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# Robust control via the computation of permuted graph bases

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**Preprint 2012/19**

**Preprint-Reihe des Instituts für Mathematik**
**Technische Universität Berlin**
http://www.math.tu-berlin.de/preprints

**Preprint 2012/19** **June 2012**

# Robust control via the computation of permuted graph bases

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08.06.12

We present a new numerical method for the $\gamma$-iteration in robust control based on the extended matrix pencil formulation of [6]. The new method bases the $\gamma$ iteration on the computation of special subspaces associated with matrix pencils. We introduce a permuted graph representation of these subspaces, which avoids the known difficulties that arise when the iteration is based on the solution of algebraic Riccati equations but at the same time makes use of the special symmetry structures that are present in the problems. We show that the new method is applicable in many situations where the conventional methods fail.

## 1 Introduction

The optimal infinite-horizon output (or measurement) feedback $H_\infty$ control problem is one of the central tasks in robust control, see, e. g., [19, 31, 39, 40]. Despite recent developments [6, 4, 14, 15, 16, 20, 25, 32, 36, 38] some of which are incorporated into software libraries like SLICOT[1] [8, 9, 21] or the MATLAB Robust Control Toolbox [2], the development of robust numerical methods for the $H_\infty$ control remains a problem [12].

Consider the linear dynamical system

$$\begin{aligned} \dot{x} &= Ax + B_1 w + B_2 u, \quad x(t_0) = x_0, \\ z &= C_1 x + D_{11} w + D_{12} u, \\ y &= C_2 x + D_{21} w + D_{22} u, \end{aligned} \tag{1}$$

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[1] See http://www.slicot.org

where $A \in \mathbb{R}^{n\times n}$, $B_i \in \mathbb{R}^{n\times m_i}$, $C_i \in \mathbb{R}^{p_i,n}$, $D_{ij} \in \mathbb{R}^{m_i,p_j}$, with $p_1 \geq m_2$ and $m_1 \geq p_2$. Here, $x(t)$, $u(t)$ and $y(t)$ are state, input and output vectors respectively, $w(t)$ is an additional input term representing noise and errors in the modeled dynamics, and $z$ is an estimation error.

The optimal infinite-horizon $\mathcal{H}_\infty$ control problem [6, 19, 40] consists in designing a controller for (1) such that the closed-loop system minimizes the magnitude of the response $z(t)$ with respect to the disturbance $w(t)$ in the worst case. Mathematically, one achieves the desired design goal of stabilizing the system and using the freedom in the design to minimize the influence of the disturbance in the transfer function $T_{zw}$ from $w$ to $z$ in the $\mathcal{H}_\infty$ norm $\|T_{zw}\|_\infty := \sup_{\omega\in\mathbb{R}} \sigma_{\max}[T_{zw}(i\omega)]$, where $\sigma_{\max}$ denotes the maximum singular value.

In the following we study problems with real coefficient matrices, however, all the results can be formulated in a similar way for complex matrices. Following [6], we shall make the following assumptions.

**A1** $(A, B_2)$ is stabilizable and $(A, C_2)$ is detectable, i. e., $\operatorname{rank}[sI - A, B_2] = \operatorname{rank}[sI - A^T, C_2^T] = n$ for all $s \in \mathbb{C}$ with nonnegative real part;

**A2** $D_{22} = 0$, and $D_{12}$, $D_{21}$ have full rank;

**A3** for each $\omega \in \mathbb{R}$, $\begin{bmatrix} A - i\omega I & B_2 \\ C_1 & D_{12} \end{bmatrix}$ has full column rank and $\begin{bmatrix} A - i\omega I & B_1 \\ C_2 & D_{21} \end{bmatrix}$ has full row rank.

When these assumptions hold, then the following theorem presents a method to estimate the $\mathcal{H}_\infty$ norm, see [6, 19, 40].

Criteria **Theorem 1.** *Consider system (1) satisfying assumptions A1–A3. Then for a fixed $\gamma > 0$, there exists an internally stabilizing controller with $\|T_{zw}\|_\infty < \gamma$ if and only if the following conditions hold.*

1. *The value $\gamma$ satisfies $\gamma > \max(\hat{\gamma}_H, \hat{\gamma}_J)$, where $\hat{\gamma}_H$, $\hat{\gamma}_J$ are the largest real values such that*

$$\begin{aligned} R_H(\hat{\gamma}_H) &= \begin{bmatrix} D_{11}^T \\ D_{12}^T \end{bmatrix} \begin{bmatrix} D_{11} & D_{12} \end{bmatrix} - \begin{bmatrix} \hat{\gamma}_H^2 I_{m_1} & 0 \\ 0 & 0 \end{bmatrix}, \\ R_J(\hat{\gamma}_J) &= \begin{bmatrix} D_{11} \\ D_{21} \end{bmatrix} \begin{bmatrix} D_{11}^T & D_{21}^T \end{bmatrix} - \begin{bmatrix} \hat{\gamma}_J^2 I_{p_1} & 0 \\ 0 & 0 \end{bmatrix}, \end{aligned}$$

   *respectively, are nonsingular (they are well-defined under our assumptions).*

2. *There exist positive semidefinite solutions $X_H(\gamma), X_J(\gamma)$ to the algebraic Riccati equations*

$$\begin{aligned} 0 &= H_{11}X_H + X_H H_{11}^T + H_{21} - X_H H_{12} X_H, \\ 0 &= J_{11}X_J + X_J J_{11}^T + J_{21} - X_J H_{12} X_J \end{aligned} \tag{2}$$

*associated with the Hamiltonian matrices*

$$\begin{aligned} H(\gamma) &= \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & -H_{11}^T \end{bmatrix} \\ &:= \begin{bmatrix} A & 0 \\ -C_1^T C_1 & -A^T \end{bmatrix} - \begin{bmatrix} B_1 & B_2 \\ -C_1^T D_{11} & -C_1^T D_{12} \end{bmatrix} R_H^{-1}(\gamma) \begin{bmatrix} D_{11}^T C_1 & B_1^T \\ D_{12}^T C_1 & B_2^T \end{bmatrix}, \\ J(\gamma) &= \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & -J_{11}^T \end{bmatrix} \\ &:= \begin{bmatrix} A^T & 0 \\ -B_1 B_1^T & -A \end{bmatrix} - \begin{bmatrix} C_1^T & C_2^T \\ -B_1 D_{11}^T & -B_1 D_{21}^T \end{bmatrix} R_J^{-1}(\gamma) \begin{bmatrix} D_{11} B_1^T & C_1 \\ D_{21} B_1^T & C_2 \end{bmatrix}. \end{aligned} \tag{3}$$

*3. If by $\rho(\cdot)$ we denote the spectral radius of a matrix, then $\rho(X_H X_J) < \gamma^2$,.*

Theorem 1 provides computationally feasible criteria to decide whether a value $\gamma$ is smaller or larger than the optimal $\gamma$ value $\gamma_{\text{opt}} = \|T_{zw}\|_\infty$ and one can then use a bisection process to determine $\gamma_{\text{opt}}$. However, as such a process approaches $\gamma_{\text{opt}}$, the failure of one of these conditions often means that either one of the two Riccati equations becomes exceedingly ill-conditioned as it approaches a non-solvable one, or one of the matrices $R_H$, $R_J$ becomes singular. Therefore, recently, different numerical methods have been designed [4, 6, 25] which avoid the formation of the Hamiltonian matrices and the solution of the Riccati equations by computing special deflating subspaces for certain matrix pencils. We will briefly recall this approach in Section 2.

Despite the described difficulties, the approach via the solution of Riccati equations is still quite popular, in particular, because the symmetry and definiteness of the solutions can be nicely monitored and allows for low-rank representations [7, 18]. To achieve a compromise between these nice properties and the avoidance of ill-conditioning, recently, in [30] a variation of the *structured doubling algorithm* (SDA), which was originally introduced in [13] for the solution of algebraic Riccati equations, has been presented that achieves improved stability and robustness by using a new representation of the relevant Lagrangian invariant subspaces of the Hamiltonian matrices in (3) which at the same time retains the Riccati solution properties. In this paper, we extend these ideas and adapt them to produce a robust implementation of the $\gamma$-iteration for $\mathcal{H}_\infty$ control. This requires, in particular, a nontrivial extension of the theory in [30] to deal with extended Lagrangian pencils, and turning the main iteration into a different form, which is not a variant of SDA but rather a version of the *inverse-free sign iteration* [3].

The paper is organized as follows. In Section 2 we recall some basic facts about algebraic Riccati equations, even pencils and Lagrangian subspaces and in Section 3 we introduce permutated graph bases of deflating subspaces of matrix pencils. We then consider structured versions of these graph bases for Lagrangian subspaces and Hamiltonian pencils in Section 4, and we extend them to a more general structure (called *partial Lagrangian*) in Section 5. In Section 6 we show how this approach applies to some special even pencils, and show how to extract a suitable Hamiltonian subpencil. In Section 7 we introduce the algorithm that we use to compute their stable deflating

subspaces, and in Section 8 we describe the full $\gamma$ iteration in this setting. Successively, we present in Section 9 an analysis of why a similar algorithm, the *doubling algorithm*, leads to a lower accuracy in the solution of our primary benchmark example. Finally, conclusions are presented.

## 2 Algebraic Riccati equations and even pencils

sec:algric

Algebraic Riccati equations are a classic tool in almost all areas of control theory. They allow for a nice and simple formulation of several results, but using them in numerical solution methods may be problematic because their solutions may be highly ill-conditioned even though the problem to be solved is well-conditioned. This is the case in particular for the $\gamma$-iteration, where frequently near the optimal $\gamma$ the solution of one the two Riccati equations in (2) becomes unbounded. Therefore, at least for small-scale dense problems, it is nowadays common numerical practice to replace the computation of the solutions to the Riccati equation by the computation of Lagrangian invariant subspaces for the Hamiltonian matrices in (3), by exploiting the following well-known result, see e. g. [29], formulated for $H(\gamma)$, with an obvious analogue for $J(\gamma)$.

riclag **Lemma 2.** *The algebraic Riccati equation with Hamiltonian matrix $H(\gamma)$ has a symmetric positive semidefinite solution $X_H(\gamma)$ if and only if $\begin{bmatrix} I \\ X_H(\gamma) \end{bmatrix}$ is the Lagrangian (semi-)stable invariant subspace of $H(\gamma)$.*

Here we call an invariant subspace *(semi-)stable* if it is associated with the eigenvalues in the (closed) left half plane. An $n$-dimensional subspace of $\mathcal{U} \subset \mathbb{R}^{2n}$ is called *Lagrangian* if $u^T \mathcal{J}_{2n} v = 0$ for all $u, v \in \mathcal{U}$, where

$$\mathcal{J}_{2n} := \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}.$$

It is an interesting observation that we do not actually need $X_H$ and $X_J$ to verify the last two conditions of Theorem 1, but rather only a basis for the invariant subspace.

thmY **Theorem 3.** *[6] Consider the Hamiltonian matrices (3) and associated Lagrangian invariant subspaces*

$$\begin{bmatrix} Y_H \\ Z_H \end{bmatrix}, \quad \begin{bmatrix} Y_J \\ Z_J \end{bmatrix}, \tag{4}$$ invsubs

*with all blocks in $\mathbb{R}^{n,n}$.*

1. *There exist symmetric matrices $X_H$, $X_J$ such that*

$$\begin{bmatrix} I_n \\ X_H \end{bmatrix}, \quad \begin{bmatrix} I_n \\ X_J \end{bmatrix}$$

*span the same invariant subspaces as (4) (respectively) if and only if $Y_H, Y_J$ are invertible. In this case, $X_H = Z_H Y_H^{-1}$ and $X_J = Z_J Y_J^{-1}$ are symmetric solutions*

*of (2). Furthermore, these are positive semidefinite if and only if the Lagrangian invariant subspaces are semistable.*

2. *Consider the matrix*

$$\mathcal{Y}(\gamma) = \begin{bmatrix} \gamma Y_H^T Z_H & Y_H^T Y_J \\ Y_J^T Y_H & \gamma Y_J^T Z_J \end{bmatrix}, \tag{5}$$

mathcalY

*constructed from the stable invariant subspaces of* (3) *partitioned as in* (4) *(thus all matrices depend implicitly on* $\gamma$*). Then,* $\mathcal{Y}(\gamma)$ *is positive semidefinite with constant rank* $r = \operatorname{rank} \mathcal{Y}(\gamma)$ *for all* $\gamma > \|T_{zw}\|_\infty$*. Conversely, if* $\gamma < \|T_{zw}\|_\infty$*, then either* $\operatorname{rank} \mathcal{Y}(\gamma) < r$ *or* $\mathcal{Y}(\gamma)$ *is not positive semidefinite.*

Theorem 3 implies that we can express the process of checking whether $\gamma$ is larger or smaller than $\gamma_{\text{opt}}$ in terms of the invariant subspaces and avoid computing the (possibly ill-conditioned) Riccati solutions.

Another problem arises if the $\gamma$ values are such that the matrices $R_H$ and $R_J$ are singular or close to singular. If one of them is singular then $\gamma \leq \gamma_{\text{opt}}$, but if one of the two matrices is close to singular, then not even the Hamiltonian matrices $H(\gamma)$ or $J(\gamma)$, respectively, should be formed. However, as has been shown in [4, 6, 25], then another formulation of the problem can be used to avoid inverting these matrices. This approach has the further advantage that it works with the original data of the problem.

**Lemma 4.** *Consider the optimal* $H_\infty$ *control problem with data as in (1). Then the matrices* $X_H(\gamma)$*,* $X_J(\gamma)$ *are the stabilizing solutions of the algebraic Riccati equations* (2) *if and only if*

$$\begin{bmatrix} X_H \\ I \\ * \\ * \\ * \end{bmatrix}, \begin{bmatrix} X_J \\ I \\ * \\ * \\ * \end{bmatrix}, \tag{6}$$

graphForm

*(for some suitable unspecified matrices denoted by* $*$*) are* $n$*-dimensional stabilizing deflating subspaces of the matrix pencils*

$$\begin{aligned} \lambda\mathcal{E} - \mathcal{A}_H &= \lambda \begin{bmatrix} 0 & I_n & 0 & 0 & 0 \\ -I_n & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & -A^T & 0 & 0 & -C_1^T \\ -A & 0 & B_1 & B_2 & 0 \\ 0 & B_1^T & \gamma^2 I_{m_1} & 0 & D_{11}^T \\ 0 & B_2^T & 0 & 0 & D_{12}^T \\ -C_1 & 0 & D_{11} & D_{12} & I_{p_1} \end{bmatrix}, \\ \lambda\mathcal{E} - \mathcal{A}_J &= \lambda \begin{bmatrix} 0 & I_n & 0 & 0 & 0 \\ -I_n & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & -A & 0 & 0 & -B_1 \\ -A^T & 0 & C_1^T & C_2^T & 0 \\ 0 & C_1 & \gamma^2 I_{p_1} & 0 & D_{11} \\ 0 & C_2 & 0 & 0 & D_{21} \\ -B_1^T & 0 & D_{11}^T & D_{21}^T & I_{m_1} \end{bmatrix}. \end{aligned} \tag{7}$$

The two pencils in (7) are so called *even pencils*, see [27], i. e., $\mathcal{E}$ is skew-symmetric and $\mathcal{A}$ is symmetric. To guarantee numerical accuracy, in particular in the neighborhood

of the optimal $\gamma$, the invariant subspaces should be computed using an algorithm that preserves this even structure exactly [5], since it is well-known that the condition number of the problem with respect to structure-preserving perturbations can be much lower than the unstructured one [10].

It is interesting to point out that the even structure is more general than the Hamiltonian structure. The main properties of interest in the context of optimal control problems associated to the Hamiltonian structure are that the eigenvalues are paired, that is, whenever $\lambda$ is an eigenvalue, then also $-\bar{\lambda}$ is an eigenvalue. Moreover, if there are no purely imaginary eigenvalues, then the stable and unstable invariant subspaces are Lagrangian, i. e., $n$-dimensional and the Riccati solutions (if they exist) are symmetric.

Even pencils also have the symmetry of the finite spectrum, but they may have odd dimension, and due to possible Kronecker blocks associated with the infinite eigenvalues, even if there are no purely imaginary eigenvalues, then the dimension of the deflating subspace may be less than $n$-dimensional. In addition, the deflating subspace need not be Lagrangian and the matrix $X$ resulting from a graph basis as in (6) may not be symmetric. However, it is known, see [26] and in an even more general context of control problems with variable coefficients [24] that one can make a change of variables to obtain a Hamiltonian substructure in each even pencil and that with respect to this substructure the same properties hold. In particular, if an even pencil of the form (7) is regular and of index at most one then there exists a $2n \times 2n$ Hamiltonian subproblem. We will exploit this fact in the following sections when computing the appropriate deflating subspaces. Note that a pencil $sE - A$ is called *regular* if it is square and $\det(sE - A)$ does not vanish identically for all complex numbers $s$. A regular pencil has *index at most one* if the Kronecker blocks associated with the eigenvalue $\infty$ have size at most one.

## 3 Permuted graph bases of (unstructured) subspaces and pencils

nstructuredPGB

In this section we recall the concept of permuted Lagrangian graph bases of [30] for the use in even pencils. In the following, we say that $U \in \mathbb{R}^{M+N,N}$ is a *basis matrix* for an $N$-dimensional subspace $\mathcal{U} \subseteq \mathbb{R}^{M+N}$ if $\operatorname{im} U = \mathcal{U}$ and $\ker U = \{0\}$.

Consider the following problem.

ProblemPGB

*Problem* 1 (Bounded permuted graph basis). Given $U \in \mathbb{R}^{M+N,N}$, with $M \geq N$, and a real number $T \geq 1$ (which represents a threshold), find $X = [x_{i,j}] \in \mathbb{R}^{M,N}$, an invertible $Y \in \mathbb{R}^{M,N}$, and a permutation matrix $\Pi \in \mathbb{R}^{M+N,M+N}$ such that

$$\Pi U = \begin{bmatrix} I \\ X \end{bmatrix} Y, \quad \text{and } |x_{i,j}| \leq T \text{ for all } i, j. \tag{8}$$

PGB

It has been shown in [30] that for any $U \in \mathbb{R}^{M+N,N}$ and $T \geq 1$, a solution to Problem 1 exists. Furthermore, if $T > 1$, then a solution can be computed in $O((M+N)^3 \log_T(M+N))$ floating point operations. The basic structure of the algorithm suggested in [30] is as follows:

A first guess for $\Pi$ is obtained via a QRP factorization of $U^T$, see e. g. [17]. If the resulting $X$ does not satisfy the threshold property, then it is improved iteratively via a sequence of rank one updates. One can prove that at most $O((M+N)\log_T(M+N))$ updates, each costing $O((M+N)^2)$ floating point operations, are sufficient to get the threshold bound, but in most cases only very few steps, or none at all, are needed in practice.

Given a matrix $U$, if we are only interested in its column space $\mathcal{U} = \operatorname{im} U$, then we can replace $U$ with

$$\hat{U} = \Pi^T \begin{bmatrix} I \\ X \end{bmatrix}. \tag{9}$$

defUhat

In other words, every subspace $\mathcal{U}$ admits a basis matrix $\hat{U} \in \mathbb{R}^{M+N,N}$ such that the identity matrix $I_N$ is a submatrix of $\hat{U}$, obtained by taking a suitable (ordered) subset of its rows, and all the other entries of $\hat{U}$ are bounded in modulus by $T$.

This special basis is well-conditioned, and can be stably computed from any other basis, in the following sense. Let $\kappa(A) := \sigma_{\max}(A)/\sigma_{\min}(A)$ denote the spectral norm condition number of $A$, which is defined also for rectangular matrices, where $\sigma_{\max}(A)$ and $\sigma_{\min}(A)$ denote the maximal and minimal singular value of $A$, see [17].

stability

**Theorem 5** ([30])**.** *For every matrix $U \in \mathbb{R}^{M+N}$, for any $\Pi, X$ satisfying* (8) *and $\hat{U}$ as in* (9) *we have the following condition estimates.*

- $\kappa(Y) \leq \kappa(U)\sqrt{MNT^2+1}$, *where $Y \in \mathbb{R}^{N\times N}$ is the basis change matrix, i. e., $U = \hat{U}Y$*
- $\kappa(\hat{U}) \leq \sqrt{MNT^2+1}$.

If one would use an orthogonal basis of $U$ in place of $\hat{U}$, then one would get the same bounds without the factor $\sqrt{MNT^2+1}$. In particular, the $\kappa(U)$ in the first bound cannot be avoided, as it represents the conditioning of the column space of the initial basis matrix $U$.

The permuted graph representation can be viewed as a sparse representation of the subspace, since it has much more zeroes than the generic orthogonal representation and if the matrix $X$ can furthermore be approximated by a low rank factorization. This representation is truly sparse in comparison with an orthogonal representation, without losing much in the conditioning of the subspace, a fact that can be exploited for large scale problems.

It is remarkable that Theorem 5 can also be used to generate different representations of matrix pencils. We say that two pencils $sE_1 - A_1$, $sE_2 - A_2$, with $E_i, A_i \in \mathbb{R}^{N,M}$, are *right-equivalent* if there exists an invertible matrix $Q \in \mathbb{R}^{N,N}$ such that $E_2 = QE_1$ and $A_2 = QA_1$. It is obvious that right-equivalent pencils have the same eigenvalues and deflating subspaces, therefore if we are interested only in these quantities then we may replace a matrix pencil with a different one in the same equivalence class. In other words, eigenvalues and right deflating subspaces depend only on the column space of the matrix

$$U = \begin{bmatrix} E^T \\ A^T \end{bmatrix}.$$

Therefore, we may replace $U$ with any other basis matrix of the same subspace, for instance

$$\hat{U} = \begin{bmatrix} \hat{E}^T \\ \hat{A}^T \end{bmatrix}$$

as in (9), and thus replace $sE - A$ with $s\hat{E} - \hat{A}$.

# 4 Permuted graph bases of Lagrangian subspaces, symplectic and Hamiltonian pencils

:LagrangianPGB

The theory of permuted graph bases presented in Section 3 has a structured analogue for Lagrangian subspaces. Given a vector $v \in \{0,1\}^N$, we define the associated *symplectic swap matrix* as

$$\Pi_v \in \mathbb{R}^{2N,2N} := \begin{bmatrix} \operatorname{diag}(1-v_i) & \operatorname{diag}(v_i) \\ -\operatorname{diag}(v_i) & \operatorname{diag}(1-v_i) \end{bmatrix}.$$

These are, up to a sign swap, all the permutation matrices generated by transpositions $(i, N+i)$, including in particular the identity ($v_i = 0$ for all $i$) and $\mathcal{J}_{2N}$ ($v_i = 1$ for all $i$). We denote the set of all symplectic swap matrices in $\mathbb{R}^{2N,2N}$ by $\mathcal{S}^N$. Note that the elements of $\mathcal{S}^N$ are symplectic and orthogonal.

Using symplectic swap matrices in place of the permutations, we can formulate a version of Problem 1 for Lagrangian subspaces.

ProblemPGBL

*Problem* 2 (Bounded Lagrangian graph basis). Given $U \in \mathbb{R}^{2N,N}$ such that $\mathcal{U} = \operatorname{im} U$ is Lagrangian, and a threshold $T \geq \sqrt{2}$, find a symmetric matrix $X = [x_{i,j}] \in \mathbb{R}^{N,N}$, an invertible matrix $Y \in \mathbb{R}^{N,N}$, and a symplectic swap matrix $\Pi_v$ such that

$$\Pi_v U = \begin{bmatrix} I \\ X \end{bmatrix} Y, \quad \text{and } |x_{i,j}| \leq T \text{ for all } i,j.$$

It was shown in [30] that for any basis matrix $U$ of a Lagrangian subspace, a solution to Problem 2 exists. Furthermore, if $T > \sqrt{2}$, then such a solution can be computed in $O(N^3 \log_T N)$ floating point operations.

The algorithm described in [30] is similar to the one for the unstructured case. In particular, one starts by taking the $\Pi_v$ obtained from a special $QR\Pi_v$ factorization, which is a variant of QRP factorization where one takes a symplectic swap matrix in place of a permutation.

In other words, every Lagrangian subspace $\mathcal{U}$ admits a basis matrix of the form

$$\hat{U} = \Pi_v^T \begin{bmatrix} I \\ X \end{bmatrix}, \quad X = X^T, \quad |x_{i,j}| \leq T \text{ for all } i,j.$$

The results of Theorem 5 hold in this case as well. While in the unstructured case an orthogonal basis matrix is slightly better conditioned but has much less sparsity, there is another clear advantage in the Lagrangian case because the Lagrangian property is

equivalent to the symmetry of $X$, so it is very easy to enforce and preserve in finite precision computations.

The matrix pencil version of this result gives bounded structure-preserving representations of regular Hamiltonian pencils. A matrix pencil $sE - A$, $A, E \in \mathbb{R}^{2n,2n}$ is called *Hamiltonian*, see e. g. [29], if $E\mathcal{J}_{2n}A^T + A\mathcal{J}_{2n}E^T = 0$ and it should be noted that this property is invariant under right equivalence.

**Theorem 6** ([30]). *For every regular Hamiltonian pencil $sE - A$, $A, E \in \mathbb{R}^{2n\times 2n}$,*

$$U = \begin{bmatrix} E^T \\ \mathcal{J}_{2n}A^T \end{bmatrix}$$

*is Lagrangian. Moreover, every regular Hamiltonian pencil has a right-equivalent pencil $s\hat{E} - \hat{A}$ such that*

$$\hat{U} = \begin{bmatrix} \hat{E}^T \\ \mathcal{J}_{2n}\hat{A}^T \end{bmatrix} \tag{10}$$

HamiltonianPGR

*is as in* (9).

In particular, the columns of $\hat{E}$ and $\hat{A}$ are, after multiplication with a symplectic swap matrix, those of an identity matrix $I_{2n}$ and those of a bounded symmetric matrix.

## 5 Partial Lagrangian subspaces

sec:parlag

It is a non-trivial task to extend the results of the previous section to deal directly with pencils of the form (7). One quickly realizes that the even structure is not invariant under right equivalence as can be seen from the results in [24, 26] the Hamiltonian structure is only contained in a subproblem. The work [35] uses so-called *generalized Lagrangian* invariant subspaces for matrix pencils in the form (7), i. e., maximal neutral subspaces $\mathcal{U}_g$ such that $v^T\mathcal{E}u = 0$ for all $u, v \in \mathcal{U}_g$, where $\mathcal{E}$ is as in (7). With this approach, however, it is not clear whether we can obtain results analogous to those in Section 6. Instead we will work with a slightly different concept. We say that a subspace $\mathcal{U}_p \subseteq \mathbb{R}^{2N+M}$ is *partial Lagrangian* if it has dimension $N + M$ and there exist $L \in \mathbb{R}^{2N\times M}$ and $L_e \in \mathbb{R}^{M\times M}$ such that with

$$\mathcal{J}_p = \begin{bmatrix} \mathcal{J}_{2N} & L \\ -L^T & L_e - L_e^T \end{bmatrix}, \tag{11}$$

defJe

we have $u^T\mathcal{J}_p v = 0$ for all $u, v \in \mathcal{U}_p$. If $U_p \in \mathbb{R}^{(2N+M)\times(N+M)}$ is a basis matrix for $\mathcal{U}_p$, then this condition takes the form $U_p^T\mathcal{J}_p U_p = 0$.

We have the following invariance property of partial Lagrangian subspaces.

Pie

**Lemma 7.** *Let $U_p \in \mathbb{R}^{(2N+M)\times(N+M)}$ and suppose that $\operatorname{im} U_p$ is partial Lagrangian. Let, moreover, $\Pi \in \mathcal{S}^N$, $Q \in \mathbb{R}^{M,M}$, $\Pi_p = \operatorname{diag}(\Pi, Q)$. Then, $\operatorname{im} \Pi_p U_p$ is partial Lagrangian as well.*

*Proof.* Since $\Pi \in \mathcal{S}^N$, we have that

$$\Pi_p^T \mathcal{J}_p \Pi_p = \begin{bmatrix} \Pi^T \mathcal{J}_{2N} \Pi & \Pi^T L Q \\ -Q^T L^T \Pi & Q^T (L_e - L_e^T) Q \end{bmatrix} = \begin{bmatrix} \mathcal{J}_{2N} & \hat{L} \\ -\hat{L}^T & \hat{L}_e - \hat{L}_e^T \end{bmatrix}$$

is still in the form (11). □

Using Lemma 7 we can extend the result in Section 4 to partial Lagrangian subspaces.

th:pdj **Theorem 8.** *Let*

$$U_p = \begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix}, \quad U_1, U_2 \in \mathbb{R}^{N,(N+M)}, \quad U_3 \in \mathbb{R}^{M,(N+M)} \tag{12}$$ Upartitioned

*be the basis matrix of a partial Lagrangian subspace $\mathcal{U} = \operatorname{im} U$. Then,*

1. *There exists $\Pi \in \mathcal{S}^N$ such that the last $N + M$ rows of*

$$\operatorname{diag}(\Pi, I_M) U_p$$

   *form an invertible matrix.*

2. *For each $T \geq \sqrt{2}$, there exist a $\Pi \in \mathcal{S}^N$ and a nonsingular $Y_p \in \mathbb{R}^{N+M,N+M}$ such that*

$$\operatorname{diag}(\Pi, I_M) U_p = \begin{bmatrix} X & X_e \\ I_N & 0 \\ 0 & I_M \end{bmatrix} Y_p, \tag{13}$$ Uext

   *with*

$$X = [x_{i,j}] = X^T, \ |x_{ij}| \leq T. \tag{14}$$ Xbound

   *In other words, $\mathcal{U}$ admits a basis matrix of the form*

$$\operatorname{diag}(\Pi, I_M)^T \begin{bmatrix} X & X_e \\ I_N & 0 \\ 0 & I_M \end{bmatrix}$$

   *with $X$ satisfying* (14).

*Proof.* Let

$$U_3^T = Q \begin{bmatrix} R \\ 0_{2N \times M} \end{bmatrix}, \quad R \in \mathbb{R}^{N \times N}$$

be a QR factorization of $U_3^T$, and set

$$U_p Q = \begin{bmatrix} U & U_e \\ 0 & R^T \end{bmatrix} \quad U \in \mathbb{R}^{2N \times N}, U_e \in \mathbb{R}^{2N \times M}.$$

Note that $R$ must be nonsingular, since otherwise $\mathcal{U}_p$ would have dimension strictly smaller than $M+N$.

The partial Lagrangian condition implies that $\operatorname{im} U$ is Lagrangian. Therefore, we can find a symplectic swap matrix $\Pi_w$ such that the trailing $N \times N$ block of $\Pi_w U$ is nonsingular (Problem 2 tells us how to find such a $\Pi_v U$ that has the leading $N \times N$ block nonsingular. Then it is easy to see that taking $w_i = 1 - v_i$ for all $i$ gives the desired result). Therefore, the last $M+N$ rows of $\operatorname{diag}(\Pi_w, I_M)U_p$ form a nonsingular matrix, as required.

Similarly, we can use Problem 2 to obtain $\Pi_w$ such that

$$\Pi_w U = \begin{bmatrix} X \\ I \end{bmatrix} Y$$

with $X$ satisfying (14). We can write

$$\operatorname{diag}(\Pi_w, I_M)U_p = \begin{bmatrix} X & Z_e \\ I & Y_e \\ 0 & R^T \end{bmatrix} \begin{bmatrix} Y & 0 \\ 0 & I \end{bmatrix}$$

for two suitable blocks $Z_e, Y_e \in \mathbb{R}^{N \times M}$. Then it suffices to take

$$Y_p = \begin{bmatrix} I & Y_e \\ 0 & R^T \end{bmatrix}^{-1} \begin{bmatrix} Y & 0 \\ 0 & I \end{bmatrix}. \qquad \square$$

Note that the proof is constructive: given a threshold $T > \sqrt{2}$, we can compute the QR factorization of $U_3^T$ and the solution of Problem 2 using the algorithm in [30], and obtain an explicit representation of the form (13).

# 6 Partial Hamiltonian pencils and extraction of Hamiltonian subpencils

sec:parham

In this section we use the ideas of the previous section to derive a method for the extraction of Hamiltonian subpencils of even pencils.

Let

$$s\mathcal{E} - \mathcal{A} = s \begin{bmatrix} \tilde{E} & 0 \\ \tilde{F} & 0 \end{bmatrix} - \begin{bmatrix} \tilde{A} & \tilde{B} \\ \tilde{C} & \tilde{R} \end{bmatrix}, \tag{15}$$

funnyForm

with $\tilde{E}, \tilde{A} \in \mathbb{R}^{N,N}, \tilde{B}, \tilde{C}^T, \tilde{F}^T \in \mathbb{R}^{N,M}, \tilde{R} \in \mathbb{R}^{M,M}$, be a matrix pencil with $M$ zero columns as in (7). We say that $s\mathcal{E} - \mathcal{A}$ is *partial Hamiltonian* if $N = 2n$ and

$$\mathcal{U}_p := \operatorname{im} U_p = \operatorname{im} \begin{bmatrix} \tilde{E} & \tilde{A}\mathcal{J}_N & \tilde{B} \\ \tilde{F} & \tilde{C}\mathcal{J}_N & \tilde{R} \end{bmatrix}^T \tag{16}$$

partlyLagrangi

is partial Lagrangian.

Note that eigenvalues and right eigenvectors of a partial Hamiltonian pencil are well-defined up to pre-multiplication by a nonsingular matrix, and thus they depend on the subspace $\mathcal{U}_p$ only (rather than on $U_p$). In particular, it is always possible to reduce to the case $\tilde{B} = 0$ by pre-multiplying with a suitable matrix. Therefore, a partial Hamiltonian matrix pencil is always equivalent to one of the form

$$sM\mathcal{E} - M\mathcal{A} = s\begin{bmatrix}\hat{E} & 0\\ \hat{F} & 0\end{bmatrix} - \begin{bmatrix}\hat{A} & 0\\ \hat{C} & \hat{R}\end{bmatrix}, \tag{17}$$

triangularPenc

with $\hat{R}$ having full column rank. Multiplying out the leading $N \times N$ block of $U_p^T \mathcal{J}_p U_p = 0$ yields $\hat{E}\mathcal{J}_N\hat{A}^T + \hat{A}\mathcal{J}_N\hat{E}^T = 0$, so the partial pencil $s\hat{E} - \hat{A}$ is Hamiltonian.

Since the pencil (17) is block triangular, this shows that partial Hamiltonian pencils have the eigenvalue $\infty$ with multiplicity at least $M$, the remaining eigenvalues have Hamiltonian eigensymmetry, and they can be associated to invariant subspaces of the form

$$\operatorname{im}\begin{bmatrix}Z_1 & 0\\ Z_2 & 0\\ 0 & I\end{bmatrix},$$

where

$$\operatorname{im}\begin{bmatrix}Z_1\\ Z_2\end{bmatrix}$$

are invariant subspaces of the Hamiltonian sub-pencil $s\hat{E} - \hat{A}$.

The following results shows that even pencils in the form (7) are partial Hamiltonian.

tlyHamiltonian

**Theorem 9.** *Let $s\mathcal{E} - \mathcal{A}$ be a pencil as in* (15) *with $\tilde{E} = \mathcal{J}_N$, $\tilde{A} = \tilde{A}^T$, $\tilde{C} = \tilde{B}^T$, $\tilde{R} = \tilde{R}^T$, $\tilde{F} = 0$. Assume that $\mathcal{E}$ and $\mathcal{A}$ have no common left or right nullspace. Then, $s\mathcal{E} - \mathcal{A}$ is partial Hamiltonian.*

*Proof.* Due to the fact that $\mathcal{E}$ and $\mathcal{A}$ have no common left or right nullspace, there exist matrices $L_1, L_2, L_e$ such that

$$\begin{bmatrix}\mathcal{J}_N & \tilde{A}\mathcal{J}_N & \tilde{B}\\ 0 & \tilde{B}^T\mathcal{J}_N & \tilde{R}\end{bmatrix}\begin{bmatrix}L_1\\ L_2\\ L_e\end{bmatrix} = \begin{bmatrix}0\\ I_M\end{bmatrix},$$

and it follows that

$$\begin{bmatrix}\mathcal{J}_N & \tilde{A}\mathcal{J}_N & \tilde{B}\\ 0 & \tilde{B}^T\mathcal{J}_N & \tilde{R}\end{bmatrix}\begin{bmatrix}0 & I_N & 0\\ -I_N & 0 & 0\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}\mathcal{J}_N & \tilde{A}\mathcal{J}_N & \tilde{B}\\ 0 & \tilde{B}^T\mathcal{J}_N & \tilde{R}\end{bmatrix}^T = \begin{bmatrix}0 & \tilde{B}\\ -\tilde{B}^T & 0\end{bmatrix},$$

$$\begin{bmatrix}\mathcal{J}_N & \tilde{A}\mathcal{J}_N & \tilde{B}\\ 0 & \tilde{B}^T\mathcal{J}_N & \tilde{R}\end{bmatrix}\begin{bmatrix}0 & 0 & L_1\\ 0 & 0 & L_2\\ -L_1^T & -L_2^T & L_e - L_e^T\end{bmatrix}\begin{bmatrix}\mathcal{J}_N & \tilde{A}\mathcal{J}_N & \tilde{B}\\ 0 & \tilde{B}^T\mathcal{J}_N & \tilde{R}\end{bmatrix}^T$$

$$= \begin{bmatrix}0 & 0 & 0\\ 0 & 0 & I_m\end{bmatrix}\begin{bmatrix}\mathcal{J}_N & \tilde{A}\mathcal{J}_N & \tilde{B}\\ 0 & \tilde{B}^T\mathcal{J}_N & \tilde{R}\end{bmatrix}^T - \begin{bmatrix}\mathcal{J}_N & \tilde{A}\mathcal{J}_N & \tilde{B}\\ 0 & \tilde{B}^T\mathcal{J}_N & \tilde{R}\end{bmatrix}\begin{bmatrix}0 & 0\\ 0 & 0\\ 0 & I_m\end{bmatrix} = \begin{bmatrix}0 & -\tilde{B}\\ \tilde{B}^T & 0\end{bmatrix}.$$

Hence, $s\mathcal{E} - \mathcal{A}$ is partial Hamiltonian with $L_e$ as given and $L = \begin{bmatrix} L_1 \\ L_2 \end{bmatrix}$. □

This results allows to construct a procedure to obtain a Hamiltonian pencil in the form (10) from every even pencil satisfying the hypotheses of Theorem 9. One first computes a permuted graph basis matrix $\hat{U}$ for the partial Lagrangian subspace (16), which exists because of Theorem 8. The associated pencil is now in the form (17), so we can restrict the invariant subspace computations to the pencil $s\hat{E} - \hat{A}$.

noinverse

*Remark* 1. An important observation is that in this extraction procedure we neither need an inversion of the matrix $R$, nor the computation of its kernel. Since even pencils with infinite eigenvalues in the form of Theorem 9 arise frequently in optimal control problems [26, 29, 23], several techniques have been developed to deflate the unwanted invariant subspace at infinity and reduce the problem to Hamiltonian matrices or pencils [11, 29, 37]. However, all these techniques use either orthogonal transformations to staircase form followed by an inversion of the matrix associated with the index one part of the eigenvalues at $\infty$ or destroy the even structure to return to a subpencil having only finite eigenvalues. When $R$ has small singular values, then a difficult numerical rank decision is needed. Here, we have shown that neither determining the kernel of $R$, nor the Kronecker structure of an even pencil, is necessary to perform this extraction of the Hamiltonian subpencil, and thus ill-conditioning in the former problem should not affect the latter. Difficulties, however, may arise if $s\mathcal{E} - \mathcal{A}$ is close to a singular pencil.

## 7 Inverse-free sign method with permuted graph bases

ec:inversefree

In the previous section we have shown how to reduce the problem of computing the semi-stable subspace of an even pencil to the computation of the stable Lagrangian subspace of a Hamiltonian pencil, we need an algorithm to do this robustly. Our first attempt was to follow the approach in [30] and extend the algorithm described there, a doubling algorithm implemented with permuted graph bases. However, when applying this concept to the benchmark examples of [6], in some of the difficult examples this algorithmic idea failed to compute reliably the smallest singular values of the matrix $\mathcal{Y}(\gamma)$ in (5). Better results can be obtained with an algorithm which is rooted in the same ideas, but differs slightly from the doubling approach by switching the inner iteration from doubling to the inverse-free sign method. We describe this method and our modifications in this section.

We recall that, for any square matrix $M$ without imaginary eigenvalues and with Jordan form $M = WJW^{-1}$, the matrix sign function [22] is defined as $\operatorname{sgn} M = WDW^{-1}$, where $D$ is the diagonal matrix obtained from $J$ by replacing a block $J_{\lambda_i}$ of size $k_i$ and eigenvalue $\lambda_i$ with $\operatorname{sgn} \lambda_i I_{k_i}$. The matrix sign function can be obtained as the limit of the iteration

$$M_{k+1} = \frac{1}{2}(\mu_k M_k + \mu_k^{-1} M_k^{-1}), \tag{18}$$

ordinarynms

for any suitable choice of the scaling parameters $\mu_k > 0$. The stable invariant subspace of $M_k$ can then obtained as $\ker(\operatorname{sgn}(M) + I)$ and the unstable one as $\ker(\operatorname{sgn}(M) - I)$.

The general *inverse-free sign method* [3] proceeds as follows. Given a regular matrix pencil $sE_k - A_k$, $E_k, A_k \in \mathbb{R}^{N,N}$, determine $C_k, S_k \in \mathbb{R}^{N,N}$ such that $C_k A_k = S_k E_k$ and $\operatorname{rank}\begin{bmatrix} C_k & S_k \end{bmatrix} = N$, and then set

$$sE_{k+1} - A_{k+1} = sS_kE_k - \frac{1}{2}\left(\mu_k S_k A_k + \mu_k^{-1} C_k E_k\right), \tag{19}$$ nms

where $\mu_k$ is a suitably-chosen scaling factor that can be tuned to achieve numerical stability and to improve the convergence speed. One can see that the iteration (19) is algebraically equivalent to (18), whenever $E_0$ and $A_0$ are nonsingular. On the other hand, it is typically more stable and reliable when they are ill-conditioned, and it works even when these matrices are singular. The convergence of the method is characterized in the following result.

**Theorem 10** ([3]). *When started from a regular pencil $sE_0 - A_0$ without purely imaginary or infinite eigenvalues, the iteration* (19) $E_k^{-1}A_k$ *converges quadratically to* $\operatorname{sgn} E_0^{-1}A_0$.

For the iteration (19) to perform stably, a crucial point is the selection of the factors $C_k$ and $S_k$. The usual choice in the general case, [1, 3, 28], is to compute a QR factorization

$$\begin{bmatrix} R_k \\ 0 \end{bmatrix} = \begin{bmatrix} Q_{11}^{(k)} & Q_{12}^{(k)} \\ Q_{21}^{(k)} & Q_{22}^{(k)} \end{bmatrix} \begin{bmatrix} A_k \\ E_k \end{bmatrix} \tag{20}$$ QRSign

and to choose $C_k = Q_{21}^{(k)}$, $S_k = -Q_{22}^{(k)}$. We suggest a different approach based on the results described in Section 3. Namely, for a given tolerance $T > 0$ we compute $\Pi$, $X = [x_{i,j}]$ with $|x_{i,j}| \le T$, as well as

$$\hat{U} = \Pi \begin{bmatrix} I \\ X \end{bmatrix} \quad \text{such that} \quad \begin{bmatrix} A_k \\ E_k \end{bmatrix} = U = \hat{U}Y, \tag{21}$$ CS1

with $Y$ nonsingular, and then choose

$$\begin{bmatrix} C_k & -S_k \end{bmatrix} = \begin{bmatrix} -X & I \end{bmatrix} \Pi^T. \tag{22}$$ CS2

Using these settings, it is simple to check that $\begin{bmatrix} C_k & -S_k \end{bmatrix} \begin{bmatrix} A_k \\ E_k \end{bmatrix} = 0$ and the resulting $C_k$ and $S_k$ have their elements bounded in modulus by $T$. Again, as mentioned before, this is a compromise between orthogonality and sparsity of the subspace representation.

Since the computation of $X$ is based on a QR decomposition with column pivoting, it may not be obvious where this approach differs from the classical sign-function iteration exactly. The following remarks discuss this point.

- In (20), the QR factorization of $U \in \mathbb{R}^{2N,N}$ is needed and the $Q$ factor has dimension $2N \times 2N$. In the solution of Problem 1, the QR factorization of $U^T$ is needed instead, with a $Q$ factor of dimension $N \times N$.

- Moreover, in Problem 1, we do not need the factor $Q$, but rather only the rectangular factor $R$. Therefore, the flop count for the factorization is of order $\frac{10}{3}N^3$ compared with $\frac{38}{3}N^3$ (using the flop counts in [22, Appendix C]).

If the starting pencil $sE_0 - A_0$ (respectively the starting matrix $M_0$) is Hamiltonian, then each $sE_k - A_k$ in (19) (resp. $M_k$ in (18)) is Hamiltonian as well. Thus, in addition, following the ideas in [30] for the doubling algorithm, after each step of the iteration we may use the solution of Problem 2 to transform the resulting pencil in the structured form (10). The resulting algorithm is as follows.

ifs

**Algorithm 1:** Inverse-free matrix sign method with permuted graph bases

**Input**: A regular $2n \times 2n$ Hamiltonian pencil $sE_0 - A_0$ that has no purely imaginary eigenvalues

**Output**: Bases $V_s, V_u$ for its stable and unstable invariant subspaces

1 **for** $k = 0, 1, 2, \ldots, nSteps$, *while a stopping criterion is not satisfied* **do**

2 Compute $\hat{E}_k, \hat{A}_k$ as in (10) (with an additional subscript $k$ for all the matrices) by solving Problem (2);

3 Compute $\hat{U} = \Pi \begin{bmatrix} I \\ X \end{bmatrix}$ such that $\begin{bmatrix} \hat{A}_k \\ \hat{E}_k \end{bmatrix} = \hat{U}Y$, by solving Problem (1);

4 Set $\begin{bmatrix} C_k & -S_k \end{bmatrix} = \begin{bmatrix} -X & I \end{bmatrix} \Pi^T$, so that $C_k A_k = S_k E_k$; products1

5 Compute $sE_{k+1} - A_{k+1} = sS_k\hat{E}_k - \frac{1}{2}\left(\mu_k S_k \hat{A}_k + \mu_k^{-1} C_k \hat{E}_k\right)$;

6 **end**

Cay1 7 Compute an SVD $A_{nSteps} - E_{nSteps} = USV^T$. Set $V_s$ to be the last $n$ columns of $V$;

Cay2 8 Compute an SVD $A_{nSteps} + E_{nSteps} = USV^T$. Set $V_u$ to be the last $n$ columns of $V$;

As a stopping criterion, we can use the distance between two consecutive iterates $\|\hat{E}_{k+1} - \hat{E}_k\| + \|\hat{A}_{k+1} - \hat{A}_k\|$.

## 8 The complete $\gamma$-iteration

sec:gammait

Based on the described inverse free sign function iteration combined with a permuted Lagrangian graph representation of the space, we have implemented a version of the $\gamma$-iteration [6, 19, 40]. This iteration is a one-parameter optimization procedure based on bisection and the secant method to compute the optimal value $\gamma_{\text{opt}}$, based on the fact that Theorem 1 gives a criterion to check if a given value $\gamma$ is larger or smaller than $\gamma_{\text{opt}}$.

- The first condition in Theorem 1 is checked by solving a generalized eigenvalue problem; it does not depend on the tested value $\gamma$, so this computation can be made once before starting the iteration.
- The second condition in Theorem 1 is checked by computing the invariant subspace of (7) using Algorithm 1. Here we take the simple approach of declaring that there is no solution if the algorithm does not converge after a sufficient number of iterations.

| Example | Our results | Results from [6] | Results `hinfopt` (from [6]) |
|---|---|---|---|
| [6, Ex. 6.1], | $a = 1$: RelErr $= 5 \cdot 10^{-14}$; $a = 10^{-8}$: RelErr $= 5 \cdot 10^{-14}$; $a = 10^{-10}$: RelErr $= 5 \cdot 10^{-14}$; $a = 10^{-12}$: RelErr $= 1 \cdot 10^{-13}$; $a = 10^{-14}$: RelErr $= 2 \cdot 10^{-8}$; | 13 correct digits up to $a = 10^{-7}$; fails for $a = 10^{-8}$ and below | $a = 1$: RelErr $= 2 \cdot 10^{-14}$; $a = 10^{-8}$: RelErr $= 6 \cdot 10^{-14}$; $a = 10^{-10}$: RelErr $= 9 \cdot 10^{-14}$; $a = 10^{-12}$: RelErr $= 1 \cdot 10^{-9}$; $a = 10^{-14}$: RelErr $= 9 \cdot 10^{-7}$; |
| [6, Ex. 6.2] | RelErr $= 1 \cdot 10^{-11}$ | 13 correct digits | fails |
| [6, Ex. 6.3] | no exact value available; the two methods agree to 13 digits | | fails |
| [6, Ex. 6.4], $\alpha = 3$ | RelErr $= 6 \cdot 10^{-13}$ | 13 correct digits | RelErr $= 4 \cdot 10^{-15}$ |

Table 1: Results for the examples in [6] with required tolerance $\epsilon = 10^{-14}$ in the $\gamma$-iteration. `Results`

- The third condition in Theorem 1 is checked with the help of Theorem 3, which converts it into computing the inertia of the symmetric matrix $\mathcal{Y}(\gamma)$. To this purpose, it is crucial that the computed $\mathcal{Y}(\gamma)$ from the Lagrangian invariant subspaces has some accuracy in its smallest eigenvalues, which is not an easy property to ensure in the neighborhood of $\gamma_{\text{opt}}$.

In the following numerical examples, for simplicity we have used the scaling factor $\mu_k = 1$. Using more sophisticated scaling strategies did not yield a significant advantage.

We implemented the method using MATLAB R2011a, and compared it with the method described in [6] and with `hinfopt` from the Robust Control Toolbox, testing it on the same test problems as in [6]. The results are displayed in Table 1.

The most demanding problem in the set of test cases is Example 6.1. It is a parametric problem in which some of the eigenvalues are of the order of unity and some are of the order of $a$. When $a$ is small, a very accurate numerical method is needed in order to deal properly with the small eigenvalues. The results show that our method can overall obtain results that are on par with or better than the other existing methods.

The code used for these experiments is published online [33]. It consists of a full set of MATLAB functions to work with permuted graph bases of matrices and pencils, both structured (Lagrangian, symplectic, Hamiltonian) and unstructured, an implementation of doubling and of the inverse-free sign algorithm, and routines to solve CAREs, compute invariant subspaces of Hamiltonian pencils and even pencils in the form (7), and for the $\gamma$-iteration. The code is not optimized for time and high-performance computing, but rather aimed to researchers who wish to change the parameters and test new iterations.

## 9 The drawbacks of doubling

`sec:baddouble`

It is interesting to discuss briefly what happens if we use the doubling method described in [30] instead of Algorithm 1 to compute the Lagrangian invariant subspaces of the Hamiltonian pencil. In Example 6.1, we get full precision for $a = 1$ and $a = 10^{-1}$, but

already for $a = 10^{-2}$ the $\gamma$-iteration fails to converge due to the inaccuracy of the small eigenvalues associated with the computed subspaces. This is surprising, since the two methods are quite similar. Indeed, we briefly present here the doubling algorithm, as Algorithm 2, in a fashion that underlines its similarities with the inverse-free sign method. Doubling and inverse-free sign are essentially the same iteration, once carried out in

doubling

**Algorithm 2:** Inverse-free doubling method with permuted graph bases

**Input**: A $2n \times 2n$ Hamiltonian pencil $sE_0 - A_0$ without purely imaginary eigenvalues

**Output**: Bases $V_s, V_u$ for its stable and unstable invariant subspaces

cayley

1 For a suitable $\alpha > 0$, compute $M_0 = A_0 + \alpha E_0$, $N_0 = A_0 - \alpha E_0$ (Cayley transform);

2 **for** $k = 0, 1, 2, \ldots, nSteps$, *while a stopping criterion is not satisfied* **do**

3 Compute a special equivalent pencil $s\hat{M}_k - \hat{N}_k$ using a symplectic variant of (10), by solving an instance of Problem (2);

rightEq

4 Compute $\hat{U} = \Pi \begin{bmatrix} I \\ X \end{bmatrix}$ such that $\begin{bmatrix} \hat{M}_k \\ \hat{N}_k \end{bmatrix} = \hat{U}Y$, by solving an instance of Problem (1);

5 Set $\begin{bmatrix} C_k & -S_k \end{bmatrix} = \begin{bmatrix} -X & I \end{bmatrix} \Pi^T$, so that $C_k M_k = S_k N_k$;

products2

6 Compute $sM_{k+1} - N_{k+1} = sS_k\hat{M}_k - C_k\hat{N}_k$;

7 **end**

8 Compute an SVD $N_{nSteps} = USV^T$. Set $V_s$ to be the last $n$ columns of $V$;

9 Compute an SVD $M_{nSteps} = USV^T$. Set $V_u$ to be the last $n$ columns of $V$;

the Hamiltonian setting and once in the symplectic setting after a Cayley transform (Line 1 of Algorithm 2). The final accuracy obtained with doubling depends also on the parameter $\alpha$ of this Cayley transform; while some heuristic strategies exist for its choice [13, 30, 34], none of them is completely satisfactory. Although good results in the subspace residual metric are attainable with doubling [30], it seems that this accuracy does not transfer completely to the more fragile accuracy of the small eigenvalues of $\mathcal{Y}(\gamma)$, thus rendering doubling inadequate for our problem.

It is instructive to see where exactly the accuracy is lost when applying Algorithm 2 to Example 6.1. First of all, we point out that the normwise backward error does not highlight any difference between the two methods, as the subspaces $\mathcal{U}_d$ and $\mathcal{U}_s$ computed by the doubling and inverse-free sign method, respectively, are both exact invariant subspaces of pencils that are within distance $\approx 10^{-15}$ from the real one $sE_0 - A_0$. Nevertheless, using $\mathcal{U}_s$, the smallest eigenvalues of $\mathcal{Y}(\gamma)$ are accurate, while using $\mathcal{U}_d$ they are perturbed by $\approx 10^{-9}$. The pencil is indeed ill-conditioned, and perturbations of order $\approx 10^{-15}$ can result in a variation of magnitude $\approx 10^{-8}$ of the invariant subspace.

Therefore, we used MATLAB's variable precision arithmetic functions to compute the exact invariant subspaces of the pencils with high precision, to check where the loss of accuracy takes place. Our first thought was that the Cayley transform (Line 1 of Algorithm 2) was to blame. However, the exact invariant subspace of $sM_0 - N_0$ is at distance $\approx 10^{-15}$ from that of $sE_0 - A_0$, so this step is safe. The loss of accuracy takes

place immediately after, Line 3 in the first iteration of the "for" cycle, that is, when we compute a right-equivalent pencil in permuted graph form. Indeed, the invariant subspace of this pencil lies at distance $\approx 10^{-9}$ from the original one. The permuted graph representation is not to blame here, however, because using an orthogonal representation (i.e., pre-multiplying by a $M$ that makes $\begin{bmatrix} M_0 & N_0 \end{bmatrix}^T$ orthogonal) results in an error of the same order.

This phenomenon does not happen in Algorithm 1. When we compute $\hat{E}_0, \hat{A}_0$ in either a permuted graph or an orthogonal representation of $sE_0 - A_0$, the stable invariant subspace is perturbed by $\approx 10^{-15}$ only. So it seems that, while the Cayley transform *per se* does not immediately cause inaccuracy, it returns a pencil that is more "fragile" with respect to small perturbations, and accuracy is lost as soon as we proceed with the algorithm and multiply it by a nonsingular matrix from the left.

As mentioned before, all these phenomena could be completely explained only by an entrywise perturbation analysis, however, such a perturbation analysis for doubling-type algorithms is an open problem.

*Remark* 2. Note that there is also a Cayley transform hidden in Algorithm 1 as well. It is in Lines 7 to 8, where we switch to $A_{nSteps} \pm E_{nSteps}$ to compute the final subspaces. However, this is performed only after convergence has been reached. While a Cayley transform on the original pencil is dangerous, since eigenvalues close to the imaginary axis could be perturbed from the stable part of the spectrum to the unstable one or vice versa, at the end of the algorithm the converged pencil $sE_{nSteps} - A_{nSteps}$ has eigenvalues only in 1 and $-1$. Therefore, we expect the two subspaces to be reasonably well separated, and the Cayley transform to be less problematic.

*Remark* 3. There is another subtle difference between the two algorithms. In Line 4 of Algorithm 1 three matrix products are needed, while in doubling two products in Line 5 suffice. A natural question is whether there is a way to rearrange the computations in order to get rid of one of those products. It is easy to come up with the identity $(\mu S + C)(\hat{A} + \mu^{-1}\hat{E}) - C\hat{A} - \hat{S}E = \mu S\hat{A} + \mu^{-1}C\hat{E}$, which reduces the formulas to two products only, since $C\hat{A} = S\hat{E}$. However, as in the Cayley transform, we would be working with linear combinations of $\hat{A}$ and $\hat{E}$ instead of the two matrices themselves, and this reformulation may be less stable.

## 10 Conclusions

The numerically robust solution of the optimal $\mathcal{H}_\infty$ control problem is difficult in finite precision arithmetic, because in the neighborhood of the optimal $\gamma_{\text{opt}}$ the necessary computational steps in the $\gamma$ iteration may be very inaccurate if the structure of the problem is not exploited properly. To this purpose, usually either unstructured algorithms such as the QZ-algorithm (in `hinfopt`) or direct algorithms based on orthogonal transformations (in [6]) are used. In this paper, we show that inverse-free iterative algorithms can be used with comparable or better accuracy. To apply them satisfactorily to this problem, we develop a deflation procedure which relies neither on the nonsingularity of $s\mathcal{E} - \mathcal{A}$ nor on

successive rank deflations. Another interesting observation is that there is a subtle but crucial difference between the doubling algorithm and the inverse-free sign algorithm, which is not easy to detect using residual computations. A premature Cayley transform may lead to large inaccuracies.

Another useful byproduct of this paper is the code released online [33], which can be useful to researchers working in this area as a quick way to test the algorithms, modify them or compare them with other solutions.

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