Linguaggi formali

Let's start from the beginning

- A program is written in a programming language
- Every programming language (as every language in general) needs to obey its own rules
- We need to formally define languages…

Strings

- An alphabet is a finite set of symbols
- Examples

Σ_1 = {a, b, c, d, …, z}: the set of letters in Italian

Σ_2 = {0, 1}: the set of binary digits

Σ_3 = { (,) }: the set of open and closed brackets

A string over alphabet Σ is a finite sequence of symbols in Σ.

- Examples

abfbz is a string over Σ1 = {a, b, c, d, …, z}

11011 is a string over Σ2 = {0, 1}

))()(() is a string over Σ3 = {(,)}

The empty string is a string having no symbol, denoted by ε.

Operations on strings: lenght

- The length of a string x, denoted by |x|, is the number of symbols which compose x.

- Examples

  |abfbz|=5

  |110010|=6

  |))()(()|=7

  |ε|=0

Operations on strings: concatenation and substrings

- The concatenation of two strings x and y is a string xy,
  i.e., x is followed by y.
  It is an associative operation that admits the neutral element ε

- s is a substring of x if there exist two strings y and z such that $x = ysz$.

> Example:
> the prefixes of abc are : ε, a, ab, abc

- In particular,
  when $x = sz$ (substring with $y=\varepsilon$), s is called a prefix of x;
  when $x = ys$ (substring with $z=\varepsilon$), s is called a suffix of x;

ε is a prefix and a suffix of any string (including ε itself)

Power of an alphabet

- We define the set of all strings over Σ of a given length.

Σ^n denotes the strings of length n whose symbols are in Σ

If $\Sigma=\{0,1\}$

$\Sigma^0 = \{\varepsilon\}$

$\Sigma^1 = \Sigma = \{0, 1\}$

$\Sigma^2 = \{00, 01, 11, 10\}$

$\Sigma^3 = \{000, 001, 010, 011, 100, 101, 110, 111\}$

$$\Sigma^+ = \Sigma^1 \cup \Sigma^2 \cup \Sigma^3 \cup \Sigma^4 \cup \ldots = \bigcup_{i>0} \Sigma^i \qquad \Sigma^* = \{\epsilon\} \cup \Sigma^+$$

$\Sigma^+ = \{0, 1, 00, 01, 11, 10, 000, 001, 010, 011, 100, 101, 110, 111\ldots.\}$

Languages

A language is a set of strings over an alphabet:

$L \subseteq \Sigma^*$ is a language over Σ

Examples

L_1 = The set of all strings over Σ_1 that contain the substring "fool"

L_2 = The set of all strings over Σ_2 that represents a binary number divisible by 7

= {111, 10001, 10101, ...}

L_3 = The set of all strings over Σ_3 where every '(' is followed exactly by 2 occurrences of ')'

={ε,),)), ()),)()), ...}

Other examples of Languages

L_4 = The set of binary numbers whose value is prime

$= \{ 10, 11, 101, 111, 1011, 1101, \ldots \}$

L_5 = The set of legal English words over the English alphabet

L_6 = The set of legal C programs over the strings of characters and punctuation symbols

Operations on Languages

Union: $A \cup B$

Intersection: $A \cap B$

Difference: $A \setminus B$ (when $B \subseteq A$)

Complement: $\overline{A} = \Sigma^* - A$ where Σ^* is the set of all strings on Σ

Concatenation: $AB = \{ab \mid a \in A \text{ and } b \in B\}$

Example: $\{0, 1\}\{1, 2\} = \{01, 02, 11, 12\}$.

Kleene Clousure

Kleene closure: $$A^* = \bigcup_{i=0}^{\infty} A^i$$

- Notation: $$A^+ = \bigcup_{i=1}^{\infty} A^i$$

More example of Languages

Examples:

- The set of strings with n 1's followed by n 0's
  $\{\varepsilon, 01, 0011, 000111, \ldots\}$

- The set of strings with an equal number of 0's and 1's
  $\{\varepsilon, 01, 10, 0011, 0101, 1001, \ldots\}$

- The empty language $\emptyset$

- The language $\{\varepsilon\}$ consisting of the empty string only

Remember $\emptyset \neq \{\varepsilon\}$

Problems

- Does the string w belong to the language L?

  Example: 11101 $\in L_4$?

We want to define a procedure to decide it!

We can try to generate all words....

We can try to recognise when a word belongs to L

The generative approach: Grammars

Starting from a particular initial symbol, using the rewriting rules of the productions,

we generate the set of all the strings belonging to the language

Definition of Grammars

We define a Grammar $G=(\Sigma, N, S, P)$ where :

- Σ is the alphabet, a set of symbols (called terminals)
- N is the set of nonterminals
- $S \in N$ is the starting symbol
- P is the set of productions, each of the form

$$U \rightarrow V$$

where $U \in (\Sigma \cup N)^+$ and $V \in (\Sigma \cup N)^*$.

Derivations of $G = (\Sigma, N, S, P)$

A string $w \in \Sigma^*$ is generated by G if there exists a derivation starting from S and resulting in w obtained by rewriting the string using the productions in P

$G = (\{a\}, \{S\}, S, P)$

$$S \rightarrow \varepsilon$$
$$S \rightarrow a$$
$$S \rightarrow aS$$

A language generated by grammar G is denoted $L(G)$ and it is the set of strings derived using G.

Example of a grammar

We want to describe L1 the language of strings with an even number of 1's

L1 can be generated by a grammar ({0,1},{S,T},S,P) with P equal to

$S \rightarrow \varepsilon$

$S \rightarrow 0S$

$S \rightarrow 1T$

$T \rightarrow 0T$

$T \rightarrow 1S$

A string belongs to L1 iff it can be generated by the grammar

Grammar Example

Does the string 01010 belong to L1?

We need to find a derivation

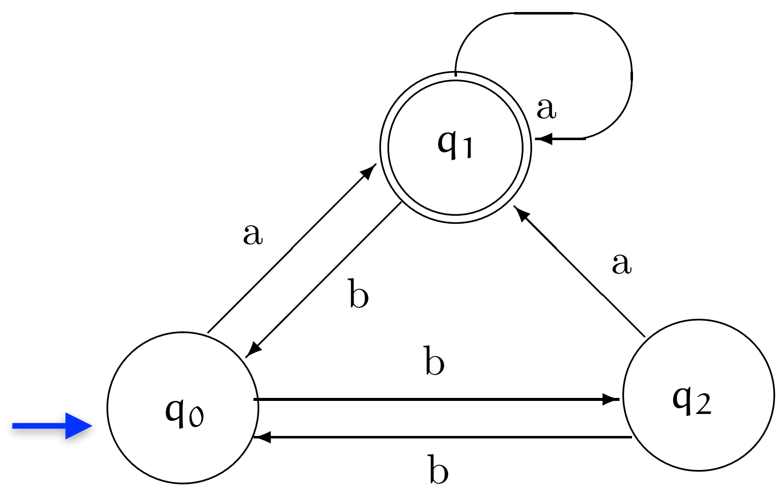
$$S \rightarrow \varepsilon \mid 0S \mid 1T$$
$$T \rightarrow 0T \mid 1S$$

S

Recognising a language: Automata

- A finite state automaton is finite state machine with an input of discrete values.
- The state machine consumes the input and possibly moves to a different state.
- The system may be in a state among a finite set of possible states. Being in a state allows to keep track of previous history.

input: baab



Back to our Problems

- Does the string w belong to the language L?

  We want to define a procedure to decide it!

- Which is the computational complexity necessary to answer to the previous question ?

It depends on the complexity of the language!!

Classification of Languages

Restrictions on productions give different types of grammars :

- Regular (type 3)
- Context-free (type 2)
- Context-sensitive (type 1)
- Phrase-structure (type 0)

$$U \rightarrow V$$

where $U \in (\Sigma \cup N)^+$ and $V \in (\Sigma \cup N)^*$.

For context-free, e.g., $U \in N$

No restrictions for phrase-structure

A language is of a type iff it admits a grammar of that type

Complexity of Languages Problems

	Regular Grammar Type 3	Context Free Grammar Type 2	Context Sensitive Grammar Type 1	Unrestricted Grammar Type 0
Is $W \in L(G)$?	P	P	PSPACE	U
Is L(G) empty?	P	P	U	U
Is $L(G1) \equiv L(G2)$?	PSPACE	U	U	U

P: decidable in polynomial time
PSPACE: decidable in polynomial space (at least as hard as NP-complete)
U: undecidable

Regular languages

All the following ways to represent regular languages are equivalent:

- Regular grammars (RG, type 3)
- Deterministic finite automata (DFA)
- Non-deterministic finite automata (NFA)
- Non-deterministic finite automata with ε transitions (ε-NFA)
- Regular expressions (RE)

Regular Grammars

A Right (or, analogously, Left) Regular Grammar is a grammar, where

- every production has the form A-> aB| a
- only for the starting symbol S we can have S$\rightarrow$ ε

Example

G=({a,b}, {S,B},S,P) where productions P are:

S-> aS|aB

B->bB|b

aaabb $\in$ L(G)??

L(G)={ $a^n b^m$| n,m>0}

S

Deterministic Finite Automata

A deterministic finite automaton (DFA) $(Q, \Sigma, \delta, qo, F)$

Q a finite set of states

Σ a finite set Σ of symbols

$\delta : Q \times \Sigma \rightarrow Q$ the transition function takes as argument a state and a symbol and returns one state

q_0 the starting state

$F \subseteq Q$ the set of final or accepting states

Deterministic Finite Automata

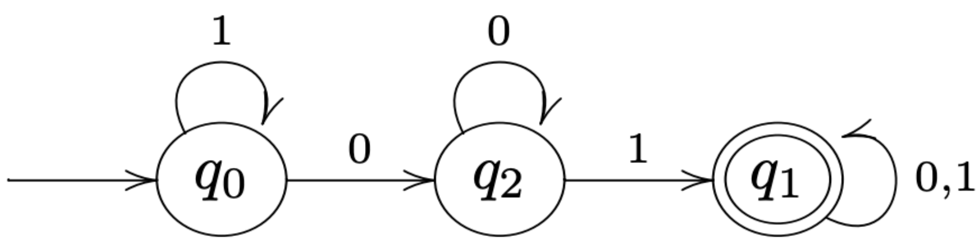
How to represent a DFA? With a transition table

	0	1
$\rightarrow q_0$	q_2	q_0
$*q_1$	q_1	q_1
q_2	q_2	q_1

-> indicates the starting state

* indicates the final states

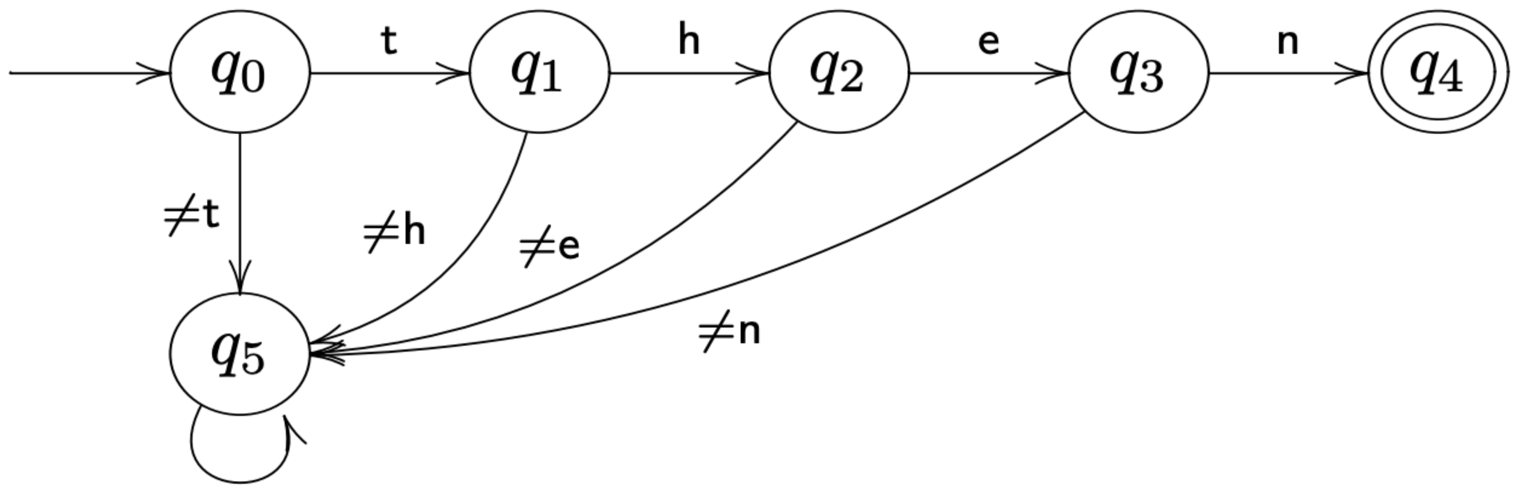
This defines the following transition system



Deterministic Finite Automata

When does an automaton accept a word?

It reads a word and accept it if it stops in an accepting state



here Q= $\{q_0, q_1, q_2, q_3, q_4, q_5\}$ F= $\{q_4\}$

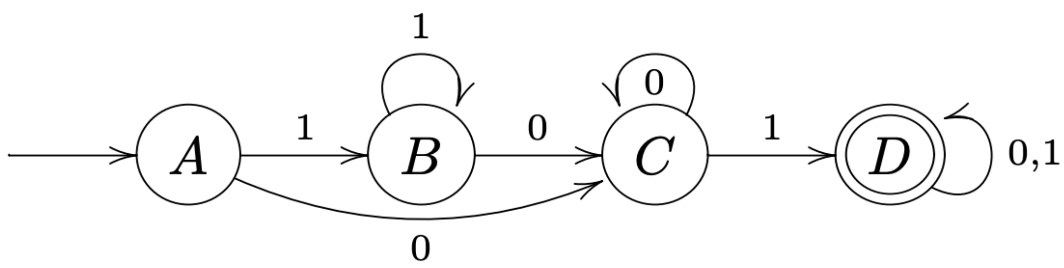
Only the word then is accepted

How DFA processes Strings

We build an automaton that accepts string containing the substring 01

$\Sigma=\{0,1\}$
$L=\{x01y \mid x,y \in \Sigma^*\}$

We get



	0	1
→A	C	B
B	C	B
C	C	D
*D	D	D

Extending the transition function to strings

We define the transitive closure of δ

$$\hat{\delta} : Q \times \Sigma^* \longrightarrow Q$$

$$\begin{cases} \hat{\delta}(q, \varepsilon) = q \\ \hat{\delta}(q, wa) = \delta(\hat{\delta}(q, w), a) \end{cases}$$

A string x is accepted by M=(Q, Σ, δ ,qo,F) iff $\widehat{\delta}(q_0, x) \in F$

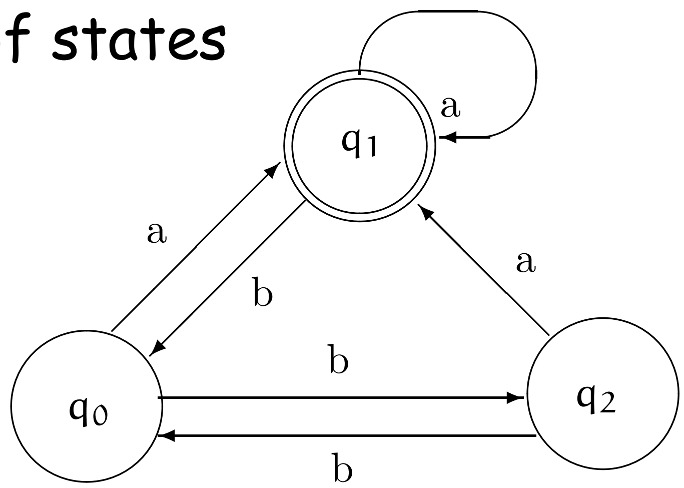
$$L(M) = \{x \in \Sigma^* | \widehat{\delta}(q_0, x) \in F\}$$

Nondeterministic Finite Automata

A nondeterministic finite automaton (NFA) allows more than one transition on the same input symbol.

Formally, a NFA is defined as $(Q, \Sigma, \delta, q_0, F)$ where the only difference is the transition function

$\delta : Q \times \Sigma \rightarrow \mathscr{P}(Q)$ a transition function that takes as argument a state and a symbol and returns a set of states



Extending the transition function to strings

We define the transitive closure of δ

$$\begin{cases} \hat{\delta}(q, \varepsilon) & = & \{q\} \\ \hat{\delta}(q, wa) & = & \bigcup_{p \in \hat{\delta}(q,w)} \delta(p, a) \end{cases}$$

A string x is accepted by M=(Q, Σ, δ ,qo,F) iff $\widehat{\delta}(q_0, x) \cap F \neq \emptyset$

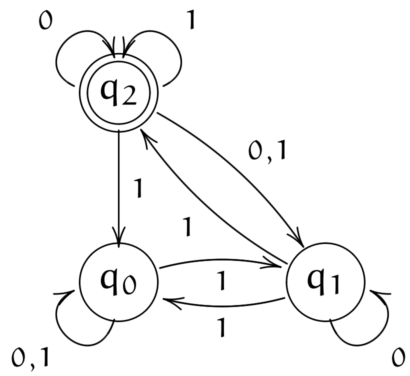
$$L(M) = \{x \in \Sigma^* | \widehat{\delta}(q_0, x) \cap F \neq \emptyset\}$$

- NFAs do not expand the class of language that can be accepted.

Example

	0	1
→ q_0	$\{q_0\}$	$\{q_0, q_1\}$
q_1	$\{q_1\}$	$\{q_0, q_2\}$
* q_2	$\{q_1, q_2\}$	$\{q_0, q_1, q_2\}$

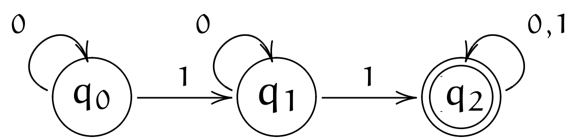
$F = \{q_2\}$.



NFA

L= $\{x \in \{0,1\}^* \mid$ x contains at least 2 occurrences of 1 }

	0	1
→ q_0	q_0	q_1
q_1	q_1	q_2
* q_2	q_2	q_2



DFA

Different characterisation of Regular Languages

There are different ways to characterise a regular language

- Regular grammars
- Deterministic Finite Automata
- Non deterministic Finite Automata
- Epsilon non deterministic Finite Automata
- Regular expression

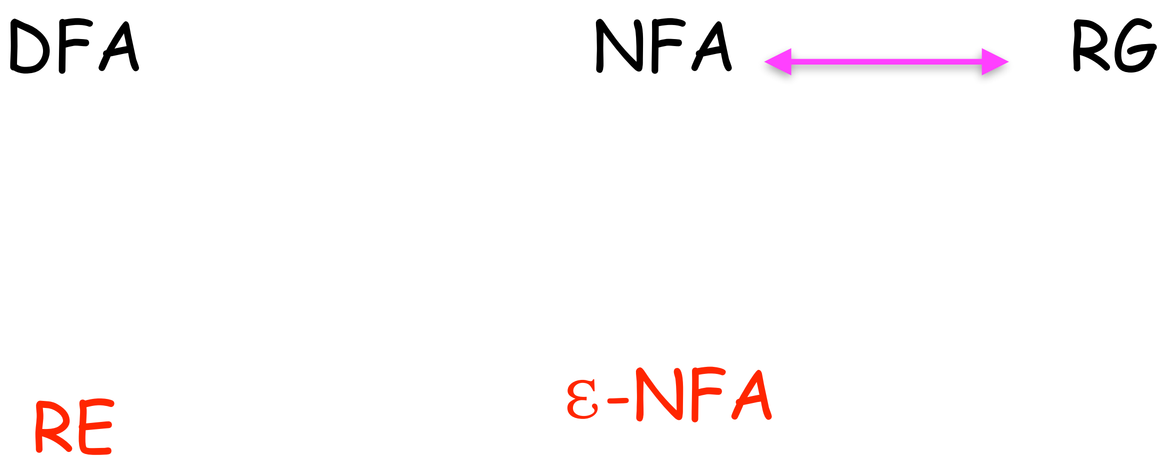
Different characterisation of Regular Languages

DFA NFA RG

RE ε-NFA

- We formally will show how to pass from one characterization to another one

Roadmap: equivalence between NFA and RG



From Regular Grammars to NFA

Theorem 1.

For each right grammar RG there is a non deterministic finite automaton NFA such that L(RG)=L(NFA).

Construction Algorithm

Given a RG=(Σ, N, S, P) construct a NFA=$(N \cup \{F\}, \Sigma, \delta, S, F')$
where F is a newly added state and
if $F' = \{F\} \cup \{S\}$ if $S \rightarrow \varepsilon$ belongs to P, $F' = \{F\}$, otherwise.

The transition function δ is defined by the following rules

1) For any $A \rightarrow a$ belonging to P, with a in Σ, set $\delta(A,a) = F$
2) For any $A \rightarrow aB$ belonging to P, with a in Σ and B in N, set $\delta(A,a)=B$

Example

$G=(\{a,b\}, \{S,B\},S,P)$ where productions P are:

$S \rightarrow aS|aB$

$B \rightarrow bB|b$ $\quad L(G)=\{ a^n b^m \mid n,m>0\}$

From NFA to Regular Grammars

Theorem 2

For each nondeterministic automaton NFA, there is one right grammar RG such that $L(RG)=L(NFA)$.

Construction Algorithm

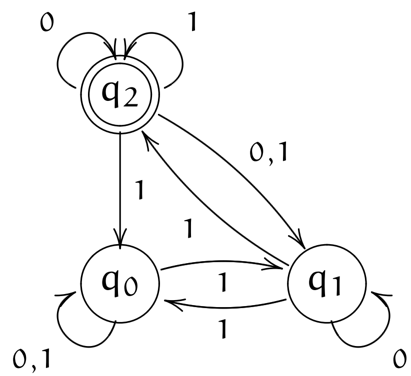
Given an automaton NFA= $(Q, \Sigma, \delta, q_0, F)$, construct a grammar RG=(Σ, Q, q_0', P) according the following steps:

1) for any $\delta(A,a)=B$ add $A \rightarrow aB$ to P,
2) if B belongs to F add also $A \rightarrow a$ to P;
3) if q_0 belongs to F then add ($q \rightarrow q_0 \mid \varepsilon$ to P and $q_0'=q$) else $q_0'=q_0$.

Example

	0	1
→ q_0	$\{q_0\}$	$\{q_0, q_1\}$
q_1	$\{q_1\}$	$\{q_0, q_2\}$
* q_2	$\{q_1, q_2\}$	$\{q_0, q_1, q_2\}$

$F = \{q_2\}$.



NFA

L= $\{x \in \{0,1\}^* \mid$ x contains at least 2 occurrences of 1 $\}$

Exercises

Write the NFA for the following languages

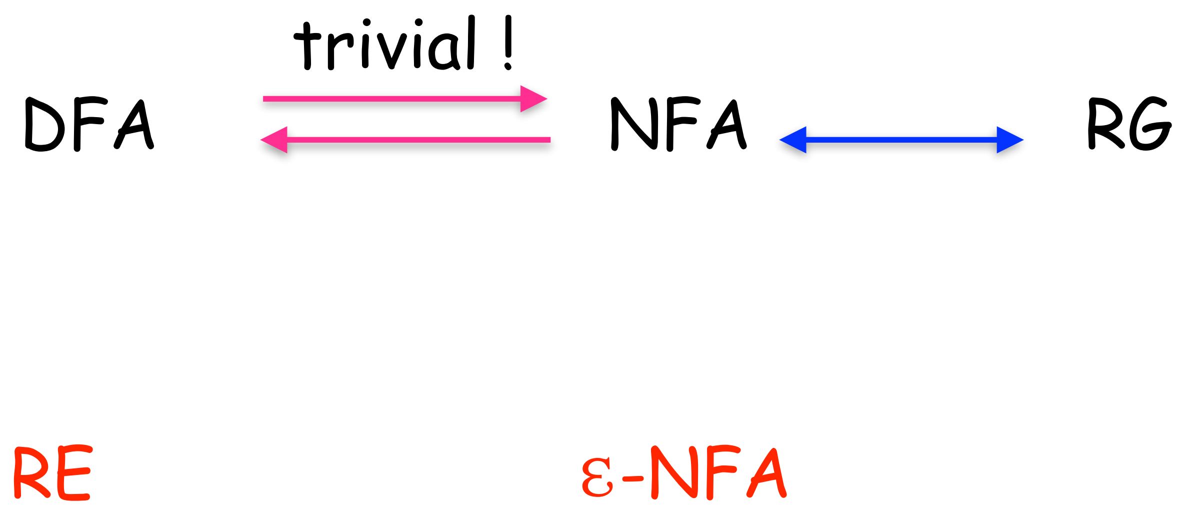
- Strings over the alphabet {a,b,c} containing at least one a and at least one b

- Strings of 0's and 1's whose tenth symbol from the right is 1

- The set of strings of 0's and 1's with at most one pair of consecutive 1's

and derive the corresponding grammars

Roadmap: equivalence between DFA and NFA



From a NFA to a DFA

The NFA are usually easier to "program".

For each NFA N there is a DFA D, such that L (D) = L (N),.

This involves a subset construction.

Given an

NFA N = $(Q_N, \Sigma, \delta_N, q_0, F_N)$

we will build a

DFA D = $(Q_D, \Sigma, \delta_D, q_0, F_D)$

such that

L (D) = L (N)

From NFA to a DFA

$$Q_D = \wp(Q_N),$$

Note that not all these state are necessary, most of them will be unreachable.

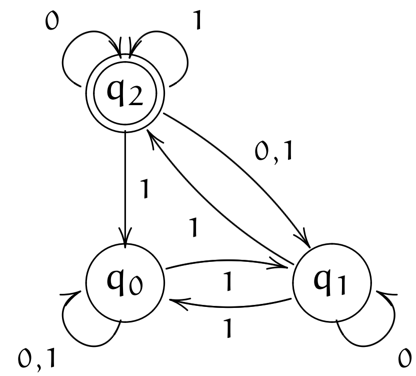
$$\forall P \in \mathcal{P}(Q_N): \qquad \delta_D(P, a) = \bigcup_{p \in P} \delta_N(p, a)$$

$$F_D = \{P \in \mathcal{P}(Q_N) \mid P \cap F \neq \emptyset\}$$

Example

NFA

	0	1
q_0	$\{q_0\}$	$\{q_0, q_1\}$
q_1	$\{q_1\}$	$\{q_0, q_2\}$
* q_2	$\{q_1, q_2\}$	$\{q_0, q_1, q_2\}$



Consider all the subsets $\mathcal{P}(Q_N)$

$\emptyset$

$\{q_0\}$ $\{q_1\}$ $\{q_2\}$

$\{q_0, q_1\}$ $\{q_0, q_2\}$ $\{q_1, q_2\}$

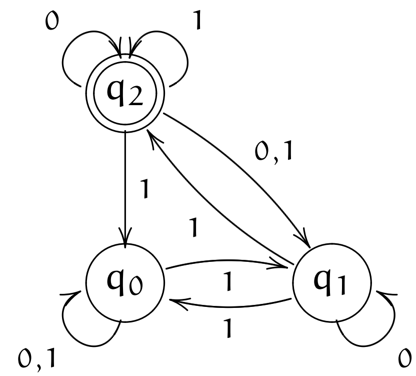
$\{q_0, q_1, q_2\}$

Which ones are final?

Example

NFA

		0	1
	q_0	$\{q_0\}$	$\{q_0, q_1\}$
	q_1	$\{q_1\}$	$\{q_0, q_2\}$
*	q_2	$\{q_1, q_2\}$	$\{q_0, q_1, q_2\}$



$\mathcal{P}(Q_N)$

$\emptyset$

$\{q_0\}$ $\{q_1\}$ $\{q_2\}$

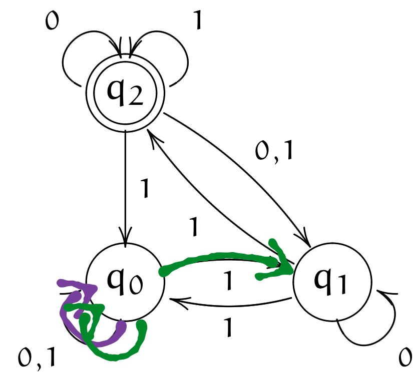
$\{q_0, q_1\}$ $\{q_0, q_2\}$ $\{q_1, q_2\}$

$\{q_0, q_1, q_2\}$

Example

NFA

	0	1
q_0	$\{q_0\}$	$\{q_0, q_1\}$
q_1	$\{q_1\}$	$\{q_0, q_2\}$
$*q_2$	$\{q_1, q_2\}$	$\{q_0, q_1, q_2\}$



		0	1
	$\emptyset$	$\emptyset$	$\emptyset$
q_0'	$\{q_0\}$	$\{q_0\}$	$\{q_0, q_1\}$

$\mathcal{P}(Q_N)$

$\emptyset$

$\{q_0\}$ $\{q_1\}$ $\{q_2\}$

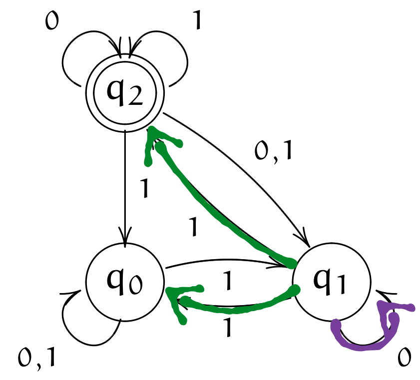
$\{q_0, q_1\}$ $\{q_0, q_2\}$ $\{q_1, q_2\}$

$\{q_0, q_1, q_2\}$

Example

NFA

	0	1
q_0	$\{q_0\}$	$\{q_0, q_1\}$
q_1	$\{q_1\}$	$\{q_0, q_2\}$
$\star q_2$	$\{q_1, q_2\}$	$\{q_0, q_1, q_2\}$



$\mathcal{P}(Q_N)$

$\emptyset$

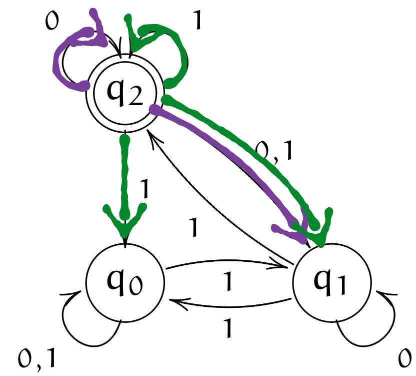
$\{q_0\}$ $\{q_1\}$ $\{q_2\}$

$\{q_0, q_1\}$ $\{q_0, q_2\}$ $\{q_1, q_2\}$

$\{q_0, q_1, q_2\}$

		0	1
	$\emptyset$	$\emptyset$	$\emptyset$
q_0'	$\{q_0\}$	$\{q_0\}$	$\{q_0, q_1\}$
q_1'	$\{q_1\}$	$\{q_1\}$	$\{q_0, q_2\}$

Example



NFA

	0	1
q_0	$\{q_0\}$	$\{q_0, q_1\}$
q_1	$\{q_1\}$	$\{q_0, q_2\}$
$* q_2$	$\{q_1, q_2\}$	$\{q_0, q_1, q_2\}$

$\mathcal{P}(Q_N)$

$\emptyset$

$\{q_0\}$ $\{q_1\}$ $\{q_2\}$

$\{q_0, q_1\}$ $\{q_0, q_2\}$ $\{q_1, q_2\}$

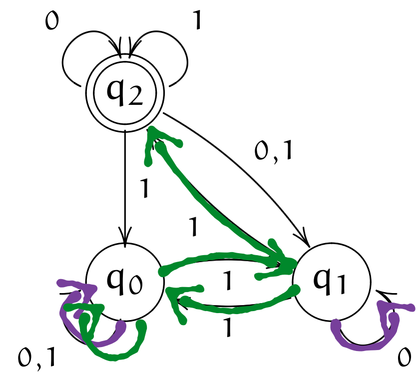
$\{q_0, q_1, q_2\}$

		0	1
	$\emptyset$	$\emptyset$	$\emptyset$
q'_0	$\{q_0\}$	$\{q_0\}$	$\{q_0, q_1\}$
q'_1	$\{q_1\}$	$\{q_1\}$	$\{q_0, q_2\}$
$* q'_2$	$\{q_2\}$	$\{q_1, q_2\}$	$\{q_0, q_1, q_2\}$

Example

NFA

	0	1
q_0	$\{q_0\}$	$\{q_0, q_1\}$
q_1	$\{q_1\}$	$\{q_0, q_2\}$
* q_2	$\{q_1, q_2\}$	$\{q_0, q_1, q_2\}$



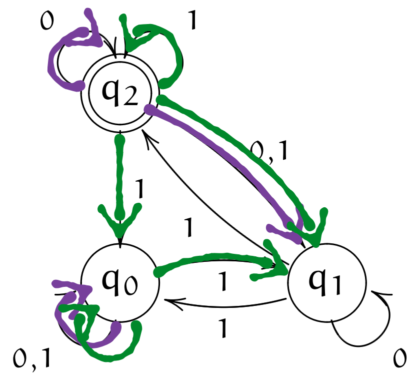
$\emptyset$ $\{q_0\}$ $\{q_1\}$ $\{q_2\}$ $\{q_0, q_1\}$ $\{q_0, q_2\}$ $\{q_1, q_2\}$ $\{q_0, q_1, q_2\}$

		0	1
	$\emptyset$	$\emptyset$	$\emptyset$
q_0'	$\{q_0\}$	$\{q_0\}$	$\{q_0, q_1\}$
q_1'	$\{q_1\}$	$\{q_1\}$	$\{q_0, q_2\}$
* q_2'	$\{q_2\}$	$\{q_1, q_2\}$	$\{q_0, q_1, q_2\}$
q_3'	$\{q_0, q_1\}$	$\{q_0, q_1\}$	$\{q_0, q_1, q_2\}$

Example

NFA

	0	1
q_0	$\{q_0\}$	$\{q_0, q_1\}$
q_1	$\{q_1\}$	$\{q_0, q_2\}$
$* q_2$	$\{q_1, q_2\}$	$\{q_0, q_1, q_2\}$



$\emptyset$

$\{q_0\}$ $\{q_1\}$ $\{q_2\}$

$\{q_0, q_1\}$ $\{q_0, q_2\}$ $\{q_1, q_2\}$

$\{q_0, q_1, q_2\}$

		0	1
	$\emptyset$	$\emptyset$	$\emptyset$
q_0'	$\{q_0\}$	$\{q_0\}$	$\{q_0, q_1\}$
q_1'	$\{q_1\}$	$\{q_1\}$	$\{q_0, q_2\}$
$* q_2'$	$\{q_2\}$	$\{q_1, q_2\}$	$\{q_0, q_1, q_2\}$
q_3'	$\{q_0, q_1\}$	$\{q_0, q_1\}$	$\{q_0, q_1, q_2\}$
$* q_4'$	$\{q_0, q_2\}$	$\{q_0, q_1, q_2\}$	$\{q_0, q_1, q_2\}$

-

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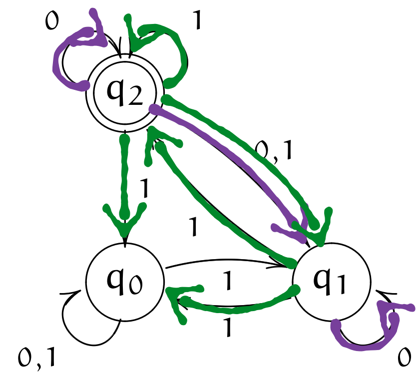
Example

NFA

	0	1
q_0	$\{q_0\}$	$\{q_0, q_1\}$
q_1	$\{q_1\}$	$\{q_0, q_2\}$
* q_2	$\{q_1, q_2\}$	$\{q_0, q_1, q_2\}$



$\emptyset$

$\{q_0\}$ $\{q_1\}$ $\{q_2\}$

$\{q_0, q_1\}$ $\{q_0, q_2\}$ $\{q_1, q_2\}$

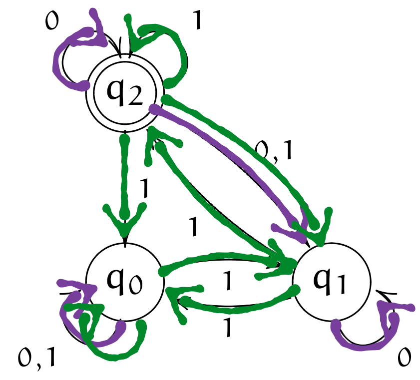
$\{q_0, q_1, q_2\}$

		0	1
	$\emptyset$	$\emptyset$	$\emptyset$
q'_0	$\{q_0\}$	$\{q_0\}$	$\{q_0, q_1\}$
q'_1	$\{q_1\}$	$\{q_1\}$	$\{q_0, q_2\}$
* q'_2	$\{q_2\}$	$\{q_1, q_2\}$	$\{q_0, q_1, q_2\}$
q'_3	$\{q_0, q_1\}$	$\{q_0, q_1\}$	$\{q_0, q_1, q_2\}$
* q'_4	$\{q_0, q_2\}$	$\{q_0, q_1, q_2\}$	$\{q_0, q_1, q_2\}$
* q'_5	$\{q_1, q_2\}$	$\{q_1, q_2\}$	$\{q_0, q_1, q_2\}$

Example

NFA

	0	1
q_0	$\{q_0\}$	$\{q_0, q_1\}$
q_1	$\{q_1\}$	$\{q_0, q_2\}$
* q_2	$\{q_1, q_2\}$	$\{q_0, q_1, q_2\}$



$\emptyset$ → (row $\emptyset$)
$\{q_0\}$ → q'_0
$\{q_1\}$ → q'_1
$\{q_2\}$ → q'_2
$\{q_0, q_1\}$ → q'_3
$\{q_0, q_2\}$ → q'_4
$\{q_1, q_2\}$ → q'_5
$\{q_0, q_1, q_2\}$ → q'_6

		0	1
	$\emptyset$	$\emptyset$	$\emptyset$
q'_0	$\{q_0\}$	$\{q_0\}$	$\{q_0, q_1\}$
q'_1	$\{q_1\}$	$\{q_1\}$	$\{q_0, q_2\}$
* q'_2	$\{q_2\}$	$\{q_1, q_2\}$	$\{q_0, q_1, q_2\}$
q'_3	$\{q_0, q_1\}$	$\{q_0, q_1\}$	$\{q_0, q_1, q_2\}$
* q'_4	$\{q_0, q_2\}$	$\{q_0, q_1, q_2\}$	$\{q_0, q_1, q_2\}$
* q'_5	$\{q_1, q_2\}$	$\{q_1, q_2\}$	$\{q_0, q_1, q_2\}$
* q'_6	$\{q_0, q_1, q_2\}$	$\{q_0, q_1, q_2\}$	$\{q_0, q_1, q_2\}$

Example

NFA

	0	1
q_0	$\{q_0\}$	$\{q_0, q_1\}$
q_1	$\{q_1\}$	$\{q_0, q_2\}$
* q_2	$\{q_1, q_2\}$	$\{q_0, q_1, q_2\}$

		0	1
	$\emptyset$	$\emptyset$	$\emptyset$
q'_0	$\{q_0\}$	$\{q_0\}$	$\{q_0, q_1\}$
q'_1	$\{q_1\}$	$\{q_1\}$	$\{q_0, q_2\}$
* q'_2	$\{q_2\}$	$\{q_1, q_2\}$	$\{q_0, q_1, q_2\}$
q'_3	$\{q_0, q_1\}$	$\{q_0, q_1\}$	$\{q_0, q_1, q_2\}$
* q'_4	$\{q_0, q_2\}$	$\{q_0, q_1, q_2\}$	$\{q_0, q_1, q_2\}$
* q'_5	$\{q_1, q_2\}$	$\{q_1, q_2\}$	$\{q_0, q_1, q_2\}$
* q'_6	$\{q_0, q_1, q_2\}$	$\{q_0, q_1, q_2\}$	$\{q_0, q_1, q_2\}$

$\emptyset$ $\{q_0\}$ $\{q_1\}$ $\{q_2\}$ $\{q_0, q_1\}$ $\{q_0, q_2\}$ $\{q_1, q_2\}$ $\{q_0, q_1, q_2\}$

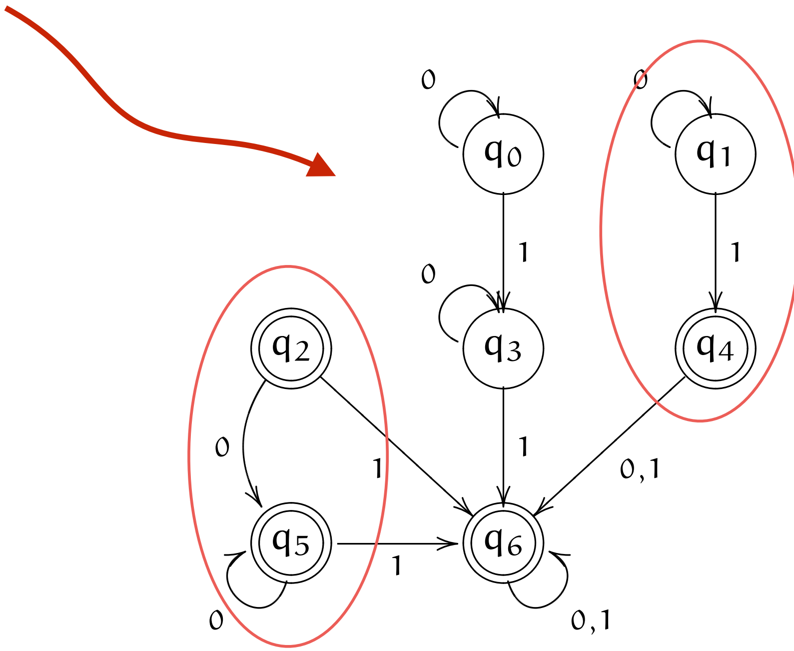
DFA

	0	1
q_0	q_0	q_3
q_1	q_1	q_4
* q_2	q_5	q_6
q_3	q_3	q_6
* q_4	q_6	q_6
* q_5	q_5	q_6
* q_6	q_6	q_6

Example

DFA

		0	1
	q_0	q_0	q_3
	q_1	q_1	q_4
*	q_2	q_5	q_6
	q_3	q_3	q_6
*	q_4	q_6	q_6
*	q_5	q_5	q_6
*	q_6	q_6	q_6



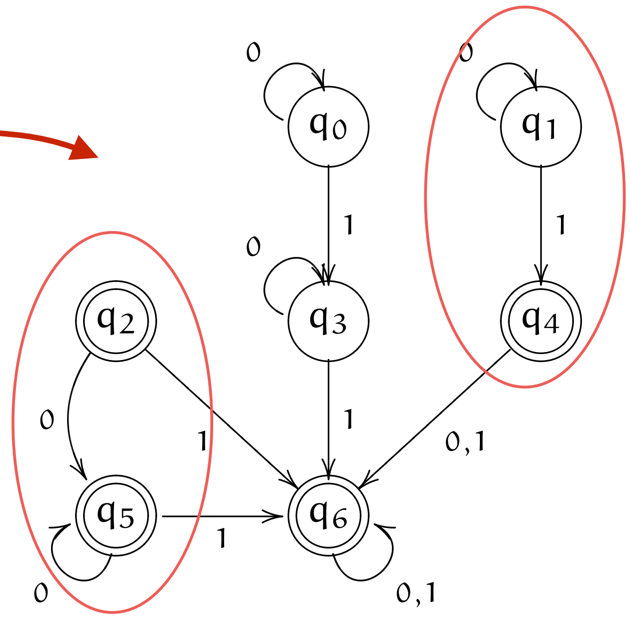
DFA with unreachable states

Example

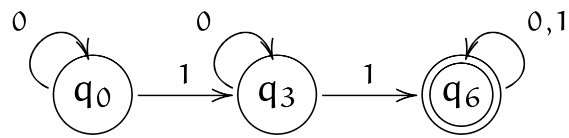
DFA

	0	1
q_0	q_0	q_3
q_1	q_1	q_4
q_2	q_5	q_6
q_3	q_3	q_6
q_4	q_6	q_6
q_5	q_5	q_6
q_6	q_6	q_6

DFA with unreachable states



minimum DFA



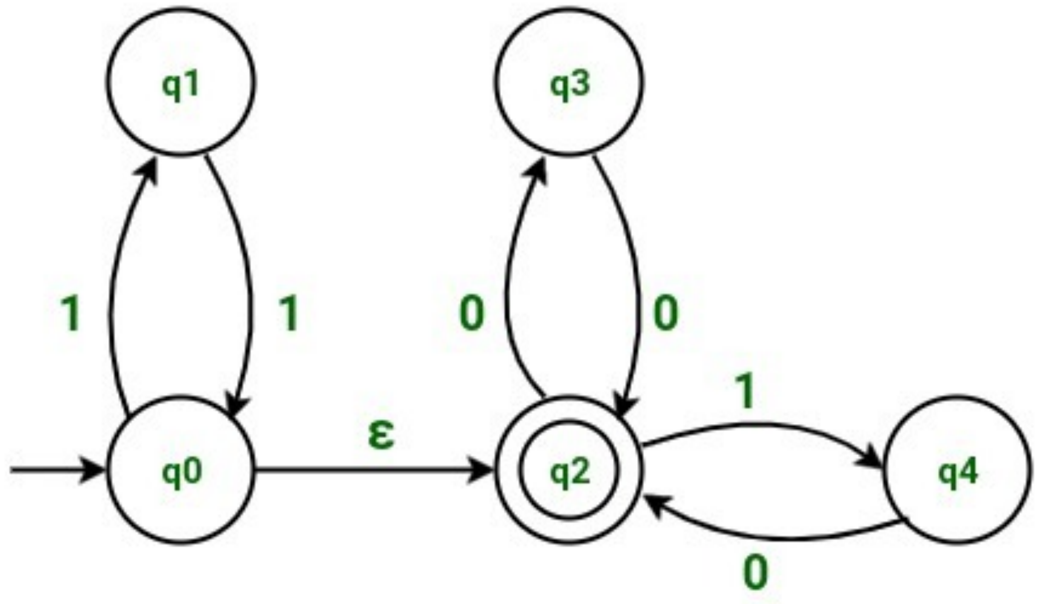
The ε-NFA: NFA with epsilon transitions

- Extension of finite automaton.
- The new feature: we allow transition on ϵ, the empty string.
- An NFA that is allowed to make transition spontaneously, without receiving any input symbol.
- As in the case of NFA w.r.t. DFA this new feature does not expand the class of languages that can be accepted.

Definition of ε-NFA

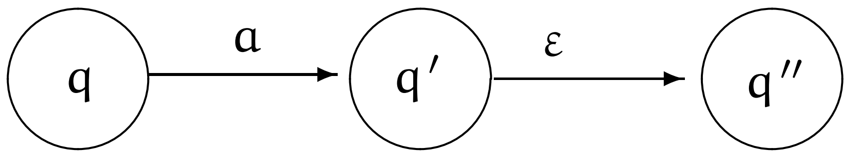
A NFA whose transition function can always choose epsilon as input symbol

$$\delta : Q \times (\Sigma \cup \{\epsilon\}) \rightarrow \wp(Q)$$



Definition of ε-closure for extending δ to Strings

We need to define the $\varepsilon\text{-}closure$ that applied to a state gives all the states reachable with ε-transitions



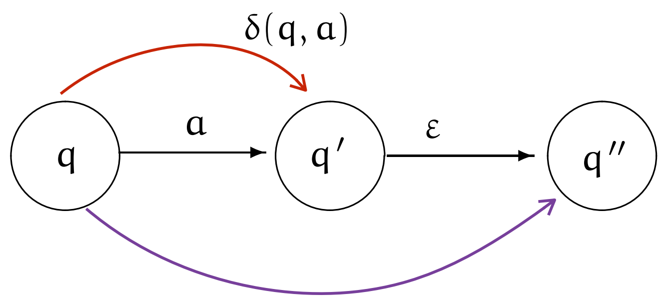
$$\epsilon\text{-closure(q)}=\{q\} \quad \epsilon\text{-closure(q')}=\{q', q''\}$$

$$\varepsilon\text{-closure}(P) = \bigcup_{p \in P} \varepsilon\text{-closure}(p)$$

The extension of δ to strings

$$\hat{\delta} : Q \times \Sigma^* \longrightarrow \wp(Q)$$

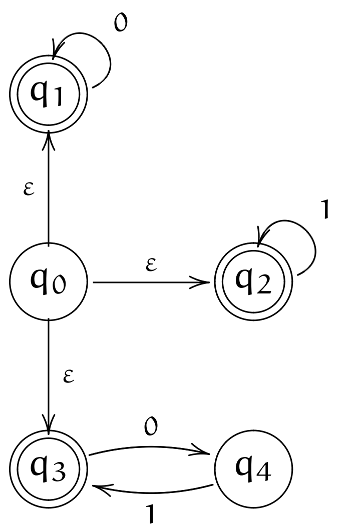
$$\begin{cases} \hat{\delta}(q, \varepsilon) & = & \varepsilon\text{-closure}(q) \\ \hat{\delta}(q, wa) & = & \bigcup_{p \in \hat{\delta}(q,w)} \varepsilon\text{-closure}(\delta(p, a)) \end{cases}$$



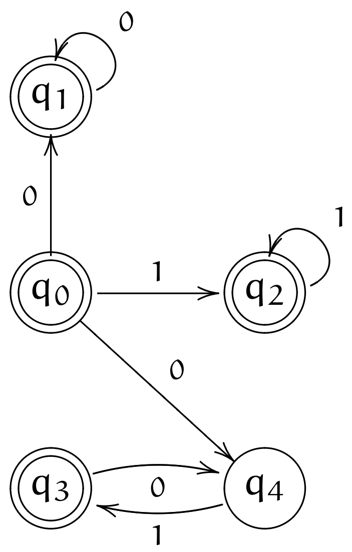
$$\hat{\delta}(q, a) = \bigcup_{p \in \hat{\delta}(q,\varepsilon)} \varepsilon\text{-closure}(\delta(p, a)) = \text{???}$$

Example

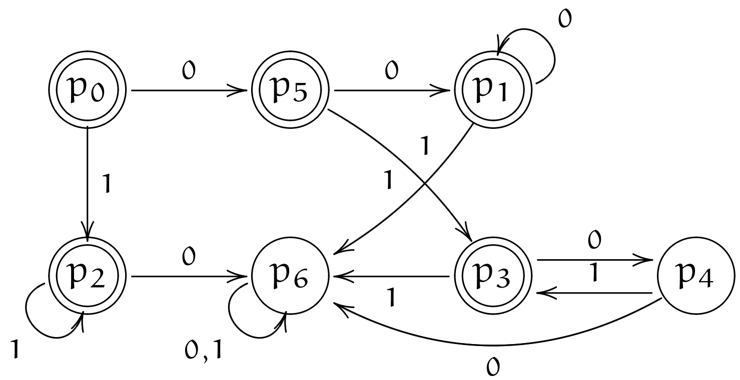
ε-NFA



NFA

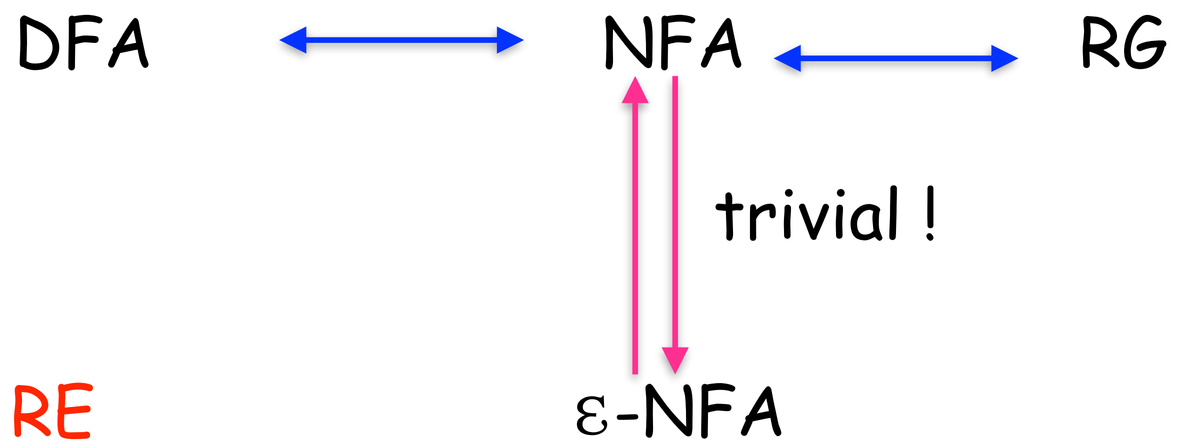


DFA



$$L = \{\ x \mid \exists n \in \mathbb{N}.\ x = 0^n \ \vee\ x = 1^n \ \vee\ x = (01)^n\ \}$$

Roadmap: equivalence between NFA and ε-NFA



From ε-NFA to NFA

For each ε-NFA E there is a NFA N, such that L (E) = L (N), and vice versa.

Given an

ε-NFA E = $(Q, \Sigma, \delta_E, q_0, F_E)$

we build a

NFA N = $(Q, \Sigma, \delta_N, q_0, F_N)$

such that

L (E) = L (N)

Equivalence between ε-NFA and NFA

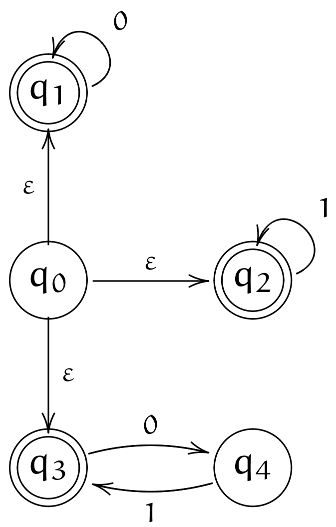
$$\delta_N(q,a) = \widehat{\delta}_E(q,a)$$

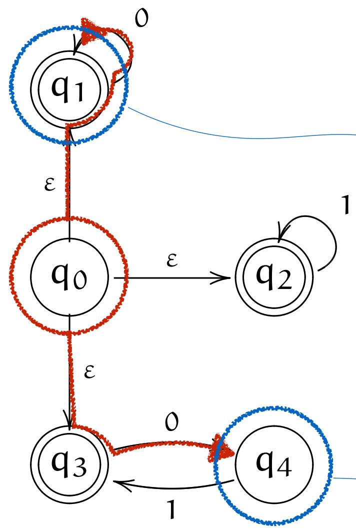
$$F_N = \begin{cases} F_E \cup \{q_0\} & \text{if } \epsilon\text{-closure}(q_0) \cap F_E \neq \emptyset \\ F_E & \text{otherwise} \end{cases}$$

if a final state can be reached with an epsilon transition from the initial state

Example

ε-*NFA*

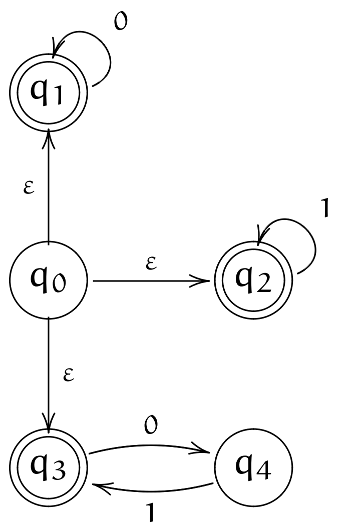


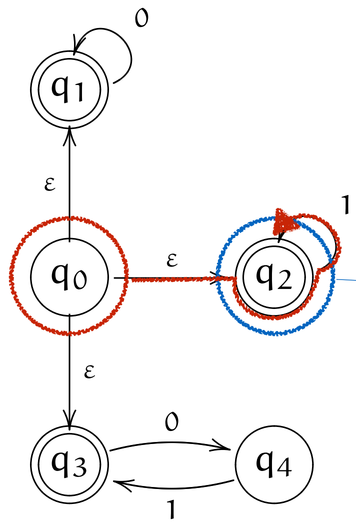


	0	1
q_0	$\{q_1, q_4\}$	$\{q_2\}$
q_1	$\{q_1\}$	$\emptyset$
q_2	$\emptyset$	$\{q_2\}$
q_3	$\{q_4\}$	$\emptyset$
q_4	$\emptyset$	$\{q_3\}$

Example

ε-*NFA*

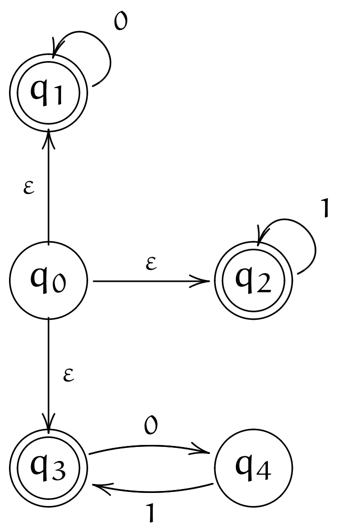


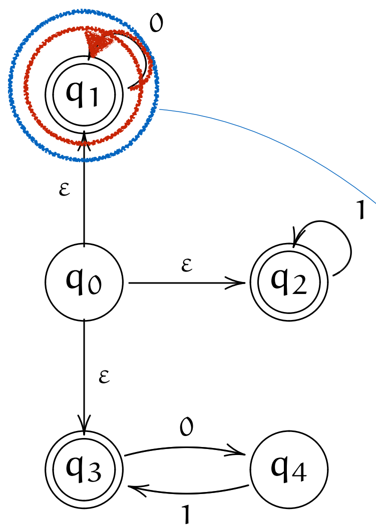


	0	1
q_0	$\{q_1, q_4\}$	$\{q_2\}$
q_1	$\{q_1\}$	$\emptyset$
q_2	$\emptyset$	$\{q_2\}$
q_3	$\{q_4\}$	$\emptyset$
q_4	$\emptyset$	$\{q_3\}$

Example

ε-*NFA*

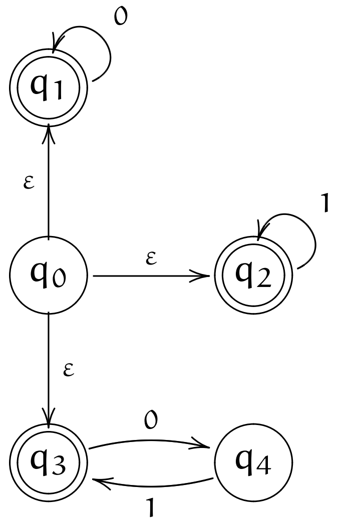


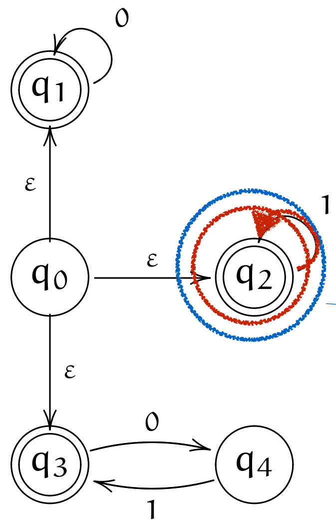


	0	1
q_0	$\{q_1, q_4\}$	$\{q_2\}$
q_1	$\{q_1\}$	$\emptyset$
q_2	$\emptyset$	$\{q_2\}$
q_3	$\{q_4\}$	$\emptyset$
q_4	$\emptyset$	$\{q_3\}$

Example

ε-*NFA*

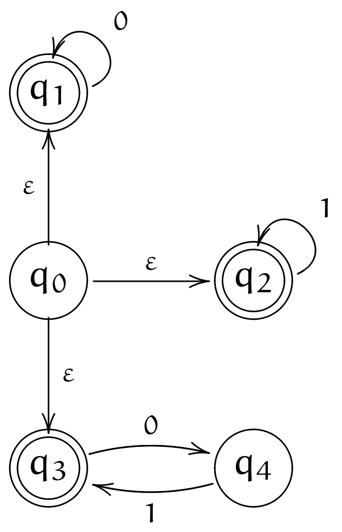


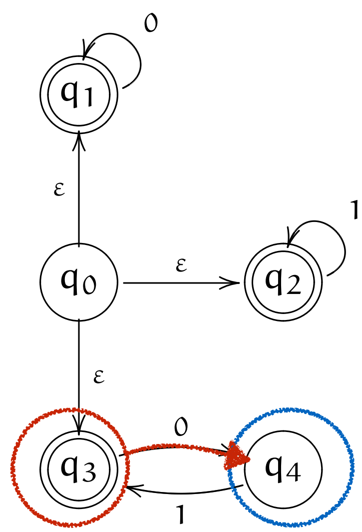


	0	1
q_0	$\{q_1, q_4\}$	$\{q_2\}$
q_1	$\{q_1\}$	$\emptyset$
q_2	$\emptyset$	$\{q_2\}$
q_3	$\{q_4\}$	$\emptyset$
q_4	$\emptyset$	$\{q_3\}$

Example

ε-*NFA*

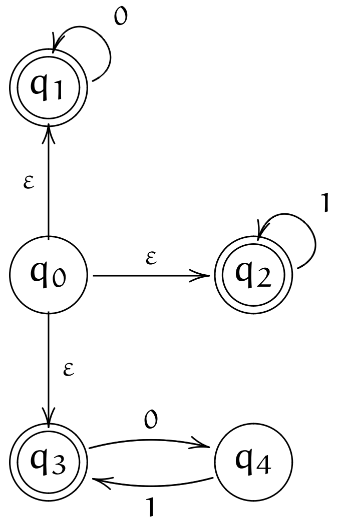


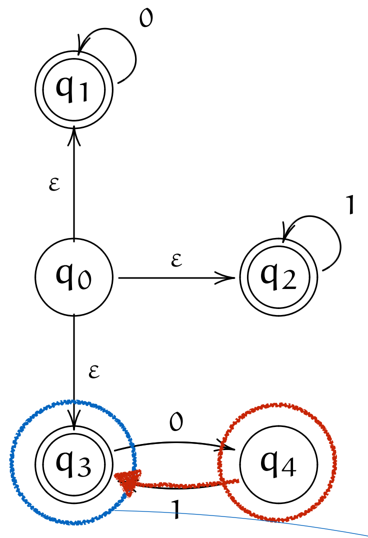


	0	1
q_0	$\{q_1, q_4\}$	$\{q_2\}$
q_1	$\{q_1\}$	$\emptyset$
q_2	$\emptyset$	$\{q_2\}$
q_3	$\{q_4\}$	$\emptyset$
q_4	$\emptyset$	$\{q_3\}$

Example

ε-*NFA*

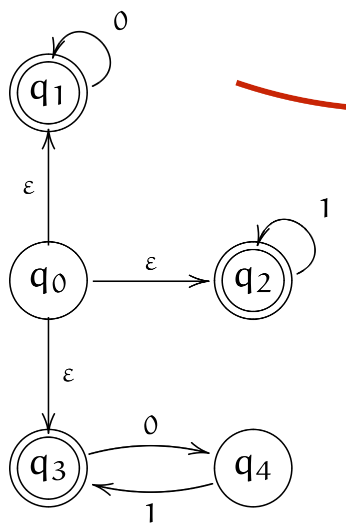




	0	1
q_0	$\{q_1, q_4\}$	$\{q_2\}$
q_1	$\{q_1\}$	$\emptyset$
q_2	$\emptyset$	$\{q_2\}$
q_3	$\{q_4\}$	$\emptyset$
q_4	$\emptyset$	$\{q_3\}$

Example

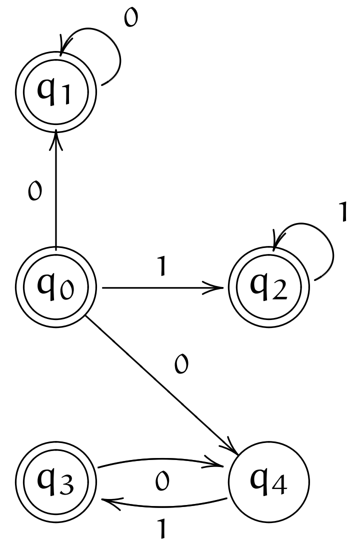
ε-NFA



NFA

	0	1
q_0	$\{q_1, q_4\}$	$\{q_2\}$
q_1	$\{q_1\}$	$\emptyset$
q_2	$\emptyset$	$\{q_2\}$
q_3	$\{q_4\}$	$\emptyset$
q_4	$\emptyset$	$\{q_3\}$

NFA



Operations on languages: recap.

Union: $A \cup B$

Intersection: $A \cap B$

Difference: $A \setminus B$

Complement: $compl(A) = \Sigma^* - A$

Concatenation: $AB = \{ab \mid a \in A, b \in B\}$

Kleene Clousure:

$$A^* = \bigcup_{i=0}^{\infty} A^i$$