Perron–based algorithms for the multilinear pagerank

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Abstract

We consider the multilinear pagerank problem studied in [Gleich, Lim and Yu, *Multilinear Pagerank*, 2015], which is a system of quadratic equations with stochasticity and nonnegativity constraints. We use the theory of quadratic vector equations to prove several properties of its solutions and suggest new numerical algorithms. In particular, we prove the existence of a certain minimal solution, which does not always coincide with the stochastic one that is required by the problem. We use an interpretation of the solution as a Perron eigenvector to devise new fixed-point algorithms for its computation, and pair them with a homotopy continuation strategy. The resulting numerical method is more reliable than the existing alternatives, being able to solve a larger number of problems.

Keywords: Multilinear pagerank; Perron vector; fixed point iteration; Newton's method

1 Introduction

Gleich, Lim and Yu [1] consider the following problem, arising as an approximate computation of the stationary measure of an order-2 Markov chain: given $\mathbf{v} \in \mathbb{R}^n$, $R \in \mathbb{R}^{n\times n^2}$, $\alpha \in \mathbb{R}$ with $\mathbf{v} \geq 0, R \geq 0, \alpha \in (0,1)$ and

$$\mathbf{1}_n^\top \mathbf{v} = 1, \quad \mathbf{1}_n^\top R = \mathbf{1}_{n^2}^\top, \tag{1}$$

compute a stochastic solution $\mathbf{s}$ to

$$\mathbf{x} = \alpha R(\mathbf{x} \otimes \mathbf{x}) + (1-\alpha)\mathbf{v}. \tag{2}$$

Here, stochastic means that $\mathbf{s} \geq 0$ and $\mathbf{1}_n^\top \mathbf{s} = 1$, where $\mathbf{1}$ denotes a column vector of all ones (with an optional subscript to specify its length), and inequalities between vectors and matrices are intended in the componentwise sense. In the paper [1], they prove some theoretical properties, consider several solution algorithms, and evaluate their performance.

This problem originally appeared in [2], and is a variation of problems related to tensor eigenvalue problems and Perron-Frobenius theory for tensors; see e.g. [3, 4, 5]. However, it also fits in the framework of quadratic vector equations derived from Markovian binary tree models introduced in [6] and later considered in [7, 8, 9]. Indeed, the paper [9] considers a more general problem, which is essentially (2) without the hypotheses (1). Hence, all of its results apply here, and they can be used in the context of multilinear pagerank. In particular, [9] considers the minimal nonnegative solution of (2) (in the componentwise sense), which is not necessarily stochastic as the one sought in [1].

In this paper, we use the theory of quadratic vector equations in [7, 8, 9] to better understand the behavior of the solutions of (2) and suggest new algorithms for computing the stochastic solution. More specifically, we show that if one considers the minimal nonnegative solution of (2) as well, the theoretical properties of (2) become clearer, even if one is only interested in stochastic solutions. Indeed we prove that there always exists a minimal nonnegative solution, which is the unique stochastic solution when $\alpha \leq 1/2$. When $\alpha > \frac{1}{2}$, the minimal nonnegative solution $\mathbf{m}$ is not stochastic and there is at least one stochastic solution $\mathbf{s} \geq \mathbf{m}$. Note that [1, Theorem 4.3] already proves that when $\alpha \leq \frac{1}{2}$ the stochastic solution is unique; our results give a broader characterization. All this is in Section 2.

When $\alpha \leq 1/2$, as already pointed out in [1], computing the stochastic solution of (2) is easy. Indeed, this is also due to the fact that the stochastic solution is the minimal solution, and for instance the numerical methods proposed in [6, 7, 8] perform very well. The most difficult case is when $\alpha > 1/2$, in particular when $\alpha \approx 1$. Since the minimal solution $\mathbf{m}$ of (2) can be easily computed, the idea is to compute and deflate it, with a similar strategy to the one developed in [7, 8], hence allowing us to compute stochastic solutions even when they are not minimal. The main tool in this approach is rearranging (2) to show that (after a change of variables) a solution $\mathbf{x}$ corresponds to the Perron eigenvector of a certain matrix that depends on $\mathbf{x}$ itself. This interpretation in terms of Perron vector allows to devise new algorithms based on fixed point iteration and on Newton's method. Sections 3 and 4 describe this deflation technique and the algorithms based on the Perron vector computation.

Finally, we propose in Section 5 a homotopy continuation strategy that allows one to solve the problem for values $\hat{\alpha} < \alpha$ in order to obtain better starting values for the more challenging cases when $\alpha \approx 1$.

We report several numerical experiments in Section 6, to show the effectiveness of these new techniques for the set of small-scale benchmark problems introduced in [1], and draw some conclusions in Section 7.

2 Properties of the nonnegative solutions

In this section, we show properties of the nonnegative solutions of the equation (2). In particular, we prove that there always exists a minimal nonnegative solution, which is stochastic when $\alpha \leq 1/2$. These properties can be derived by specializing the results of [9], which apply to more general vector equations defined by bilinear forms.

We introduce the map

$$G(\mathbf{x}) := \alpha R(\mathbf{x} \otimes \mathbf{x}) + (1-\alpha)\mathbf{v},$$

and its Fréchet derivative

$$G'_{\mathbf{x}} := \alpha R(\mathbf{x} \otimes I_n) + \alpha R(I_n \otimes \mathbf{x}).$$

We have the following result.

Lemma 1. *Consider the fixed-point iteration*

$$\mathbf{x}_{k+1} = G(\mathbf{x}_k), \quad k = 0, 1, \ldots, \tag{3}$$

started from $\mathbf{x}_0 = 0$*. Then the sequence of vectors* $\{\mathbf{x}_k\}$ *is such that* $0 \le \mathbf{x}_k \le \mathbf{x}_{k+1} \le \mathbf{1}$*, there exists* $\lim_{k\to\infty} \mathbf{x}_k = \mathbf{m}$ *and* $\mathbf{m}$ *is the minimal nonnegative solution of* (2)*, i.e., equation* (2) *has a (unique) solution* $\mathbf{m} \ge 0$ *such that* $\mathbf{m} \le \mathbf{x}$ *for any other possible solution* $\mathbf{x} \ge 0$*.*

Proof. The map $G(\mathbf{x})$ is weakly positive, i.e., $G(\mathbf{x}) \ge 0$, $G(\mathbf{x}) \ne 0$ whenever $\mathbf{x} \ge 0$, $\mathbf{x} \ne 0$. Moreover, if $0 \le \mathbf{x} \le \mathbf{1}$ then $0 \le G(\mathbf{x}) \le \mathbf{1}$. Therefore Condition A1 of [9] is satisfied which, according to Theorem 4 of [9], implies that the sequence of vectors $\{\mathbf{x}_k\}$ is bounded and converges monotonically to a vector $\mathbf{m}$, which is the minimal nonnegative solution of (2). □

2.1 Sum of entries and criticality

Moreover, in this specific problem, the hypotheses (1) enforce a stronger structure on the iterates of (3): the sum of the entries of $G(\mathbf{x})$ is a function of the sum of the entries of $\mathbf{x}$ only.

Lemma 2. *Let* $g(u) := \alpha u^2 + (1-\alpha)$*. Then,* $\mathbf{1}^\top G(\mathbf{x}) = g(\mathbf{1}^\top \mathbf{x})$ *for any* $\mathbf{x} \in \mathbb{R}^n$*.*

Proof.

$$\begin{aligned}\mathbf{1}^\top G(\mathbf{x}) &= \mathbf{1}^\top(\alpha R(\mathbf{x}\otimes\mathbf{x}) + (1-\alpha)\mathbf{v}) = \alpha\mathbf{1}^\top R(\mathbf{x}\otimes\mathbf{x}) + (1-\alpha)\mathbf{1}^\top\mathbf{v}\\ &= \alpha\mathbf{1}^\top(\mathbf{x}\otimes\mathbf{x}) + (1-\alpha) = \alpha(\mathbf{1}^\top\mathbf{x})^2 + (1-\alpha).\end{aligned}$$ □

This fact has important consequences for the sum of the entries of the solutions of (2).

Corollary 3. *For each solution* $\mathbf{x}$ *of* (2)*,* $\mathbf{1}^\top\mathbf{x}$ *is one of the two solutions of the quadratic* $u = g(u)$*, i.e.,* $u = 1$ *or* $u = \frac{1-\alpha}{\alpha}$*.*

Let u be one of the solutions of $u = g(u)$ and define the level set $\ell_u = \{\mathbf{x} : \mathbf{1}^\top\mathbf{x} = u, \mathbf{x} \ge 0\}$. Since ℓ_u is convex and compact, and since $G(\mathbf{x})$ maps ℓ_u to itself by Lemma 2, then the Brouwer fixed-point theorem implies the following result.

Corollary 4. *There exists at least a solution* $\mathbf{x} \ge 0$ *to* (2) *with* $\mathbf{1}^\top\mathbf{x} = 1$ *and a solution* $\mathbf{x} \ge 0$ *with* $\mathbf{1}^\top\mathbf{x} = \frac{1-\alpha}{\alpha}$*.*

Hence we can have two different settings, for which we borrow the terminology from [9].

Subcritical case $\alpha \le \frac{1}{2}$, hence the minimal nonnegative solution $\mathbf{m} = \mathbf{s}$ is the unique stochastic solution.

Supercritical case $\alpha > \frac{1}{2}$, hence the minimal nonnegative solution $\mathbf{m}$ satisfies $\mathbf{1}^\top\mathbf{m} = \frac{1-\alpha}{\alpha} < 1$ and there is at least one stochastic solution $\mathbf{s} \ge \mathbf{m}$.

Note that [1, Theorem 4.3] already proves that when $\alpha \le \frac{1}{2}$ the stochastic solution is unique; these results give a broader characterization.

The tools that we have introduced can already be used to determine the behavior of simple iterations such as (3).

Theorem 5. *Consider the fixed-point iteration* (3)*, with a certain initial value* $\mathbf{x}_0 \ge 0$*, for the problem* (2) *with* $\alpha > \frac{1}{2}$*. Then,*

- *If* $\mathbf{1}^\top\mathbf{x}_0 \in (0, \frac{1-\alpha}{\alpha}]$*, then* $\lim_{k\to\infty} z_k = \frac{1-\alpha}{\alpha}$*, and the iteration* (3) *can converge only to the minimal solution* $\mathbf{m}$ *(if it converges).*

- *If* $\mathbf{1}^\top \mathbf{x}_0 \in (\frac{1-\alpha}{\alpha}, 1]$ *or* $z_0 = 1$*, then* $\lim_{k\to\infty} z_k = 1$*, hence the iteration* (3) *can converge only to a stochastic solution (if it converges).*
- *If* $\mathbf{1}^\top \mathbf{x}_0 \in (1, +\infty)$*, then* $\lim_{k\to\infty} z_k = +\infty$*, hence the iteration diverges.*

Proof. Thanks to Lemma 2, the quantity $z_k := \mathbf{1}^\top \mathbf{x}_k$ evolves according to $z_{k+1} = g(z_k)$. So the result follows by the theory of scalar fixed-point iterations, since this iteration converges to $\frac{1-\alpha}{\alpha}$ for $z_0 \in (0, \frac{1-\alpha}{\alpha}]$, to 1 for $z_0 \in (\frac{1-\alpha}{\alpha}, 1]$, and diverges for $z_0 \in (1, +\infty)$. □

An analogous result holds for the subcritical case.

The papers [7, 8, 9] describe several methods to compute the minimal solution $\mathbf{m}$. In particular, all the ones described in [9] exhibit monotonic convergence, that is, $0 = \mathbf{x}_0 \le \mathbf{x}_1 \le \mathbf{x}_2 \le \cdots \le \mathbf{x}_k \le \cdots \le \mathbf{m}$. Due to the uniqueness and the monotonic convergence properties, computing the minimal solution $\mathbf{m}$ is typically simple, fast, and free of numerical issues. Hence in the subcritical case the multilinear pagerank problem is easy to solve. The supercritical case is more problematic.

Among all available algorithms to compute the minimal solution $\mathbf{m}$, we recall Newton's method, which is one of the most efficient ones. The Newton–Raphson method applied to the function $F(\mathbf{x}) = \mathbf{x} - G(\mathbf{x})$ generates the sequence

$$(I - G'_{\mathbf{x}_k})\mathbf{x}_{k+1} = (1-\alpha)\mathbf{v} - \alpha R(\mathbf{x}_k \otimes \mathbf{x}_k), \quad k = 0, 1, \dots. \tag{4}$$

The following result holds [9, Theorem 13].

Lemma 6. *Suppose that* $\mathbf{m} > 0$*, and that* $G'_{\mathbf{m}}$ *is irreducible. Then, Newton's method* (4) *starting from* $\mathbf{x}_0 = 0$ *is well defined and converges monotonically to* $\mathbf{m}$ *(i.e.,* $0 = \mathbf{x}_0 \le \mathbf{x}_1 \le \mathbf{x}_2 \le \cdots \le \mathbf{x}_k \le \cdots \le \mathbf{m}$*).*

Algorithm (1) shows a straightforward implementation of Newton's method as described above.

Input: R, α, $\mathbf{v}$ as above, a tolerance ε
Output: An approximation to the minimal solution $\mathbf{m}$ of (2).
$\mathbf{x} \leftarrow 0$;
while $\|G(\mathbf{x}) - \mathbf{x}\|_1 > \varepsilon$ **do**
  $G'_{\mathbf{x}} \leftarrow \alpha R(\mathbf{x} \otimes I_n) + \alpha R(I_n \otimes \mathbf{x})$;
  $\mathbf{x} \leftarrow (I_n - G'_{\mathbf{x}})^{-1}((1-\alpha)\mathbf{v} - \alpha R(\mathbf{x} \otimes \mathbf{x}))$
end
$\mathbf{m} = \mathbf{x}$;

Algorithm 1: Newton's method for the computation of the minimal solution $\mathbf{m}$ to (2).

Note that the theory in [9, Section 9] shows how one can predict where zeros appear in $\mathbf{m}$ and eliminate them reducing the problem to a smaller one. Indeed, in view of the probabilistic interpretation of multilinear pagerank, zero entries in $\mathbf{m}$ can appear only when the second-order Markov chain associated to R is not irreducible. So we can assume in the following that $\mathbf{m} > 0$, and that the nonnegative matrix $G'_{\mathbf{m}}$ is irreducible. In particular, in this case $G'_{\mathbf{m}}$ also substochastic (as proved in [9, Theorem 6]).

3 Deflating the minimal solution

From now on, we assume to be in the supercritical case, i.e. $\alpha > 1/2$, and that $\mathbf{m}$ has been already computed and is explicitly available.

We wish adapt to this setting the deflation strategy introduced in [8]. Since all solutions $\mathbf{s}$ to (2) satisfy $\mathbf{s} \geq \mathbf{m}$, it makes sense to change variable to obtain an equation in $\mathbf{y} := \mathbf{s} - \mathbf{m} \geq 0$. After a few manipulations, using bilinearity of $(\mathbf{m}+\mathbf{y}) \otimes (\mathbf{m}+\mathbf{y})$ and the fact that $\mathbf{m} = \alpha R(\mathbf{m} \otimes \mathbf{m}) + (1-\alpha)\mathbf{v}$, one gets

$$\mathbf{y} = \alpha R(\mathbf{y}\otimes\mathbf{y}) + \alpha R(\mathbf{y}\otimes\mathbf{m}) + \alpha R(\mathbf{m}\otimes\mathbf{y}) = \alpha R(\mathbf{y}\otimes\mathbf{y}) + G'_{\mathbf{m}}\mathbf{y} = (\alpha R(\mathbf{y}\otimes I_n) + G'_{\mathbf{m}})\mathbf{y}. \tag{5}$$

Moreover,

$$\mathbf{1}^\top \mathbf{y} = \mathbf{1}^\top(\mathbf{s} - \mathbf{m}) = 1 - \frac{1-\alpha}{\alpha} = \frac{2\alpha - 1}{\alpha}. \tag{6}$$

We set $P_{\mathbf{y}} := \alpha R(\mathbf{y} \otimes I_n) + G'_{\mathbf{m}}$. Note that $P_{\mathbf{y}} \geq G'_{\mathbf{m}}$ for each $\mathbf{y} \geq 0$, hence it is irreducible. In addition, if $\mathbf{y}$ is chosen such that $\mathbf{1}^\top \mathbf{y} = \frac{2\alpha-1}{\alpha}$ as in (6), then

$$\begin{aligned} \mathbf{1}^\top P_{\mathbf{y}} &= \alpha \mathbf{1}^\top R(\mathbf{y} \otimes I) + \alpha \mathbf{1}^\top R(I \otimes \mathbf{m}) + \alpha \mathbf{1}^\top R(\mathbf{m} \otimes I) \\ &= \alpha \mathbf{1}^\top (\mathbf{y} \otimes I) + \alpha \mathbf{1}^\top (I \otimes \mathbf{m}) + \alpha \mathbf{1}^\top (\mathbf{m} \otimes I) \\ &= \alpha \frac{2\alpha-1}{\alpha}\mathbf{1}^\top + \alpha\frac{1-\alpha}{\alpha}\mathbf{1}^\top + \alpha\frac{1-\alpha}{\alpha}\mathbf{1}^\top = \mathbf{1}^\top, \end{aligned} \tag{7}$$

so $P_{\mathbf{y}}$ is a stochastic matrix.

Let us introduce the map $\mathcal{PV}(A)$ that gives the Perron vector $\mathbf{w}$ of an irreducible matrix $A \geq 0$, normalized such that $\mathbf{1}^\top \mathbf{w} = 1$. Then, since $P_{\mathbf{y}}$ is irreducible and stochastic, (5) and (6) show that

$$\mathbf{y} = \frac{2\alpha-1}{\alpha}\mathcal{PV}(P_{\mathbf{y}}), \tag{8}$$

i.e., the sought vector $\mathbf{y}$ is the Perron vector of the matrix $P_{\mathbf{y}}$, multiplied by the constant $\frac{2\alpha-1}{\alpha}$.

4 Perron-based algorithms

Equation (8) suggests a new fixed-point iteration for computing $\mathbf{y}$, which is analogous to the one appearing in [8],

$$\mathbf{y}_{k+1} = \frac{2\alpha-1}{\alpha}\mathcal{PV}(P_{\mathbf{y}_k}), \tag{9}$$

starting from a given nonnegative vector $\mathbf{y}_0$ such that $\mathbf{1}^\top \mathbf{y}_0 = \frac{2\alpha-1}{\alpha}$. This iteration is implemented in Algorithm 2

We may also apply Newton's method to find a solution to (8), following [7]. To this end, we first compute the Jacobian of the function $\mathbf{w}(\mathbf{y}) := \mathcal{PV}(P_{\mathbf{y}})$.

Lemma 7. *Let $\mathbf{w}(\mathbf{y}) := \mathcal{PV}(P_{\mathbf{y}})$, with $\mathbf{y} \geq 0$ such that $\mathbf{1}^\top \mathbf{y} = \frac{2\alpha-1}{\alpha}$. Then, its Jacobian is given by*

$$\mathbf{w}'(\mathbf{y}) = \alpha\left(\mathbf{w}(\mathbf{y})\mathbf{1}^\top - (I - P_{\mathbf{y}} + \mathbf{w}(\mathbf{y})\mathbf{1}^\top)^{-1} R(I \otimes \mathbf{w}(\mathbf{y}))\right). \tag{10}$$

Input: R, α, $\mathbf{v}$ as above, with $\alpha > \frac{1}{2}$, a tolerance ε, an initial value $\mathbf{x}_0$.
Output: An approximation to a stochastic solution $\mathbf{s}$ of (2).
Compute $\mathbf{m}$ with Algorithm 1;
Normalize $\mathbf{x}_0$ (if needed): $\mathbf{x}_0 \leftarrow \max(\mathbf{x}_0, \mathbf{0})$, $\mathbf{x}_0 \leftarrow \frac{\mathbf{x}_0}{\mathbf{1}^\top \mathbf{x}_0}$;
$\mathbf{y} \leftarrow \mathbf{x}_0 - \mathbf{m}$;
Normalize $\mathbf{y}$ (if needed): $\mathbf{y} \leftarrow \max(\mathbf{y}, \mathbf{0})$, $\mathbf{y} \leftarrow \frac{2\alpha-1}{\alpha}\frac{\mathbf{y}}{\mathbf{1}^\top \mathbf{y}}$;
$G'_{\mathbf{m}} \leftarrow \alpha R(\mathbf{m} \otimes I_n) + \alpha R(I_n \otimes \mathbf{m})$;
while $\|G(\mathbf{x}) - \mathbf{x}\|_1 > \varepsilon$ **do**
  $P_{\mathbf{y}} \leftarrow \alpha R(\mathbf{y} \otimes I_n) + G'_{\mathbf{m}}$;
  $\mathbf{y} \leftarrow \frac{2\alpha-1}{\alpha}\mathcal{PV}(P_{\mathbf{y}})$;
  $\mathbf{x} \leftarrow \mathbf{m} + \mathbf{y}$
end
$\mathbf{s} = \mathbf{x}$;

Algorithm 2: The Perron-based iteration for the computation of a stochastic solution $\mathbf{s}$ to (2).

Proof. Let us compute its directional derivative along the direction $\mathbf{z}$. We set $\mathbf{y}(h) = \mathbf{y} + h\mathbf{z}$; hence, $P'_{\mathbf{y}(h)} = \alpha R(\mathbf{z} \otimes I)$. Since $P_{\mathbf{y}}$ is irreducible, its Perron eigenvalue is a simple eigenvalue, and hence we can define locally smooth functions $\lambda(h)$ as the Perron eigenvalue of $P_{\mathbf{y}(h)}$ and $\mathbf{v}(h) = \mathcal{PV}(P_{\mathbf{y}(h)})$. A computation analogous to (7) shows that $\mathbf{1}^\top P_{\mathbf{y}(h)} = (1 + h\alpha\mathbf{1}^\top\mathbf{z})\mathbf{1}^\top$, hence $\lambda(h) = 1 + h\alpha\mathbf{1}^\top\mathbf{z}$ and $\lambda'(h) = \alpha(\mathbf{1}^\top\mathbf{z})$.

By differentiating $P_{\mathbf{y}(h)}\mathbf{v}(h) = \lambda(h)\mathbf{v}(h)$ and evaluating at $h = 0$, we get

$$\alpha R(\mathbf{z} \otimes I)\mathbf{v}(0) + P_{\mathbf{y}}\mathbf{v}'(0) = \alpha(\mathbf{1}^\top\mathbf{z})\mathbf{v}(0) + \mathbf{v}'(0),$$

or, rearranging terms,

$$(I - P_{\mathbf{y}})\mathbf{v}'(0) = \alpha(\mathbf{v}(0)\mathbf{1}^\top - R(I \otimes \mathbf{v}(0)))\mathbf{z},$$

where $\mathbf{v}(0) = \mathbf{w}(\mathbf{y})$. Since we defined $\mathbf{v}(h)$ so that $\mathbf{1}^\top\mathbf{v}(h) = 1$, we have $\mathbf{1}^\top\mathbf{v}'(0) = 0$, and hence also

$$(I - P_{\mathbf{y}} + \mathbf{v}(0)\mathbf{1}^\top)\mathbf{v}'(0) = \alpha\left(\mathbf{v}(0)\mathbf{1}^\top - R(I \otimes \mathbf{v}(0))\right)\mathbf{z}.$$

The matrix in the left-hand side is nonsingular, since it can be obtained by replacing the eigenvalue 0 with 1 in the eigendecomposition of the singular irreducible M-matrix $I - P_{\mathbf{y}}$. Thus we can write

$$\mathbf{v}'(0) = \alpha(I - P_{\mathbf{y}} + \mathbf{v}(0)\mathbf{1}^\top)^{-1}\left((\mathbf{v}(0)\mathbf{1}^\top - R(I \otimes \mathbf{v}(0)))\right)\mathbf{z}.$$

Since $(I - P_{\mathbf{y}} + \mathbf{v}(0)\mathbf{1}^\top)\mathbf{v}(0) = \mathbf{v}(0)$, we can simplify this expression further to

$$\mathbf{v}'(0) = \alpha\left(\mathbf{v}(0)\mathbf{1}^\top - (I - P_{\mathbf{y}} + \mathbf{v}(0)\mathbf{1}^\top)^{-1}R(I \otimes \mathbf{v}(0))\right)\mathbf{z},$$

from which we get (10). □

From the above result, the Jacobian of the function $H(\mathbf{y}) = \mathbf{y} - \frac{2\alpha-1}{\alpha}\mathcal{PV}(P_{\mathbf{y}})$ is given by

$$H'_{\mathbf{y}} = I_n - (2\alpha - 1)\mathbf{w}(\mathbf{y})\mathbf{1}^\top + (2\alpha - 1)(I - P_{\mathbf{y}} + \mathbf{w}(\mathbf{y})\mathbf{1}^\top)^{-1}R(I \otimes \mathbf{w}(\mathbf{y})) \quad (11)$$

and Newton's method consists in generating the sequence of vectors

$$\mathbf{y}_{k+1} = \mathbf{y}_k - (H'_{\mathbf{y}_k})^{-1} H(\mathbf{y}_k)$$

The Perron-Newton method is described in Algorithm 3.

Input: R, α, $\mathbf{v}$ as above, with $\alpha > \frac{1}{2}$, a tolerance ε, an initial value $\mathbf{x}_0$.
Output: An approximation to a stochastic solution $\mathbf{s}$ of (2).
Compute $\mathbf{m}$ with Algorithm 1;
Normalize $\mathbf{x}_0$ (if needed): $\mathbf{x}_0 \leftarrow \max(\mathbf{x}_0, \mathbf{0})$, $\mathbf{x}_0 \leftarrow \frac{\mathbf{x}_0}{\mathbf{1}^\top \mathbf{x}_0}$;
$\mathbf{y} \leftarrow \mathbf{x}_0 - \mathbf{m}$;
Normalize $\mathbf{y}$ (if needed): $\mathbf{y} \leftarrow \max(\mathbf{y}, \mathbf{0})$, $\mathbf{y} \leftarrow \frac{2\alpha-1}{\alpha} \frac{\mathbf{y}}{\mathbf{1}^\top \mathbf{y}}$;
$G'_{\mathbf{m}} \leftarrow \alpha R(\mathbf{m} \otimes I_n) + \alpha R(I_n \otimes \mathbf{m})$;
while $\|G(\mathbf{x}) - \mathbf{x}\|_1 > \varepsilon$ **do**
  $P_{\mathbf{y}} \leftarrow \alpha R(\mathbf{y} \otimes I_n) + G'_{\mathbf{m}}$;
  $\mathbf{w} \leftarrow \mathcal{PV}(P_{\mathbf{y}})$;
  $H'_{\mathbf{y}} \leftarrow I_n - (2\alpha - 1)\mathbf{w}\mathbf{1}^\top + (2\alpha - 1)(I - P_{\mathbf{y}} + \mathbf{w}\mathbf{1}^\top)^{-1} R(I \otimes \mathbf{w})$;
  $\mathbf{y} \leftarrow \mathbf{y} - {H'_{\mathbf{y}}}^{-1}(\mathbf{y} - \frac{2\alpha-1}{\alpha}\mathbf{w})$;
  Normalize $\mathbf{y}$ (if needed): $\mathbf{y} \leftarrow \frac{2\alpha-1}{\alpha} \frac{\mathbf{y}}{\mathbf{1}^\top \mathbf{y}}$;
  $\mathbf{x} \leftarrow \mathbf{m} + \mathbf{y}$
end
$\mathbf{s} = \mathbf{x}$;

Algorithm 3: The Perron-Newton method for the computation of a stochastic solution $\mathbf{s}$ to (2).

The standard theorems [10] on local convergence of Newton's method imply the following result.

Theorem 8. *Suppose that the matrix $H'_{\mathbf{s}-\mathbf{m}}$ is nonsingular. Then the Perron-Newton method is locally convergent to a stochastic solution $\mathbf{s}$ of (2).*

Remark 9. *Since $\mathbf{1}^\top \left(\mathbf{w}(\mathbf{y})\mathbf{1}^\top + (I - P_{\mathbf{y}} + \mathbf{w}(\mathbf{y})\mathbf{1}^\top)^{-1} R(I \otimes \mathbf{w}(\mathbf{y}))\right) = 0$, the matrix $H'_{\mathbf{y}}$ has an eigenvalue 1 with left eigenvector $\mathbf{1}^\top$, for each value of $\mathbf{y}$.*

5 Continuation techniques

The above algorithms, as well as those in [1], are sufficient to solve most of the test problems that are explored in [1]. However, especially when $\alpha \approx 1$, the algorithms may converge very slowly or stagnate far away from the minimal solution. For this reason, we explore an additional technique, based loosely on the ideas of homotopy continuation [11], which is a well-known strategy to derive approximate solutions for a parameter-dependent equation.

The main result is the following.

Lemma 10. *Let $\mathbf{s}$ be a stochastic solution of problem (2) (for a certain $\alpha > \frac{1}{2}$), and suppose that $I - G'_{\mathbf{s}}$ is nonsingular. Then, there is a stochastic solution $\mathbf{s}_{\hat\alpha}$ to (2) with the parameter α replaced by $\hat\alpha$ such that*

$$\mathbf{s}_{\hat\alpha} = \mathbf{s} + (I - G'_{\mathbf{s}})^{-1}(\mathbf{v} - R(\mathbf{s} \otimes \mathbf{s}))(\hat\alpha - \alpha) + O((\hat\alpha - \alpha)^2). \tag{12}$$

Proof. Let us make the dependence of the various quantities on the parameter α explicit, i.e., we write $\mathbf{s}_\alpha$ to denote a stochastic solution of (2) for a certain value of the parameter α, and similarly $G_\alpha, G'_{\alpha,\mathbf{x}}$ and F_α.

We apply the implicit function theorem [12, Theorem 9.28] to

$$0 = F_\alpha(\mathbf{s}_\alpha) = \mathbf{s}_\alpha - \alpha R(\mathbf{s}_\alpha \otimes \mathbf{s}_\alpha) - (1-\alpha)\mathbf{v},$$

obtaining

$$\begin{aligned}
\frac{\partial F_\alpha}{\partial \mathbf{s}_\alpha} &= I - \alpha R(s_\alpha \otimes I) - \alpha R(I \otimes s_\alpha) = I - G'_{\alpha,\mathbf{s}_\alpha},\\
\frac{\partial F_\alpha}{\partial \alpha} &= \mathbf{v} - R(\mathbf{s}_\alpha \otimes \mathbf{s}_\alpha),\\
\frac{d}{d\alpha}\mathbf{s}_\alpha &= \left(\frac{\partial F_\alpha}{\partial \mathbf{s}_\alpha}\right)^{-1} \frac{\partial F_\alpha}{\partial \alpha} = (I - G'_{\alpha,\mathbf{s}_\alpha})^{-1}(\mathbf{v} - R(\mathbf{s}_\alpha \otimes \mathbf{s}_\alpha)).
\end{aligned}$$

Note that the function $\mathbf{s}_{\hat\alpha}$ obtained by the theorem must satisfy $\mathbf{1}^\top \mathbf{s}_{\hat\alpha} = 1$ for each $\hat\alpha > \frac{1}{2}$: indeed, by Corollary 3 for a solution $\mathbf{s}_{\hat\alpha}$ of the equation $\mathbf{x} = G_{\hat\alpha}(\mathbf{x})$ it must hold either $\mathbf{1}^\top \mathbf{s}_{\hat\alpha} = 1$ or $\mathbf{1}^\top \mathbf{s}_{\hat\alpha} = \frac{1-\hat\alpha}{\hat\alpha} < 1$, and the continuous function $\mathbf{1}^\top \mathbf{s}_{\hat\alpha}$ cannot jump from 1 to $\frac{1-\hat\alpha}{\hat\alpha}$ without taking any intermediate value.

The formula (12) now follows by Taylor expansion. □

This result suggests a further solution strategy: we start solving the problem with a small value of $\alpha = \alpha_0$, e.g. 0.6, then we solve it repeatedly for increasing values $\alpha_0 < \alpha_1 < \alpha_2 < \cdots < \alpha_k = \alpha$; at each step h we use (12) to obtain from $\mathbf{s}_{\alpha_{h-1}}$ a first-order approximation of $\mathbf{s}_{\alpha_h}$ to use as initial value.

The only missing part is designing an effective strategy to choose the intermediate values α_h. We adopt the following one. Once $\mathbf{s}_{\alpha_h}$ is computed, from the relation

$$\mathbf{s}_{\alpha_h} \approx \mathbf{s}_{\alpha_{h-1}} + \frac{d\mathbf{s}_\alpha}{d\alpha}|_{\alpha_{h-1}}(\mathbf{s}_{\alpha_h} - \mathbf{s}_{\alpha_{h-1}}) + \frac{1}{2}\frac{d^2\mathbf{s}_\alpha}{d\alpha^2}|_{\alpha_{h-1}}(\mathbf{s}_{\alpha_h} - \mathbf{s}_{\alpha_{h-1}})^2$$

one can derive an estimate of the second derivative $\frac{d^2\mathbf{s}_\alpha}{d\alpha^2}|_{\alpha_{h-1}}$. With this estimate at hand, one can choose α_{h+1} such that the second-order term that has been neglected in the approximation (12)

$$\frac{1}{2}\frac{d^2\mathbf{s}_\alpha}{d\alpha^2}|_{\alpha_h}(\mathbf{s}_{\alpha_{h+1}} - \mathbf{s}_{\alpha_h})^2 \approx \frac{1}{2}\frac{d^2\mathbf{s}_\alpha}{d\alpha^2}|_{\alpha_{h-1}}(\mathbf{s}_{\alpha_{h+1}} - \mathbf{s}_{\alpha_h})^2$$

is below a absolute threshold, e.g., 0.01.

The resulting homotopy continuation algorithm is described in Algorithm 4. Note that we start from $\alpha_0 = 0.6$, to steer clear of the double solution for $\alpha = 0.5$. Convergence for such a value of α_0 is typically not problematic.

6 Numerical experiments

We have implemented the described methods using Matlab, and compared them to some of the algorithms in [13] on the same benchmark set of small-size models ($n \in \{3, 4, 6\}$) used in [1]. We have used tolerance $\varepsilon = \sqrt{\mathbf{u}}$, where $\mathbf{u}$ is the machine precision, $\mathbf{x}_0 = \mathbf{v}$ as an initial value unless differently specified, and $\tau = 0.01$ for Algorithm 4. To compute Perron vectors, we have used the output of Matlab's `eig`. For problems with

Input: R, α, $\mathbf{v}$ as above, with $\alpha > 0.6$, a tolerance ε, a threshold τ for the second derivative.
Output: An approximation to a stochastic solution $\mathbf{s}$ of (2).
$\alpha_0 \leftarrow 0.6$;
$h = 0$;
$\mathbf{x} \leftarrow$ a stochastic solution to (2) with parameters R, α_0, $\mathbf{v}$;
while $\alpha_h < \alpha$ **do**
    if *at the first step* **then**
        SecondDerivativeEstimate $= 0.01$;
    else
        SecondDerivativeEstimate $= 2\|\mathbf{x} - \mathbf{x}_{old}\|_1/(\alpha_h - \alpha_{h-1})^2$;
    end
    StepSize $= \sqrt{2\tau/\text{SecondDerivativeEstimate}}$;
    $\alpha_{h+1} = \alpha_h + \text{StepSize}$;
    if $\alpha_{h+1} > \alpha$ **then**
        $\alpha_{h+1} \leftarrow \alpha$;
    end
    $\mathbf{x}_{old} \leftarrow \mathbf{x}$;
    $\mathbf{x}_0 \leftarrow \mathbf{x} + (I - G'_{\mathbf{x}})^{-1}(\mathbf{v} - R(\mathbf{x} \otimes \mathbf{x}))(\alpha_{h+1} - \alpha_h)$;
    $\mathbf{x} \leftarrow$
     a stochastic solution to (2) with parameters R, α_{h+1}, $\mathbf{v}$ and initial approximation $\mathbf{x}_0$;

    $h \leftarrow h + 1$;
end
$\mathbf{s} \leftarrow \mathbf{x}$;

Algorithm 4: The homotopy continuation method for the computation of a stochastic solution $\mathbf{s}$ to (2).

larger n, different methods (`eigs`, inverse iteration, Gaussian elimination for kernel computation ...) can be considered [14].

The results, for various values of α, are reported in Tables 1 to 6. Each of the 29 rows represents a different experiment in the benchmark set, and the columns stand for the different algorithms, according to the following legend.

F The fixed-point iteration (3), from an initial value such that $\mathbf{1}^\top \mathbf{x}_0 = 1$ (and with renormalization to $\mathbf{1}^\top \mathbf{x}_k = 1$ at each step).

IO The inner-outer iteration method, as described in [1].

N Newton's method with normalization to $\mathbf{1}^\top \mathbf{x}_k = 1$ at each step, as described in [1]. Note that this is *not* Algorithm 1, which would converge instead to the minimal solution $\mathbf{m}$.

P The Perron method (Algorithm 2).

PN The Perron-Newton method (Algorithm 3).

CPN The homotopy continuation method (Algorithm 4), where the inner problems are solved with method PN.

CN The homotopy continuation method (Algorithm 4), where the inner problems are solved with method N.

For each experiment, we report on the left the number of iterations required, and on the right the CPU times in seconds (obtained on Matlab R2017a on a computer equipped with a 64-bit Intel core i5-6200U processor). The value NaN inside a table represents failure to converge after 10,000 iterations. For P and PN, the number of iterations is defined as the sum of the number of Newton iterations required to compute $\mathbf{m}$ with Algorithm 1 and the number of iterations in the main algorithm. For CPN and CN, we report the number of inner iterations, that is, the total number of iterations performed by PN or N (respectively), summing along all calls to the subroutine.

Note that neither the number of iterations nor the CPU time are particularly indicative of the true performance of the algorithms: indeed, the iterations in the different methods amount to different quantity of work, since some require merely linear system solutions and some require eigenvalue computations. Moreover, Matlab introduces overheads which are difficult to predict for several instructions (such as variable access and function calls). In order to make the time comparisons more fair, for methods IO and N we did not use the code in [13], but rather rewrote it without making use of Matlab's object-oriented features, which made the original functions significantly slower. The only change introduced with respect to their implementations is that we changed the stopping criterion to $\|G(\mathbf{x}) - \mathbf{x}\|_1 \leq \sqrt{\mathbf{u}}$, so that the stopping criterion is the same one for all tested methods.

In any case, the performance comparison between the various methods should be taken with a pinch of salt. For this reason, we prefer to report the raw timings instead of graphs or plots that would convey the wrong message.

We comment briefly on the results. Newton-based methods typically require a constant number of iterations to converge, but on some of the benchmarks they fail to converge or require a much larger number of iterations. From the point of view of reliability, combining the Perron-Newton algorithm with the continuation strategy gives a definite advantage: the resulting method CPN is the only one (among the ones considered) that can solve successfully all the problems in the original benchmark set in [1], which contained the experiments with all values of α up to 0.99. Raising α to the more extreme value of 0.999 reveals a failure case for this method as well, the

F	IO	N	P	PN	CPN	CN	F	IO	N	P	PN	CPN	CN
39	54	4	16	9	24	6	1.7e−3	2.2e−2	2.8e−4	1.1e−3	7.3e−4	1.7e−3	5.1e−4
12	49	5	16	9	26	7	3.3e−4	2.4e−2	3.2e−4	1.3e−3	8.7e−4	2.0e−3	5.5e−4
12	49	5	13	9	26	7	3.0e−4	1.8e−2	3.3e−4	1.0e−3	1.0e−3	2.4e−3	5.8e−4
32	38	4	15	9	24	5	7.0e−4	1.5e−2	2.8e−4	1.0e−3	1.2e−3	2.6e−3	6.5e−4
64	143	5	29	9	26	6	1.3e−3	7.0e−2	3.8e−4	2.8e−3	8.3e−4	1.9e−3	5.1e−4
42	105	5	25	9	26	7	9.4e−4	4.4e−2	3.4e−4	2.1e−3	8.7e−4	2.0e−3	5.5e−4
57	101	4	23	9	26	6	1.2e−3	4.1e−2	2.9e−4	1.8e−3	7.6e−4	2.0e−3	5.9e−4
37	88	4	21	9	26	6	8.2e−4	3.4e−2	2.9e−4	1.6e−3	8.5e−4	2.2e−3	5.1e−4
59	94	4	20	9	26	6	1.7e−3	3.7e−2	2.7e−4	1.5e−3	8.6e−4	2.3e−3	5.4e−4
39	87	5	21	9	26	6	9.2e−4	3.7e−2	3.1e−4	1.6e−3	7.7e−4	3.4e−3	5.6e−4
48	92	4	21	9	26	6	1.2e−3	3.9e−2	2.7e−4	1.6e−3	7.6e−4	2.5e−3	6.0e−4
39	83	4	19	9	26	6	9.1e−4	3.4e−2	5.8e−4	1.5e−3	7.4e−4	2.0e−3	5.2e−4
51	88	5	22	9	26	7	1.1e−3	3.7e−2	3.4e−4	2.2e−3	8.1e−4	1.9e−3	5.6e−4
40	92	5	19	9	26	7	9.2e−4	3.9e−2	4.9e−4	1.5e−3	7.7e−4	2.0e−3	5.6e−4
68	89	4	20	9	25	6	1.6e−3	3.5e−2	2.9e−4	1.5e−3	8.4e−4	2.4e−3	5.4e−4
39	87	4	24	9	24	5	8.5e−4	3.1e−2	2.7e−4	1.9e−3	1.0e−3	2.0e−3	4.9e−4
29	74	5	17	9	26	7	6.8e−4	3.1e−2	3.6e−4	1.3e−3	9.5e−4	2.3e−3	5.6e−4
44	98	5	22	9	26	7	9.0e−4	4.2e−2	3.3e−4	1.8e−3	7.5e−4	1.9e−3	5.5e−4
42	89	5	21	9	26	7	9.8e−4	3.6e−2	4.0e−4	1.9e−3	8.0e−4	2.5e−3	6.2e−4
52	95	4	22	9	25	5	1.3e−3	4.2e−2	3.2e−4	1.8e−3	1.4e−3	2.4e−3	5.2e−4
42	94	4	23	9	26	6	1.1e−3	4.2e−2	2.8e−4	1.8e−3	1.7e−3	2.7e−3	5.5e−4
40	86	5	24	9	26	7	9.0e−4	3.3e−2	3.3e−4	1.7e−3	9.9e−4	2.0e−3	5.4e−4
39	92	5	22	9	26	7	1.5e−3	3.9e−2	3.1e−4	1.7e−3	9.0e−4	1.9e−3	8.4e−4
36	82	4	21	9	26	6	7.9e−4	3.4e−2	3.5e−4	1.6e−3	8.3e−4	2.0e−3	5.0e−4
30	77	5	18	9	26	7	8.2e−4	3.2e−2	4.2e−4	1.6e−3	1.5e−3	2.8e−3	6.5e−4
37	79	4	20	9	26	6	8.3e−4	3.5e−2	3.5e−4	1.9e−3	1.3e−3	3.2e−3	5.7e−4
26	69	4	19	9	26	6	6.2e−4	2.8e−2	3.2e−4	1.8e−3	8.8e−4	2.1e−3	5.5e−4
30	83	5	23	9	25	7	9.5e−4	3.7e−2	3.5e−4	2.1e−3	8.9e−4	2.0e−3	6.0e−4
22	66	5	18	9	25	7	7.2e−4	2.5e−2	3.5e−4	1.5e−3	8.2e−4	2.0e−3	6.4e−4

Tab. 1: Iteration counts and times for the 29 benchmark tensors and $\alpha = 0.7$

F	IO	N	P	PN	CPN	CN	F	IO	N	P	PN	CPN	CN
97	47	4	19	8	24	6	6.4e−3	4.6e−2	3.0e−4	1.7e−3	2.0e−3	3.5e−3	9.1e−4
16	41	5	20	8	32	9	6.9e−4	3.3e−2	5.8e−4	2.8e−3	1.3e−3	3.0e−3	7.9e−4
16	41	5	14	8	32	9	4.4e−4	1.7e−2	3.3e−4	1.2e−3	7.4e−4	2.6e−3	7.7e−4
54	30	4	17	8	24	5	1.1e−3	1.1e−2	5.1e−4	1.7e−3	1.1e−3	3.0e−3	7.9e−4
80	127	6	39	10	27	8	2.8e−3	7.3e−2	3.6e−4	2.8e−3	9.0e−4	2.0e−3	6.9e−4
101	127	6	38	9	26	8	2.1e−3	6.7e−2	3.9e−4	3.1e−3	8.4e−4	2.1e−3	6.0e−4
196	123	5	45	9	25	6	4.2e−3	6.4e−2	3.5e−4	3.7e−3	9.7e−4	2.2e−3	5.1e−4
75	115	5	42	8	25	7	1.6e−3	5.9e−2	3.1e−4	3.3e−3	8.5e−4	2.1e−3	7.2e−4
203	130	5	41	9	26	7	4.4e−3	6.9e−2	4.3e−4	3.9e−3	1.0e−3	3.1e−3	5.5e−4
122	125	6	50	9	26	7	2.5e−3	6.9e−2	3.6e−4	5.2e−3	9.3e−4	2.4e−3	5.7e−4
168	119	5	51	9	25	6	3.4e−3	6.2e−2	4.8e−4	4.2e−3	8.8e−4	2.3e−3	5.5e−4
92	112	5	34	9	25	7	2.0e−3	5.6e−2	3.1e−4	3.1e−3	9.3e−4	2.2e−3	6.0e−4
227	106	6	47	9	31	8	4.8e−3	5.7e−2	3.8e−4	3.8e−3	9.5e−4	2.7e−3	7.2e−4
109	114	6	38	9	32	10	2.3e−3	5.7e−2	3.8e−4	3.1e−3	9.0e−4	2.9e−3	1.1e−3
278	115	5	43	9	25	6	6.2e−3	5.8e−2	7.1e−4	4.5e−3	8.4e−4	2.0e−3	5.7e−4
94	131	5	56	8	24	6	2.0e−3	6.2e−2	3.5e−4	4.6e−3	7.4e−4	3.2e−3	5.9e−4
66	108	6	29	9	26	7	1.5e−3	5.7e−2	6.2e−4	3.0e−3	8.8e−4	2.0e−3	5.4e−4
115	137	6	44	9	25	7	2.4e−3	7.9e−2	4.1e−4	3.9e−3	9.9e−4	3.5e−3	5.7e−4
100	104	6	37	9	32	9	2.3e−3	5.9e−2	4.0e−4	3.0e−3	9.9e−4	2.6e−3	1.0e−3
178	120	5	41	9	25	6	5.1e−3	6.3e−2	5.3e−4	4.0e−3	8.8e−4	2.0e−3	5.0e−4
113	131	5	50	8	26	7	2.7e−3	7.9e−2	6.5e−4	7.8e−3	8.3e−4	2.4e−3	6.9e−4
158	130	6	56	8	26	8	6.4e−3	6.9e−2	3.7e−4	4.2e−3	8.0e−4	2.3e−3	6.6e−4
106	162	6	54	9	26	8	2.1e−3	8.3e−2	3.7e−4	4.0e−3	1.1e−3	2.4e−3	6.5e−4
90	98	5	38	8	25	6	1.9e−3	5.2e−2	3.4e−4	3.6e−3	1.4e−3	3.2e−3	5.5e−4
78	123	6	27	9	33	10	1.7e−3	6.6e−2	3.9e−4	2.4e−3	1.0e−3	4.5e−3	1.8e−3
75	102	5	37	9	26	7	1.9e−3	5.8e−2	3.8e−4	3.6e−3	1.0e−3	2.3e−3	6.7e−4
49	82	5	29	8	25	6	2.4e−3	4.1e−2	3.4e−4	2.3e−3	8.0e−4	2.5e−3	1.2e−3
64	98	6	39	8	25	8	1.4e−3	5.4e−2	5.4e−4	4.1e−3	2.3e−3	2.7e−3	7.6e−4
46	85	5	29	8	24	8	1.1e−3	4.4e−2	3.8e−4	2.5e−3	8.8e−4	2.2e−3	7.6e−4

Tab. 2: Iteration counts and times for the 29 benchmark tensors and $\alpha = 0.85$

F	IO	N	P	PN	CPN	CN	F	IO	N	P	PN	CPN	CN
184	45	4	19	7	23	6	1.2e−2	3.6e−2	3.5e−4	1.6e−3	8.0e−4	3.5e−3	9.5e−4
18	40	5	22	9	32	9	4.5e−4	2.3e−2	5.8e−4	2.0e−3	8.7e−4	5.7e−3	1.4e−3
18	40	5	14	8	32	9	5.0e−4	2.1e−2	3.3e−4	1.1e−3	1.0e−3	4.9e−3	1.4e−3
67	28	4	17	7	23	6	1.4e−3	9.2e−3	2.9e−4	1.9e−3	1.5e−3	3.4e−3	5.8e−4
54	83	6	29	10	30	13	2.1e−3	5.2e−2	3.5e−4	2.4e−3	1.1e−3	6.0e−3	1.6e−3
227	169	7	59	9	32	10	8.6e−3	1.0e−1	4.0e−4	5.5e−3	1.0e−3	2.8e−3	7.9e−4
680	142	5	63	8	25	7	1.4e−2	7.9e−2	3.3e−4	5.3e−3	8.2e−4	2.4e−3	5.7e−4
117	153	6	63	8	25	7	2.8e−3	8.2e−2	3.8e−4	5.2e−3	9.2e−4	2.0e−3	5.6e−4
512	166	5	63	8	25	7	1.2e−2	9.8e−2	3.1e−4	5.7e−3	9.0e−4	2.3e−3	6.5e−4
407	160	6	95	9	25	7	8.3e−3	1.1e−1	4.9e−4	9.1e−3	1.3e−3	2.3e−3	5.5e−4
654	140	5	97	8	24	6	1.5e−2	8.0e−2	3.2e−4	8.2e−3	8.4e−4	2.0e−3	5.4e−4
186	141	6	51	9	31	8	3.9e−3	8.0e−2	3.7e−4	4.4e−3	1.0e−3	3.0e−3	7.4e−4
NaN	118	6	73	8	31	9	NaN	7.8e−2	4.2e−4	7.0e−3	8.6e−4	2.8e−3	7.7e−4
248	134	6	58	8	32	11	5.4e−3	7.8e−2	3.5e−4	4.9e−3	8.6e−4	3.3e−3	8.6e−4
2,259	151	5	79	8	24	7	4.7e−2	8.2e−2	3.7e−4	6.2e−3	7.9e−4	2.1e−3	5.8e−4
171	186	5	92	7	23	6	3.8e−3	9.0e−2	3.8e−4	7.1e−3	7.4e−4	2.0e−3	6.1e−4
114	155	6	43	8	31	10	2.4e−3	8.2e−2	3.6e−4	3.4e−3	9.3e−4	2.7e−3	8.6e−4
240	175	6	68	8	25	8	4.9e−3	9.9e−2	3.9e−4	6.1e−3	9.2e−4	2.3e−3	6.0e−4
204	124	6	52	8	32	10	4.5e−3	7.0e−2	3.9e−4	4.5e−3	9.0e−4	2.6e−3	8.2e−4
594	140	5	57	8	24	6	1.2e−2	7.7e−2	3.2e−4	4.7e−3	8.4e−4	2.4e−3	6.1e−4
227	167	5	79	8	25	7	4.8e−3	9.6e−2	3.3e−4	6.7e−3	9.0e−4	2.8e−3	6.2e−4
1,514	207	8	103	7	25	8	3.2e−2	2.0e−1	1.8e−3	3.4e−2	2.1e−3	1.4e−2	1.6e−3
119	127	8	55	10	27	10	7.1e−3	2.4e−1	2.4e−3	9.6e−3	9.2e−3	6.3e−3	5.2e−3
164	109	5	51	7	24	7	1.5e−2	1.5e−1	4.7e−4	1.4e−2	9.5e−4	2.9e−3	3.3e−3
170	225	7	46	8	33	11	1.2e−2	2.8e−1	6.7e−4	7.1e−3	7.1e−3	7.5e−3	1.7e−3
91	110	6	40	9	26	8	3.5e−3	1.6e−1	1.3e−3	6.3e−3	1.9e−3	1.0e−2	1.6e−3
69	99	5	38	7	24	6	5.9e−3	1.1e−1	2.0e−3	1.0e−2	1.3e−3	4.3e−3	3.6e−3
112	131	6	58	8	31	10	7.3e−3	1.6e−1	5.3e−4	1.4e−2	2.1e−3	4.1e−3	1.2e−3
70	106	6	38	7	24	8	2.1e−3	1.1e−1	5.4e−4	7.9e−3	4.1e−3	4.2e−3	9.5e−4

Tab. 3: Iteration counts and times for the 29 benchmark tensors and $\alpha = 0.90$

F	IO	N	P	PN	CPN	CN	F	IO	N	P	PN	CPN	CN
1,386	43	4	21	7	23	6	3.3e−2	2.0e−2	5.8e−4	3.6e−3	2.0e−3	8.5e−3	2.5e−3
20	42	6	24	8	36	12	9.1e−4	4.5e−2	7.1e−4	4.3e−3	2.3e−3	5.5e−3	1.2e−3
20	42	6	13	7	36	12	5.6e−4	2.6e−2	3.7e−4	1.1e−3	1.2e−3	3.2e−3	9.4e−4
88	26	4	19	7	23	6	1.9e−3	9.6e−3	2.7e−4	1.4e−3	6.4e−4	1.6e−3	5.0e−4
35	56	55	22	9	27	16	7.6e−4	3.5e−2	2.8e−3	2.9e−3	1.0e−3	2.2e−3	9.3e−4
NaN	263	7	145	NaN	34	12	NaN	1.9e−1	4.1e−4	1.4e−2	NaN	4.2e−3	9.0e−4
NaN	174	5	117	8	25	7	NaN	1.5e−1	3.2e−4	1.0e−2	8.6e−4	2.1e−3	5.7e−4
313	273	7	149	8	25	8	7.0e−3	1.9e−1	4.1e−4	1.2e−2	8.3e−4	2.4e−3	6.0e−4
NaN	263	5	150	8	25	7	NaN	1.8e−1	4.6e−4	1.4e−2	8.8e−4	2.1e−3	1.1e−3
NaN	226	6	656	8	31	9	NaN	1.6e−1	4.5e−4	5.9e−2	9.5e−4	4.4e−3	7.5e−4
NaN	181	5	1,152	8	24	6	NaN	1.4e−1	3.2e−4	1.1e−1	9.5e−4	1.9e−3	5.2e−4
3,027	207	6	101	10	32	9	6.4e−2	1.4e−1	4.1e−4	8.6e−3	1.1e−3	4.1e−3	7.5e−4
NaN	135	6	168	8	31	9	NaN	9.3e−2	3.8e−4	1.4e−2	9.3e−4	2.9e−3	8.3e−4
NaN	195	6	146	10	32	11	NaN	1.5e−1	5.1e−4	1.4e−2	1.2e−3	3.1e−3	8.5e−4
NaN	254	6	735	9	24	7	NaN	1.7e−1	4.6e−4	6.4e−2	9.3e−4	1.9e−3	6.1e−4
683	416	5	258	7	24	6	1.6e−2	2.3e−1	3.5e−4	2.0e−2	7.5e−4	2.7e−3	5.3e−4
155	153	8	79	14	33	12	3.1e−3	9.7e−2	5.2e−4	6.0e−3	1.8e−3	3.7e−3	9.0e−4
NaN	271	6	152	10	32	10	NaN	1.9e−1	3.6e−4	1.4e−2	1.7e−3	2.8e−3	7.7e−4
NaN	174	7	94	9	32	10	NaN	1.4e−1	4.2e−4	9.6e−3	1.2e−3	3.1e−3	8.1e−4
NaN	175	6	94	8	25	7	NaN	1.5e−1	4.0e−4	9.5e−3	1.6e−3	4.2e−3	9.6e−4
3,793	260	6	200	9	26	8	8.6e−2	1.7e−1	3.8e−4	1.7e−2	1.0e−3	3.2e−3	7.1e−4
NaN	1,037	NaN	586	19	97	NaN	NaN	8.1e−1	NaN	5.6e−2	2.4e−3	1.2e−2	NaN
647	212	NaN	127	NaN	29	NaN	2.0e−2	1.5e−1	NaN	1.3e−2	NaN	5.7e−3	NaN
657	128	5	82	7	25	7	2.3e−2	8.1e−2	3.1e−4	7.7e−3	7.6e−4	2.2e−3	5.9e−4
203	193	77	75	NaN	43	30	4.3e−3	1.4e−1	3.8e−3	6.8e−3	NaN	8.3e−3	3.8e−3
394	222	9	128	NaN	33	11	1.4e−2	1.7e−1	5.7e−4	1.4e−2	NaN	7.6e−3	1.8e−3
124	150	6	65	8	24	7	6.6e−3	1.7e−1	4.1e−4	7.4e−3	9.2e−4	4.5e−3	2.8e−3
322	196	6	109	10	32	11	1.2e−2	1.6e−1	6.5e−4	1.2e−2	1.4e−3	3.2e−3	9.4e−4
134	168	6	65	8	30	10	2.9e−3	9.3e−2	4.6e−4	6.4e−3	9.5e−4	3.2e−3	9.5e−4

Tab. 4: Iteration counts and times for the 29 benchmark tensors and $\alpha = 0.95$

F	IO	N	P	PN	CPN	CN	F	IO	N	P	PN	CPN	CN
NaN	42	5	21	6	22	6	NaN	2.3e−2	3.7e−4	1.6e−3	6.4e−4	2.0e−3	9.4e−4
23	48	6	26	7	36	13	5.5e−4	2.4e−2	3.6e−4	1.9e−3	7.2e−4	3.1e−3	1.0e−3
23	48	6	10	7	36	13	5.4e−4	2.5e−2	3.8e−4	8.0e−4	7.5e−4	3.0e−3	1.5e−3
114	24	4	19	6	22	6	2.3e−3	7.7e−3	2.7e−4	1.4e−3	6.6e−4	2.4e−3	9.5e−4
22	41	17	15	7	26	29	5.2e−4	2.5e−2	7.9e−4	1.1e−3	6.9e−4	2.1e−3	2.6e−3
NaN	639	7	NaN	NaN	39	14	NaN	5.4e−1	4.1e−4	NaN	NaN	4.4e−3	1.3e−3
NaN	234	5	369	8	24	7	NaN	1.7e−1	3.3e−4	3.2e−2	1.0e−3	2.1e−3	5.7e−4
NaN	1,221	6	1,845	9	26	9	NaN	7.8e−1	3.9e−4	1.5e−1	1.1e−3	2.2e−3	6.4e−4
NaN	532	5	NaN	7	24	7	NaN	3.4e−1	3.2e−4	NaN	2.2e−3	3.4e−3	9.3e−4
NaN	339	6	NaN	7	30	9	NaN	2.3e−1	3.6e−4	NaN	1.2e−3	2.9e−3	7.5e−4
NaN	228	5	NaN	7	23	7	NaN	2.1e−1	3.6e−4	NaN	1.3e−3	2.3e−3	6.2e−4
NaN	357	6	362	NaN	31	9	NaN	2.9e−1	4.2e−4	3.5e−2	NaN	3.4e−3	9.1e−4
NaN	156	6	7,509	6	31	10	NaN	1.3e−1	4.1e−4	6.6e−1	9.7e−4	3.5e−3	8.0e−4
NaN	329	6	NaN	NaN	32	11	NaN	2.5e−1	3.7e−4	NaN	NaN	6.6e−3	9.0e−4
NaN	550	6	NaN	8	23	7	NaN	4.5e−1	4.3e−4	NaN	1.3e−3	2.4e−3	6.1e−4
NaN	NaN	5	NaN	7	23	7	NaN	NaN	1.1e−3	NaN	1.2e−3	2.3e−3	1.1e−3
NaN	1,173	NaN	NaN	71	32	11	NaN	9.2e−1	NaN	NaN	1.4e−2	3.6e−3	1.8e−3
NaN	541	6	1,595	NaN	31	10	NaN	3.9e−1	3.9e−4	1.4e−1	NaN	8.0e−3	1.7e−3
NaN	273	NaN	248	8	37	12	NaN	2.0e−1	NaN	2.6e−2	1.2e−3	3.8e−3	9.8e−4
NaN	230	7	196	7	24	7	NaN	1.7e−1	4.4e−4	1.7e−2	8.6e−4	2.2e−3	9.9e−4
NaN	533	6	NaN	9	31	10	NaN	3.6e−1	3.7e−4	NaN	2.1e−3	4.8e−3	1.1e−3
NaN	NaN	7	205	6	32	16	NaN	NaN	4.9e−4	1.6e−2	1.2e−3	3.6e−3	1.0e−3
NaN	435	NaN	504	NaN	35	NaN	NaN	3.3e−1	NaN	5.7e−2	NaN	4.9e−3	NaN
NaN	147	5	137	7	29	8	NaN	1.1e−1	3.2e−4	1.2e−2	7.8e−4	2.3e−3	7.4e−4
NaN	835	NaN	NaN	44	59	36	NaN	6.1e−1	NaN	NaN	1.3e−2	1.2e−2	3.8e−3
NaN	654	NaN	NaN	NaN	33	12	NaN	4.9e−1	NaN	NaN	NaN	7.4e−3	1.7e−3
NaN	NaN	NaN	8,025	1,326	903	NaN	NaN	NaN	NaN	8.3e−1	2.6e−1	2.0e−1	NaN
NaN	334	6	307	NaN	37	13	NaN	3.1e−1	4.5e−4	4.0e−2	NaN	6.3e−3	2.2e−3
2,005	2,100	9	909	10	33	14	5.0e−2	1.1e+0	5.6e−4	9.2e−2	1.5e−3	4.6e−3	1.3e−3

Tab. 5: Iteration counts and times for the 29 benchmark tensors and $\alpha = 0.99$

F	IO	N	P	PN	CPN	CN	F	IO	N	P	PN	CPN	CN
NaN	41	5	21	6	22	6	NaN	2.7e−2	1.0e−3	1.5e−2	2.3e−3	4.2e−3	5.0e−3
29	51	6	28	7	36	13	1.5e−3	6.7e−2	8.5e−4	3.7e−3	1.4e−3	5.3e−3	1.9e−3
29	51	6	9	6	36	13	1.2e−3	3.2e−2	5.2e−4	7.9e−4	8.2e−4	3.3e−3	1.0e−3
122	24	4	19	6	22	6	2.5e−3	7.7e−3	2.7e−4	1.4e−3	8.5e−4	1.9e−3	6.9e−4
18	38	336	13	8	26	71	4.3e−4	2.2e−2	1.5e−2	1.1e−3	1.3e−3	3.0e−3	3.2e−3
NaN	NaN	7	NaN	NaN	39	14	NaN	NaN	4.5e−4	NaN	NaN	3.9e−3	1.2e−3
NaN	251	5	723	8	24	7	NaN	1.6e−1	3.7e−4	5.8e−2	9.7e−4	1.9e−3	5.9e−4
NaN	NaN	6	NaN	9	26	9	NaN	NaN	3.7e−4	NaN	1.5e−3	2.6e−3	6.8e−4
NaN	NaN	5	NaN	7	24	7	NaN	NaN	3.5e−4	NaN	9.9e−4	2.0e−3	6.1e−4
NaN	383	6	NaN	7	31	9	NaN	2.7e−1	3.7e−4	NaN	1.5e−3	3.4e−3	8.7e−4
NaN	243	5	NaN	7	23	7	NaN	1.6e−1	3.3e−4	NaN	1.2e−3	2.3e−3	5.8e−4
NaN	442	6	824	NaN	31	9	NaN	3.0e−1	3.6e−4	6.8e−2	NaN	5.2e−3	1.7e−3
NaN	164	6	NaN	6	31	10	NaN	1.2e−1	3.6e−4	NaN	8.1e−4	2.7e−3	9.3e−4
NaN	396	6	NaN	NaN	36	12	NaN	2.5e−1	3.6e−4	NaN	NaN	4.0e−3	1.0e−3
NaN	812	6	NaN	8	23	7	NaN	4.5e−1	4.2e−4	NaN	1.3e−3	2.4e−3	7.4e−4
NaN	NaN	5	NaN	7	23	7	NaN	NaN	3.4e−4	NaN	9.3e−4	1.9e−3	6.0e−4
NaN	NaN	NaN	NaN	NaN	33	12	NaN	NaN	NaN	NaN	NaN	3.5e−3	1.1e−3
NaN	848	6	NaN	NaN	31	10	NaN	7.2e−1	3.8e−4	NaN	NaN	5.6e−3	1.6e−3
NaN	323	NaN	388	8	37	12	NaN	2.6e−1	NaN	3.8e−2	1.1e−3	3.5e−3	1.0e−3
NaN	253	7	258	7	24	7	NaN	1.7e−1	4.4e−4	2.1e−2	7.7e−4	2.0e−3	5.7e−4
NaN	857	6	NaN	9	31	10	NaN	5.0e−1	3.6e−4	NaN	1.5e−3	3.1e−3	7.9e−4
NaN	334	7	165	6	32	16	NaN	2.3e−1	4.1e−4	1.2e−2	5.9e−4	2.7e−3	1.0e−3
NaN	577	NaN	1,185	NaN	35	NaN	NaN	3.7e−1	NaN	1.1e−1	NaN	4.5e−3	NaN
NaN	150	5	160	7	29	8	NaN	1.0e−1	3.9e−4	1.3e−2	7.9e−4	2.4e−3	7.6e−4
NaN	NaN	NaN	NaN	NaN	60	37	NaN	NaN	NaN	NaN	NaN	8.4e−3	2.7e−3
NaN	NaN	NaN	NaN	NaN	38	14	NaN	NaN	NaN	NaN	NaN	8.7e−3	3.5e−3
348	358	50	276	192	64	54	9.6e−3	2.6e−1	2.5e−3	2.4e−2	2.7e−2	7.8e−3	2.9e−3
NaN	406	7	522	NaN	38	13	NaN	2.7e−1	5.6e−4	4.6e−2	NaN	7.6e−3	1.9e−3
205	224	12	108	NaN	NaN	17	8.6e−3	1.9e−1	7.3e−4	9.7e−3	NaN	NaN	2.5e−3

Tab. 6: Iteration counts and times for the 29 benchmark tensors and $\alpha = 0.999$

last problem R6_5. However, if one lowers the parameter τ to 0.001, method CPN converges in 652 iterations on this problem, too.

These results confirm the findings of [1] that problem R6_3 (the third to last one) for $\alpha = 0.99$ is the hardest problem; Perron-based algorithms can solve it successfully in the end, but (like the algorithms in [1]) they stagnate for a large number of iterations around a point which is far away from the true solution.

7 Conclusions

We have used the theory of quadratic vector equations in [7, 8, 9] to attack the multilinear pagerank problem described in [1]. Considering all nonnegative solutions instead of only the stochastic ones reveals some new properties on the structure of these solutions, and allows one to use a broader array of algorithms, with computational advantage. The new algorithms achieve better results when $\alpha \approx 1$, which is the most interesting and computationally challenging case.

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