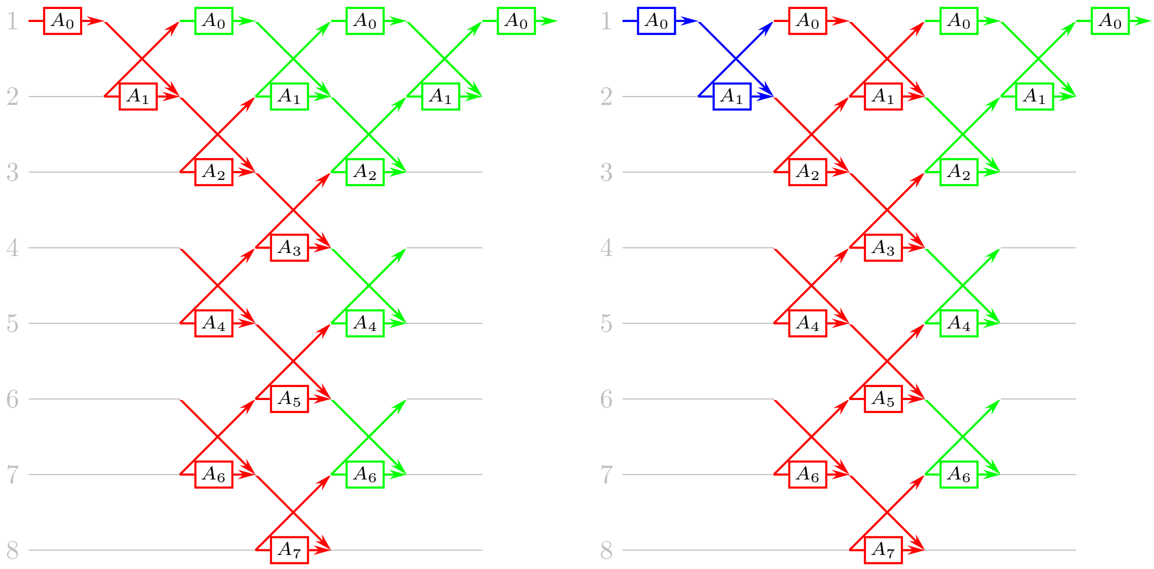
A family of palindromic linearizations obtained from FPRs is introduced and studied in [4]. Here we introduce what is essentially the same family, but with a different approach which has, in our opinion, a simpler structure, when it comes to considering signs. Note that a matrix polynomial $A(x)$ is palindromic if and only if $A(-x)$ is antipalindromic. Hence we first consider antipalindromic Fiedler linearizations of antipalindromic polynomials, and then one can obtain palindromic linearizations of antipalindromic matrix polynomial by making the substitution $x \mapsto -x$.

Let $A(x)$ be a $\star$-antipalindromic matrix polynomial of degree d. We say that a FPR $\mathcal{A}^+ - \mathcal{A}^- x$ is *antipalindromic* if $\mathcal{A}^+ = (\mathcal{A}^-)^\star$ for each antipalindromic polynomial $A(x)$. In view of Theorem 9, this is equivalent to $L^+L^-Q^+R^+R^- \sim (L^+L^-Q^-R^+R^-)^\star$.

The first question to address is if these linearizations exist. Unfortunately, the answer is negative, apart from the trivial case $d = 1$.

Theorem 25. *Let $A(x)$ be a $\star$-antipalindromic matrix polynomial of degree d. It is not possible to construct a Fiedler pencil with repetitions $\mathcal{A}^+ - \mathcal{A}^- x \in \mathbb{C}^{nd\times nd}$ such that $\mathcal{A}^+ = (\mathcal{A}^-)^\star$, apart from the trivial case $d = 1$.*

Proof. Let $\mathcal{A}^+ - \mathcal{A}^- x$ be an antipalindromic FPR, i.e., $\mathcal{A}^+ = (\mathcal{A}^-)^\star$. Then, since $\mathcal{A}^+$ and $\mathcal{A}^-$ have block diagonal structures (10) and (11) with different sizes, both block row g and block column g of $\mathcal{A}^+$ must contain only zeros apart from the diagonal block (g, g). Moreover, if one applies Lemma 7 to $\mathcal{F}^+ = \mathcal{M}(L^+Q^+R^+)$, we see that the matrix $\mathcal{B}$ appearing in the lemma coincides with $\mathcal{A}_U^+$ in (10). In particular, its last block row is block row (g, g) in $\mathcal{A}^+$, and by Lemma 7 it contains at least two nonzero blocks unless $\mathcal{B}$ is composed of only one $n \times n$ block, which means that $g = 1$. Similarly, the analogous of Lemma 7 for inverse Fiedler products lets us predict the content of the first block row of $\mathcal{A}_L^-$, and the only possibility for it to have only one nonzero block is that it is 1×1 and hence $g = d$. So we have shown that $d = g = 1$. $\square$



$L = I$, $Q = F_6F_7F_4F_5F_0F_1F_2F_3$,
$R = F_6F_4F_0F_1F_2F_0F_1F_0$

$L = F_0F_1$, $Q = F_6F_7F_4F_5F_2F_3F_0F_1$,
$R = F_6F_4F_2F_0F_1F_0$

Figure 29: The diagram of the first element of the infix pair corresponding to $w_1 r_{w_1}$ in [3, Example 3.9], with colorings corresponding to two different choices of the generators. The coloring on the left represents the generators in the canonical form suggested in [3], the one on the right is the canonical form suggested here.

Following [4], we consider instead a slightly different definition that allows one to construct a nontrivial family of structured FPRs for antipalindromic polynomials. We say that a pencil $\mathcal{A}^+ - \mathcal{A}^- x$ is *J-antipalindromic*, if $J(\mathcal{A}^+ - \mathcal{A}^- x)$ is antipalindromic, where $J \in \mathbb{R}^{nd \times nd}$ is the matrix

$$J = \begin{bmatrix} 0 & & I_n \\ & \iddots & \\ I_n & & 0 \end{bmatrix}.$$

This means that $(J\mathcal{A}^+)^\star = (J\mathcal{A}^-)$, or equivalently $\mathcal{A}^- = (J\mathcal{A}^+J)^\star$. We define for notational convenience $\mathcal{S}(M) = (JMJ)^\star = JM^\star J$, where M is either a matrix or a formal matrix product, so that we can write the last equality as $\mathcal{A}^- = \mathcal{S}(\mathcal{A}^+)$. Note that this operator enjoys many of the properties of transposition: $\mathcal{S}(\mathcal{S}(M)) = M$, and $\mathcal{S}(M_1M_2\cdots M_k) = \mathcal{S}(M_k)\cdots\mathcal{S}(M_2)\mathcal{S}(M_1)$.

If the matrix polynomial $A(x)$ is antipalindromic, then $\mathcal{S}(\cdot)$ acts nicely on elementary Fiedler matrices:

$$\mathcal{S}(F_i) = F_{-(d-i)}, \quad i = 0, 1, \ldots, d-1, \tag{12}$$

and, conversely,

$$\mathcal{S}(F_{-i}) = F_{d-i}, \quad i = 1, \ldots, d. \tag{13}$$

For example for $d = 7$ and $F = F_1F_0F_3F_5F_{-6}F_{-7}$, we have $\mathcal{S}(F) = F_{7-7}F_{7-6}F_{5-7}F_{3-7}F_{0-7}F_{1-7} = F_0F_1F_{-2}F_{-4}F_{-7}F_{-6}$.

There is a simple interpretation of $\mathcal{S}(\cdot)$ in terms of the associated diagrams.

Lemma 26. *Assume that $A_i = -A_{d-i}^\star$ for each i. Then a diagram representing $\mathcal{S}(F)$ can be obtained flipping horizontally and then vertically a diagram representing F.*

Proof. Let us ignore initially the content of the boxes in the diagrams. If M is a $nd \times nd$ block matrix, the operation $M \mapsto JMJ$ corresponds to reversing the order of its blocks; hence, in terms of our diagrams, reflecting along a horizontal axis (flipping vertically). We have already established in Lemma 11 that the operation $M \mapsto M^\star$ corresponds to reflecting along the diagram a vertical axis.

As for the content of the boxes, we have shown in (12) and (13) that $\mathcal{S}$ maps each elementary Fiedler pencil to a corresponding inverse Fiedler pencil and vice versa, so they are transformed correctly. ◻

An example is shown in Figure 30.

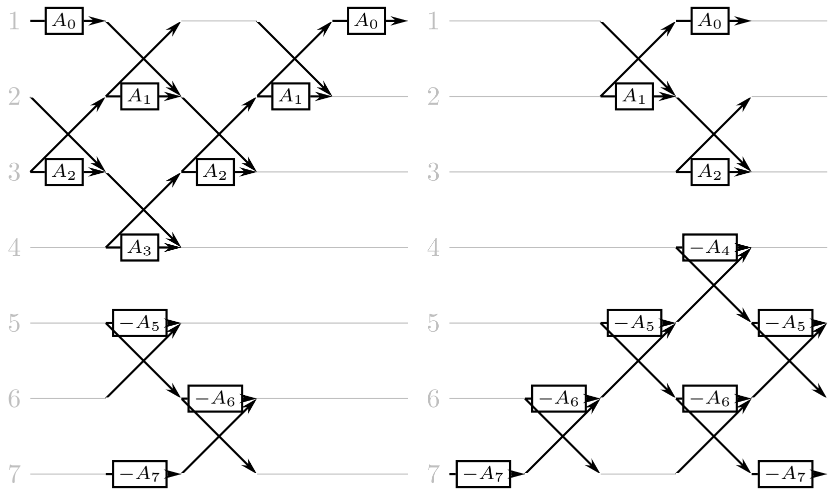


Figure 30: On the left a possible diagram for $F = F_0F_2F_1F_3F_2F_1F_0F_{-5}F_{-7}F_{-6}$, and on the right the diagram of $\mathcal{S}(F) = F_1F_0F_2F_{-7}F_{-6}F_{-5}F_{-4}F_{-6}F{-}5F_{-7}$ obtained by flipping it vertically and horizontally (or *vice versa*) and relabelling the boxes.

A FPR $\mathcal{A}^+ - \mathcal{A}^- x$ is J-antipalindromic for every choice of the antipalindromic matrix polynomial $A(x)$ if and only if $L^+L^-Q^+R^+R^- \sim \mathcal{S}(L^+L^-Q^-R^+R^-)$, in view of Theorem 9. It is possible to construct nontrivial J-antipalindromic FPRs, but only under some conditions on d and g.

Lemma 27. *Let $\mathcal{A}^+ - \mathcal{A}^- x$ be a J-antipalindromic FPR. Then, d must be odd and $g = \frac{d+1}{2}$.*

Proof. It is sufficient to count the number of elementary factors in $\mathcal{A}^+$ and $\mathcal{A}^-$. By the definition of FPR, Q^+ contains g elementary factors, and Q^- contains $d - g + 1$. Let m be the total number of elementary factors present in L^+, L^-, R^+, R^-. Then $L^+L^-Q^+R^+R^-$ contains $m + g$ elementary factors, and $L^+L^-Q^-R^+R^-$ contains $m + d - g + 1$. If $\mathcal{A}^- = \mathcal{S}(\mathcal{A}^+)$, in particular the two products must contain the same number of factors: hence $g = d - g + 1$, and thus $d + 1 = 2g$. □

It has already been established in [12] that palindromic linearizations can be constructed with a general process that works for all palindromic matrix polynomials only if d is odd, so the result above is not surprising.

Theorem 28. *A FPR $\mathcal{A}^+ - \mathcal{A}^- x$ is J-antipalindromic if and only if $\mathcal{S}(L^+Q^+R^+) \sim L^-\,Q^-\,R^-$ and $\mathcal{S}(L^+R^+) \sim L^-\,R^-$.*

Proof. We rely on the block diagonal decompositions (10) and (11). If $g = \frac{d+1}{2}$, then $\mathcal{S}(\mathcal{A}^-)$ is partitioned conformably with $\mathcal{A}^+$, and hence they are equal if and only if the two diagonal blocks are equal, or equivalently

$$\begin{bmatrix} \mathcal{A}_U^+ & 0 \\ 0 & I_{n(d-g)} \end{bmatrix} = \mathcal{S}\left(\begin{bmatrix} I_{n(g-1)} & 0 \\ 0 & \mathcal{A}_L^- \end{bmatrix}\right), \quad \begin{bmatrix} I_{ng} & 0 \\ 0 & \mathcal{A}_L^+ \end{bmatrix} = \mathcal{S}\left(\begin{bmatrix} \mathcal{A}_U^- & 0 \\ 0 & I_{n(d-g+1)} \end{bmatrix}\right),$$

which means that $L^-Q^-R^- = \mathcal{S}(L^+Q^+R^+)$ and $L^-R^- = \mathcal{S}(L^+R^+)$. □

So in this case $(L^-Q^-R^-, L^-R^-)$ is an inverse infix pair equivalent to $(\mathcal{S}(L^+Q^+R^+), \mathcal{S}(L^+R^+))$. It can be verified directly that the matrices $\mathcal{S}(R^+), \mathcal{S}(Q^+), \mathcal{S}(L^+)$ are a triple of generators for it. This fact allows us to express a J-antipalindromic FPR with a single set of generators instead of two.

Corollary 29. *In every J-antipalindromic $\mathcal{A}^+ - \mathcal{A}^- x$, one can replace L^-, Q^-, R^- with $\mathcal{S}(R^+), \mathcal{S}(Q^+), \mathcal{S}(L^+)$ without altering $\mathcal{A}^+$ and $\mathcal{A}^-$.*

In other words, a J-antipalindromic FPR is uniquely determined by the generators (L^+, Q^+, R^+), and we can use this result to count them.

Corollary 30. *For each d odd, there are $\Pi_{\frac{d+1}{2}}$ distinct J-antipalindromic FPRs of degree d.*

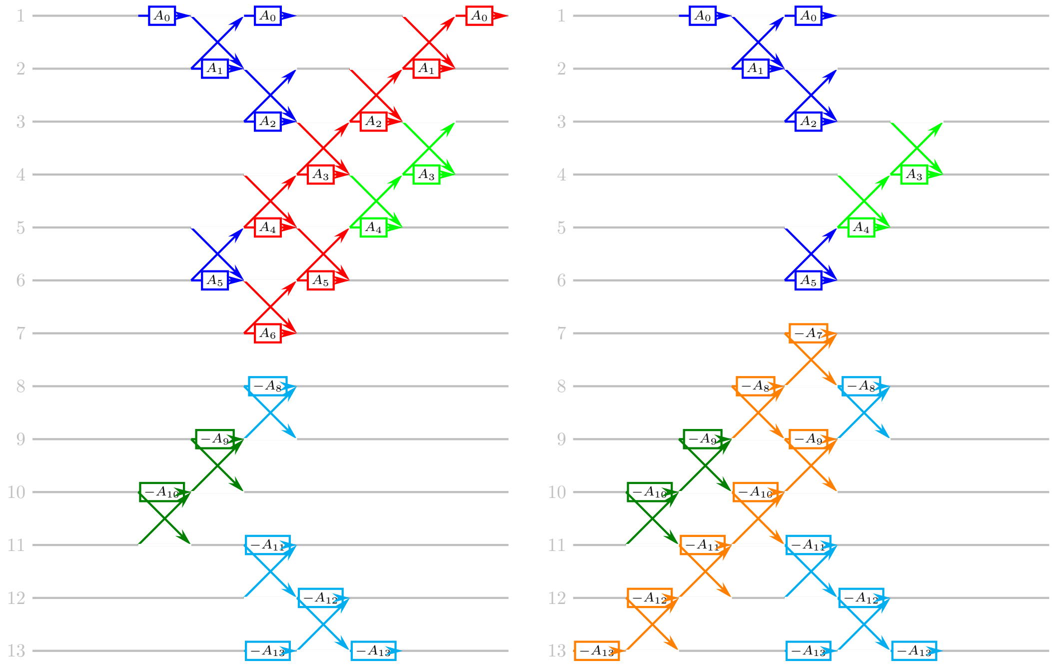


Figure 31: The MSF diagrams of a palindromic Fiedler pencil with repetitions, drawn in our canonical form with $R^- = \mathcal{S}(L^+)$, $Q^- = \mathcal{S}(Q^+)$, $L^- = \mathcal{S}(R^+)$. Note that the two diagrams are obtained one from another by flipping horizontally and vertically.

Recall that by Lemma 27 there are no J-antipalindromic FPRs when d is even.

Moreover, note that if a Fiedler product (resp., an inverse Fiedler product) F is in MSF, then $\mathcal{S}(F)$ is an inverse Fiedler product (resp., a Fiedler product) in MSF, too, and the same holds for the associated diagrams. Thus, the MSF diagrams for $\mathcal{A}^+$ and $\mathcal{A}^-$ can be obtained one from another by flipping vertically and then horizontally. An example is in Figure 31. In other words, the MSF diagrams for $\mathcal{A}^+$ and $\mathcal{A}^-$ display visually the symmetry of the associated linearization.

Finally, we show how to convert these results into results on palindromic matrix polynomials. We say that the matrix pencil $\mathcal{A}^+ + \mathcal{A}^- x$ (with the plus sign) is J-palindromic if $J\mathcal{A}^+ = (J\mathcal{A}^-)^\star$, i.e., if the matrix pencil $J(\mathcal{A}^+ + \mathcal{A}^- x)$ is palindromic.

Lemma 31. *Let $A(x)$ be a palindromic matrix polynomial, i.e., $A_i = A_{d-i}^\star$ for each i. Then, $\mathcal{A}^+ + \mathcal{A}^- x$ is a J-palindromic linearization for $A(x)$ if and only if $\mathcal{A}^+ - \mathcal{A}^- x$ be a J-antipalindromic linearization for the antipalindromic matrix polynomial $A(-x)$.*

This shows how to construct a large family of J-palindromic linearizations that are derived from Fiedler pencils with repetitions.

In the paper [4], a similar approach is used to construct palindromic and antipalindromic linearizations of the form $SJ(\mathcal{A}^+ - \mathcal{A}^- x)$, where $\mathcal{A}^+ - \mathcal{A}^- x$ is a FPR and S is a quasi-identity matrix, i.e., a $nd \times nd$ block diagonal matrix in which each diagonal block is either I_n or $-I_n$. In contrast, our approach does not require sign changes for the antipalindromic case, and contains only predictable sign changes on the coefficient matrices A_i in the palindromic case.

14 Conclusions

We have introduced new notation and diagrams to work with Fiedler pencils, and we have used them to solve several combinatorial counting problems for Fiedler pencils with repetitions. It is interesting to note that in some cases these results return sequences that have already appeared in other fields, and for which asymptotics and other properties are known.

We believe that our approach can be used in future to solve other problems in the study of Fiedler pencils, and we hope that the visualization opportunities that it gives are useful to other researchers approaching this field of study as well.

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