Barter models*

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Abstract

This paper presents a family of models that involve a pair of actors that aim at bartering the goods from two privately owned pools of heterogeneous goods. The barter can occur only once or can be a repeated process with possibilities of retaliation and can involve either a single good or a basket of goods from each actor. We are indeed going to examine both the basic symmetric model (one-to-one barter) and its extensions (one-to-many, many-to-one and many-to-many barters), none of which reproduces a symmetric situation. The paper opens with some basic criteria and a brief description of some classical solutions, then it gives the basic motivation of the models followed by some definitions and then switch to the descriptions of the models in an increasing complexity order. The paper closes with a section devoted to some applications, some sections devoted to two more "hybrid" models and a section devoted to conclusions and future plans.

1 Introduction

This paper presents a family of models that involve a pair of actors[1] that aim at bartering the goods from two privately owned pools of heterogeneous goods. The barter can occur only once or can be a repeated process with

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[1]We use the terms **actors** and **players** as synonyms in this paper.

possibilities of retaliation[2] and can involve either a single good or a basket of goods from each actor. We are indeed going to examine both the basic symmetric model (one-to-one barter) and its extensions (one-to-many, many-to-one and many-to-many barters), none of which reproduces a symmetric situation.
The paper is structured as follows. We start with some basic criteria and a brief description of some classical solutions, then we give the basic motivation of the models followed by some definitions and then switch to the descriptions of the models in an increasing complexity order. The paper closes with a section devoted to some applications, some sections devoted to two more "hybrid" models and a section devoted to conclusions and future plans.

2 The basic criteria

In this section we introduce some basic criteria (from Brams and Taylor (1996), Brams and Taylor (1999) and Young (1994)) that allow us to frame the models we propose in a general context. Our aim is twofold: to define "objective" criteria and to use them to evaluate the goodness of the proposed models of barter.
The starting point is to have **fair** barters. As a measure of fairness we refer to Brams and Taylor (1999) where a procedure is defined as fair if it satisfies the criteria of **envy-freeness**, **equitability** and **efficiency**[3] so that each party's level of satisfaction is fully independent from the levels of satisfaction of the other parties.
In our context we have two players[4] each possessing a pool of private heterogeneous goods and each aiming at a barter that satisfies all the aforesaid criteria so to be fair.
Generally speaking, we say an agreement turns into an allocation of the goods between the players that is **envy-free** if (Brams and Taylor (1996), Brams and Taylor (1999) and Young (1994)) none of the actors involved in an agreement would prefer somebody's else portion, how it derives to him from the agreement, to his own. If an agreement involves the sharing of benefits it is considered envy-free if none of the participants believes his share to be lower than somebody's else share, whereas if it involves the share of burdens or

[2] With this term we denote spiteful and vindictive attitudes of one player with respect to the other.

[3] In the present paper we consider only Pareto efficiency.

[4] We use the term player or actor with a meaning analogous but not identical to that it has in Game Theory so that every player has the possibility to perform some choices (or moves) not always being guided by some strategies and not always trying to obtain the best possible outcome (something very similar to optimizing an expected utility).

chores it is considered envy-free if none of the participants believes his share to be greater that somebody's else share.
If an allocation is envy-free then (Brams and Taylor (1999)) it is proportional (so that each of the n players thinks to have received at least $1/n$ of the total value) but the converse is true only if $n = 2$.
As to **equitability** we say (according to Brams and Taylor (1999)) that an allocation is equitable if each player thinks to have received the same fraction of the total value of the goods to be allocated.
Last but not least, as to **efficiency**, we say (according to Brams and Taylor (1999)) that an allocation is efficient if there is no other allocation where one of the players is better off and none of them is worse off.
Such criteria, to be used in our context[5], must be adapted, if it is possible, or must be redefined someway so to be in agreement both with their classical definitions and with intuition.
We start with **envy-freeness**. If we denote[6] with a_A and l_A the values for A himself, respectively, of what A gets and loses from the barter[7] we say that the allocation deriving from a barter (or a barter tout court) is **envy-free** if we have for A:

$$\frac{a_A}{l_A} \geq 1 \tag{1}$$

and for B:

$$\frac{a_B}{l_B} \geq 1 \tag{2}$$

As will be shown from section 6 on, if a barter actually occurs it is guaranteed to be envy-free.
Since, in the case of two players, we want to maintain the equivalence between proportionality and envy-freeness we must derive from this definition a definition that mirrors the classical definition of proportionality.
For player A we may define a barter as proportional if it satisfies the following condition:

$$\frac{a_A}{a_A + l_A} \geq \frac{l_A}{a_A + l_A} \tag{3}$$

so that the percentage value of what A gets from the barter is at least equal to that of what he loses from it. A similar condition holds also for B:

$$\frac{a_B}{a_B + l_B} \geq \frac{l_B}{a_B + l_B} \tag{4}$$

[5]We recall that we have an in kind barter involving two players (A and B) and two pools of privately possessed heterogeneous indivisible goods.

[6]Such notations will be specialized in the single models.

[7]Similar quantities a_B and l_B can be defined also for player B and with the same caveat.

It is easy to see how from equation (3) it is possible to derive equation (1) and vice versa. The same holds also for equations (4) and (2).
As to **equitability** we must adapt its definition to our framework in this way. We need firstly some definitions. We define I and I', respectively, as the ex-ante and ex-post sets of goods[8] of A and J and J', respectively, as the ex-ante and ex-post sets of goods of B. If (i, j) denotes the bartered goods[9] in a one-to-one barter, we have[10]:

$$I' = I \setminus \{i\} \cup \{j\} \tag{5}$$

$$J' = J \setminus \{j\} \cup \{i\} \tag{6}$$

On these sets we define, for player A, the quantities that represent the values, after the bater, of his goods and B's goods for A himself, respectively, as $v_A(I')$ and $s_A(J')$. We therefore define a barter as equitable for A himself if the fractional value of what he gets is at least equal to the fractional value he gives to what B gets from the barter or[11]:

$$\frac{v_A(j)}{v_A(I')} \geq \frac{s_A(i)}{s_A(J')} \tag{7}$$

On the other hand the barter is equitable for B if[12]:

$$\frac{v_B(i)}{v_B(J')} \geq \frac{s_B(j)}{s_B(I')} \tag{8}$$

whereas if both relations hold we say that the barter is equitable.
As to **efficiency** we say that a barter of the two subsets[13] I_0 and J_0 is efficient if there is not another pair of subsets that gives to each player a better result. Formally, the barter (I_0, J_0) (to which there correspond l_A and a_A) is efficient for A if $\nexists\, (I'_0, J'_0)$ (to which there correspond l'_A and a'_A) such that:

$$\frac{a_A}{l_A} < \frac{a'_A}{l'_A} \tag{9}$$

[8] The terms ex-ante and ex-post refer to the occurrence of the barter itself.

[9] i from A to B and j from B to A.

[10] In the case of other kind of barters we must appropriately replace single goods with subsets.

[11] As it will be explained in the following sections we are under an additivity hypothesis where the value of a set is given by the sum of the values of its elements.

[12] In equation (8) we find the corresponding quantities we find in equation (7) but referred to player B.

[13] All the subsets we deal with in this definition are referred to the two "main" sets I and J and that $I_0 = \{i\}$ whereas $J_0 = \{j\}$.

whereas for B the condition is that $\nexists\,(I_0', J_0')$ such that:

$$\frac{a_B}{l_B} \leq \frac{a_B'}{l_B'} \tag{10}$$

In this way if the barter is such that both players attain[14]:

$$\frac{a_{A_{max}}}{l_{A_{min}}} \tag{11}$$

$$\frac{a_{B_{max}}}{l_{B_{min}}} \tag{12}$$

we have an **efficient barter** whereas if both attain[15]:

$$\frac{a_{A_{min}}}{l_{A_{max}}} \tag{13}$$

$$\frac{a_{B_{min}}}{l_{B_{max}}} \tag{14}$$

the barter is surely inefficient. We note, from the above equations, how **efficiency** of a barter cannot be always guaranteed and must be verified case by case.

3 A brief tracking shot of some classical solutions

Our starting point is Brams and Taylor (1996). In this book, the authors propose a lot of tools and algorithms for the allocation of goods for both divisible and indivisible cases: They start from $n = 2$ players and then extend their results to the general cases with $n > 2$. A common characteristic of such models is that players aim at more or less fair sharing of a common pool of goods on which they state preferences that can be compared in some way, even on common cardinals scales.

Another good reference is Brams and Taylor (1999), where authors present various methods for the allocation of the goods from a single pool, starting with (strict and balanced) alternation methods to switch to divide-and-choose and to end with adjusted winner method.

[14] In (11) with $a_{A_{max}}$ we denote the maximum value A can get form the barter and with $l_{A_{min}}$ we denote the minimum value A can lose form the barter. In (12) we have the same quantities for player B.

[15] In (13) with $a_{A_{min}}$ we denote the minimum value A can get form the barter and with $l_{A_{max}}$ we denote the maximum value A can lose form the barter. In (14) we have the same quantities for player B.

Also all these methods are devised to allow more or less fair divisions between two players of the goods belonging to a common pool (though extensions to more than two players are provided for all the methods).
We note, moreover, how adjusted winner method requires the use of a common cardinal scale among the players since it requires that each of them assigns to each good some points on 100 and that such points are compared (either directly or as ratios) so to determine to which player every good is assigned.
A short analysis of classical solutions for the division of goods can be found also in Fragnelli (2005a) again with regard to either one or more divisible goods or a pool of indivisible goods. Again the presence of a common pool of goods among the players makes such tools inappropriate as solutions to our problem.
From the comments made in Fragnelli (2005a) about auctions, moreover, it is also evident how such tools are not suitable to solve our problem.
Other solutions to division problems that can be found in the literature involve **market games** (Fragnelli (2005b) and Shubik (1959)), **assignment games** (Fragnelli (2005b)) and **cost games** (Fragnelli (2005b)).
In market games each player has an initial endowment and a preference relation on it. Each player has an utility function defined from such relation. Players aim at a redistribution of their initial endowments so to attain efficient redistributions. A redistribution is termed efficient if no player prefers any other distribution to this one. The main point here is the merging that assumes the use of common scales for the evaluation of the endowments.
In **assignment games** players are subdivided in two groups: buyers and sellers. Every seller owns only one good (of which he knows the evaluation) and each buyer can buy one good (of which she knows the evaluation). Prices of the objects depend on these evaluations and on the ability to bargaining of the players. In these games players aim at obtaining their maximum gain with regard to each one's evaluation. Our models owe much to these games but for the fact that every player is both a buyer and a seller so that the gain each player obtains strictly depends on two simultaneous exchanges. Moreover we have no numerary good so there is no real possibility to sell or buy.
In **cost games** we must define a division of the costs of a project among the many involved users so to take care of their roles and interests. It is easy to see how this family of games has nothing to do with the problem we aim at solving.

4 The basic motivation

The basic motivation of the models we propose is the need to describe how an exchange of goods can happen without the intervention of any transferable utility such that represented by money or by any other numerary good. In this way all actors involved do not need to share anything[16] but the will to propose pool of goods that they present each other so to perform some barters.
All barters are in kind and are essentially based on a very simple basic scheme, in case we have only two actors in the simplest setting (see section 6.1): the two actors show each other the goods, each of them chooses one of the goods of the other and, if they both consent, they have a barter otherwise some rearrangement is needed and the process is repeated until either a barter occurs or both agree to give up.
The presence of more that two actors and the use of more complex schemes do not really greatly modify the above scheme since in any case the basic module is the one involving a pair of actors at a time. We note, indeed, how within this framework there is no numerary good so no auction like scheme is possible. Possible extensions will be examined briefly in section 10.
Lastly we underline the fact that our approach will be more **descriptive** than **normative** since we are more interested in giving a framework that allows the description of actors' possible behaviours in various abstract settings than in giving (more or less detailed) recipes through which players can attain their best outcomes.
Within this perspective it should be obvious why we do not explicitly describe detailed optimal strategies that the players can follow. Though it may seem strange we think that, given the purposes of the models, a normative approach would prove as too restrictive. Anyway some comments about possible strategies will be made when we introduce the single models.

5 Some definitions

With the term **barter** we mean an exchange of goods for other goods without any involvement of money or any other numerary good. It usually involves two players[17] that act as peers in a peer-to-peer relationship. There

[16] Such as preferences or utilities as shared information.

[17] If more than two players are present we do not admit auction like interactions since we do not admit any common numerary good so things can be very complicated because we must consider not only all the possible barters among all the possible pairs but also the fact that one actor can perform a barter only to let his goods get a higher value in

may be variants such as more than two actors or not peer-to-peer[18] relations and in section 10 we examine briefly only those of the former type.
As to the barter we note that we can have either a **one shot barter** or a repeated or **multi shot barter**.
In the former case the two actors execute the barter only once by using a potentially multi stage process that aims at a single exchange of goods and can involve a reduction of the sets of goods to be bartered.
In the latter case they repeatedly execute the preceding process, every time either with a new set of goods or with the same set partially renewed but usually excluding previously bartered goods.
In this paper we are going to examine only one shot barter between two actors so that there is no possibility of retaliation owing to repetitions of the barter.
We introduce the following simplifications:

1. the values of the goods the two actors want to barter cover two overlapping intervals[19] so that a one shot barter is always possible (at least theoretically);

2. such goods and the associated values are chosen privately by each actor without any information on the goods and associated values of the other actor[20];

3. such values are fixed and cannot be changed as a function of the request from the other actor;

4. such values must be truthfully revealed upon request from an independent third party after both requests have been made.

The last two assumptions have been made only to simplify the analysis and will be relaxed in future developments.

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detriment of the goods of another player.

[18]In the case of not peer-to-peer relations we think we are not in presence of a real barter mainly if one of the actors cannot refuse to accept the proposed barter.

[19]To avoid interpersonal comparisons and the use of a common scale we can proceed as follows: we let the two players show each other their goods and ask separately to each of them if he thinks the goods of the other are worth bartering. If both answer affirmatively we are sure that such interval exists otherwise we cannot be sure of its existence. Anyway the bartering process can go on, though with a lower possibility of successful termination.

[20]Obviously each actor can make guesses on the goods and the associated values of the other actor and such guesses can determine in some way the composition of each set of goods to be bartered.

6 Barter models

We suppose two actors[21]:

1. an actor A with a pool $I = \{i_1, \ldots, i_n\}$ of n heterogeneous goods,
2. an actor B with a pool $J = \{j_1, \ldots, j_m\}$ of m heterogeneous goods.

A assigns a private[22] vector v_A of n values to his goods in I and this vector is fixed and cannot be modified. Also B assigns a private vector v_B of m values to her goods in J and this vector is fixed and cannot be modified. From these hypotheses, for any subset K either of I or of J we can evaluate, once for all[23]:

$$v_X(K) = \sum_{k \in K} v_X(k) \tag{15}$$

with $X = A$ or $X = B$.
In a similar way[24] we can define two more private vectors:

1. s_A of m values of the appraisals of the goods of B from A,
2. s_B of n values of the appraisals of the goods of A from B,

so that it is possible to evaluate:

$$s_X(H) = \sum_{k \in K} s_X(k) \tag{16}$$

(again with $X = A$ or $X = B$) for any subset H of J or I respectively. The basic hypothesis is that A can see the goods of B but does not know v_B and the same holds for B with respect to A.
We have four types of barter:

1. **one-to-one** or one good for one good;
2. **one-to-many** or one good for a basket of goods;
3. **many-to-one** or a basket of goods for one good;

[21] We use for the former actor male syntactic forms and female for the latter.

[22] With the term **private** we denote information known only to one player and not to both.

[23] In this way we introduce a property of additivity. The same holds in all the other similar cases where we have an equality. If we had a $\geq$ sign we would be in a super additivity case whereas if we had $\leq$ we would be in a sub additivity case.

[24] We note that in this case a_A is specialized as $s_A(j)$ and l_A as $v_A(i)$. Similar considerations hold also for B and, mutatis mutandis, in all the models we are going to present in subsequent sections of the paper.

4. **many-to-many** or a basket of goods for a basket of goods.

The second and the third case are really two symmetric cases. We are going to examine such types one after the other, starting with the simplest or the one-to-one type.

6.1 One-to-one barter

Even in this simple type of barter there must be a pre-play agreement between the two actors that freely and independently agree that each other's goods are suitable for a one-to-one barter. We have two sub-types:

1. with simultaneous (or "blind") requests,

2. with sequential requests.

In the case of **simultaneous requests**, at the moment of having a barter we can imagine that the two actors privately write the identifier of the desired good on a piece of paper and reveal such information at a fixed time after both choices have been made. In this case we have that A requires $j \in J$ and B requires $i \in I$ so that:

1. A has a gain $s_A(j)$ but suffers a loss $v_A(i)$;

2. B has a gain $s_B(i)$ but suffers a loss $v_B(j)$.

The two actors can, therefore, evaluate the two changes of value of their goods[25]:

$$u_A(i, j) = s_A(j) - v_A(i) \tag{17}$$

$$u_B(i, j) = s_B(i) - v_B(j) \tag{18}$$

since all the information is available to both actors after the two requests have been made and revealed. Equations (17) and (18) are privately evaluated by each player that only declares **acceptance** or **refusal** of the barter, declaration that can be verified to be true by an independent third party upon request. We note that a possible strategy for both players is to maximize the value they get form the barter (and so $s_A(j)$ and $s_B(i)$). This however is not a guarantee for each player of maximizing his own utility since in equations

[25] With a little misuse of terminology we are going to denote such changes as **utilities**. In equation (17) and (18) we use differences and not ratios (see section 2) essentially because in this way we think to describe better the evaluation strategy of the players when they decide to accept or refuse a barter whereas, after the barter has occurred, they tend to use ratios to evaluate its fairness. Anyway it is easy to see how, for instance, from equation (17) and rule (19), it is possible to derive equation (1) and vice versa.

(17) and (18) we have a loss due to what the other player asks for himself (and so $v_A(i)$ and $v_B(j)$).
The basic rule for A is the following[26]:

$$\textbf{if}(u_A \geq 0)\,\textbf{then}\,accept_A\,\textbf{else}\,refuse_A \tag{19}$$

and a similar rule holds also for B.
We have therefore the following four cases:

1. $accept_A$ and $accept_B$,
2. $refuse_A$ and $accept_B$,
3. $accept_A$ and $refuse_B$,
4. $refuse_A$ and $refuse_B$.

The first case is really trivial. In this case the barter occurs since none of the two actors is worse off and at least one may be better off.
In the fourth case both A and B refuse so both may modify their set of goods by excluding some of the goods and precisely those who gave rise to the refusals. In this way we have:

1. $I = I \setminus \{i\}$
2. $J = J \setminus \{j\}$

and the barter process starts again on the two new reduced sets[27]. This occurs because in this case they both suffer a loss so both will be in a better condition if they exclude such goods from future rounds.
The second and the third case are symmetric so we analyse only the former of the two.
In this case A refuses whereas B accepts. There are two mutually exclusive possibilities[28]:

[26]In the general case we have $u_A \geq \varepsilon$ with $\varepsilon > 0$ if there is a guaranteed minimum gain or with $\varepsilon < 0$ if there is an acceptable minimum loss.

[27]This is surely true under the hypothesis that both players tend to choose the good of the other that each value the most. In all the other cases the plausibility of these reduction operations must be verified case by case.

[28]We can imagine other possibilities that we disregard so to keep the structure of the barter simple though flexible and expressive. For instance we can imagine A keeps fixed his choice and B changes her, possibly within a subset of goods that A suggests. Also in this simple extension we have to deal with many complications such as: the structure of this subset, the possibility for B to choose outside of it and how to manage a refusal from B to use that set as a basis for her next choice.

1. A takes i off his bartering set,

2. the request of B is kept fixed but A repeats his request, changes his choice and affects B's utility so that B can now either accept or refuse.

In the first case we have $I = I \setminus \{i\}$ and the process starts again with a new simultaneous request. In the second case:

1. if B accepts, the barter occurs since both are satisfied with the outcome,

2. if B refuses, then there is a reversing of the situation and a new phase with B playing the role formerly played by A.

All this can go on until:

1. a situation of common acceptance occurs (positive outcome),

2. there is no possibility of a common acceptance so that both actors agree to give up and no barter occurs.

In the case of **sequential requests** we can imagine that there is a chance move to choose who moves first and makes a public request. In this way both A and B have a probability of 0.5 to move first.
If A moves first (the other case is symmetric) and requires $j \in J$, B (since she knows her possible request $i \in I$) may evaluate her utility in advance as:

$$u_B(i,j) = s_B(i) - v_B(j) \tag{20}$$

whereas the same does not hold for A that, when he makes the request, does not know $v_A(i)$. At this level B can either explicitly refuse (if $u_b < 0$) or implicitly accept (if $u_b \geq 0$).
In the former case B can only take the good j off her set and the process restart with B moving first. Though the truthfulness of B's refusal may be checked by A upon request in this way both actors risk the exclusion of each one's best goods from the barter since the same attitude can be adopted also by A. There are however cases in which no better solution is available.
In the latter case the implicit acceptance is revealed by the fact that B makes a request. In this case he may be tempted to evaluate $max u_B(i,j)$ but, acting this way, may harm A by causing $u_A < 0$ and this would prevent the barter from occurring at this pass. Anyway B makes a request of $i \in I$ so that also A can evaluate:

$$u_A(i,j) = s_A(j) - v_A(i) \tag{21}$$

Now, using rules such as (19), we may have only the following cases[29]:

[29] In the symmetric case where B moves first at the very start we can have only:

1. $accept_A$ and $accept_B$,

2. $refuse_A$ and $accept_B$.

In the first case the barter occurs. In the second case A suffers a loss and has two possibilities:

1. can take i off his barter set and the barter goes on with B making another choice,

2. can make another choice with B keeping fixed her.

In this second case we have a new evaluation of both u_A and u_B so that at this step we can have:

1. $accept_A$ and $accept_B$,

2. $accept_A$ and $refuse_B$.

So that the process either ends with a barter or goes on with B acting as A at the previous step.
All this goes on until when both accepts so the barter occurs or one of them empties his set of goods or both decide to give up since no barter is possible.

6.2 Formalization of the models

In this section we present a point-to-point concise listing of the two models of the one-to-one barter, starting from the case of simultaneous or "blind" requests.
In this case the algorithm is based on the following steps:

1. both A and B show each other their goods;

2. both players negotiate if the barter is [still] possible or not[30];

    (a) if it is not possible (double refusal) then go to step 6;

    (b) if it is possible then continue;

3. both simultaneously perform their choice;

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1. $accept_A$ and $accept_B$,

2. $accept_A$ and $refuse_B$.

[30]At the very beginning of the process we suppose the barter is possible though this does not necessarily hold at successive interactions.

4. when the choices have been made and revealed both A and B can make an evaluation (using equations (17) and (18)) and say if each accepts or refuses (using rules such as (19));

5. we can have one of the following cases:

   (a) if ($accept_A$ and $accept_B$) then go to step 6;

   (b) if ($refuse_A$ and $accept_B$) then

      i. either A performs $I = I \setminus \{i\}$ and if ($I \neq \emptyset$) then go to step 2 else go to step 6;

      ii. or A only performs a new choice and then go to step 4;

   (c) if ($accept_A$ and $refuse_B$)

      i. either B performs $J = J \setminus \{j\}$ and if ($J \neq \emptyset$) then go to step 2 else go to step 6;

      ii. or B only performs a new choice and then go to step 4;

   (d) if ($refuse_A$ and $refuse_B$) then

      i. $I = I \setminus \{i\}$;

      ii. $J = J \setminus \{j\}$;

      iii. if ($I \neq \emptyset$ and $J \neq \emptyset$) then go to step 2 else go to step 6;

6. end of the barter.

We now give the same concise description of the model with sequential requests.
In this case the algorithm is based on the following steps[31]:

1. both players show each other their goods;

2. both players negotiate if the barter is [still] possible or not;

   (a) if it is not possible (double refusal) then go to step 10;

   (b) if it is possible then continue;

3. there is a chance move (such as the toss of a fair coin) to decide who moves first and makes a choice;

4. 1 reveals his choice $i_2 \in I_2$;

[31] In this case we denote the player who moves first as 1 (it can be either A or B) and the player who moves second as 2 (it can be either B or A). With a similar convention we denote as I_1 the set of goods and i_1 a single good of 1 whereas for 2 we have I_2 and i_2. We use male syntactic forms for both players

5. 2 can now perform an evaluation of all his possibilities;
6. if 2 refuses he takes i_2 off his barter set then go to 2;
7. if 2 accepts he can reveal his choice $i_1 \in I_1$;
8. both 1 and 2 can make an evaluation (using equations such as (17) and (18)) and say if each accepts or refuses (using rules such as (19));
9. we can have one of the following cases:
   (a) if ($accept_1$ and $accept_2$) then go to step 10;
   (b) if ($refuse_1$ and $accept_2$) then
      i. either 1 performs $I_1 = I_1 \setminus \{i_1\}$ and if ($I_1 \neq \emptyset$) then go to step 2 else go to step 10;
      ii. or 1 only performs and reveals a new choice and then go to step 8;
   (c) if ($accept_1$ and $refuse_2$) then
      i. either 2 performs $J_1 = J_1 \setminus \{j_1\}$ and if ($J_1 \neq \emptyset$) then go to step 2 else go to step 10;
      ii. or 2 only performs and reveals a new choice and then go to step 8;
10. end of the barter.

6.3 One-to-many and many-to-one barters

In these cases one of the two actors requires one good whereas the other requires a basket of goods (that can even contain a single good) and so any proper subset[32] of the goods offered by the former. This kind of barter must be agreed on by both actors and can occur only if one of the two actor thinks he is offering a large pool of "light" goods whereas the other thinks she is offering a small pool of "heavy" goods.
The meaning of the terms "light" and "heavy" may depend on the application and must be agreed on during a pre-barter phase by the actors themselves. The aim of this preliminary phase is to give one of the two actors the possibility of asking for any set of goods whereas this same possibility is denied to the other. If there is no agreement during this phase, three possibilities are left: they may decide either to give up (so the bather process neither starts)

[32] A proper subset of a generic set A is a set that is neither empty nor coincident with A itself.

or to switch to a one-to-one barter or to a many-to-many barter.
If there is a pre-barter agreement we can have the two symmetrical cases we mentioned in the section's title so we are going to examine only the former or "one-to-many" barter. In this case we have:

1. A owns "light" goods and requires a single good $j \in J$,
2. B owns "heavy" goods and requires a proper subset $\hat{I}_0 \subset I$ of goods with $|\hat{I}_0| < n$,

and the two requests may be either simultaneous or sequential.
If we have **simultaneous requests** both actors can evaluate their respective utilities, soon after the requests have been revealed, as[33]:

1. $u_A(\hat{I}_0, j) = s_A(j) - v_A(\hat{I}_0)$
2. $u_B(\hat{I}_0, j) = s_B(\hat{I}_0) - v_B(j)$

For possible strategies we refer to what we have noticed in section 6.1. Again, using rules such as (19), we have four possible cases:

1. $accept_A$ and $accept_B$,
2. $refuse_A$ and $accept_B$,
3. $accept_A$ and $refuse_B$,
4. $refuse_A$ and $refuse_B$.

In all these cases the barter goes on as in the $one-to-one$ case with simultaneous requests.
In the case of **sequential requests** the procedure does not use a chance move to assign one of the two actors the right to move first but gives this right to the actor that owns the pool of "light" goods. After this first move the barter goes on as in the $one-to-one$ case with sequential requests.

6.4 Many-to-many barter

In this case both actors require a proper subset of the goods offered by the other or:

1. A requires a generic $\hat{J}_0 \subset J$

[33] In this case a_A is specialized as $s_A(j)$ and l_A as $v_A(\hat{I}_0)$. Similar considerations hold also for a_B and l_B. We recall the additivity hypothesis so that, for instance, $v_A(\hat{I}_0) = \sum_{i \in \hat{I}_0} v_A(i)$.

2. B requires a generic $\hat{I}_0 \subset I$

and the two requests may be either simultaneous or sequential.
Also this kind of barter must be agreed on by both actors during a pre-barter phase.
Since also in this case we can have either simultaneous or sequential requests the algorithms are basically the same that in cases of one-to-one barter. The main differences are about:

1. the use of the subsets,

2. the way in which is managed the case of the double refusal.

As to the first issue we note that in the algorithms we must replace single elements with subsets of the pool of goods so that the evaluations must be performed on such subsets by using the additivity hypothesis.
As to the second issue, in the one-to-one barter (with simultaneous requests) the solution we adopted was a symmetric pruning of the two sets by the two actors but this solution cannot be applied in the present case since this policy would empty one of the two initial pools or both in a few steps. To get a solution in this case we can imagine an independent partitioning of the two sets of goods from both actors A and B.
The solution is implemented as:

1. if($refuse_A$ and $refuse_B$) then
   (a) $I = \mathit{partitioning}_A(I)$
   (b) $J = \mathit{partitioning}_B(J)$

Such “code” must replace the analogous piece of “code” we saw in section 6.1.
In this case A (the case of B is symmetric) uses procedure $\mathit{partitioning}_A(I)$:

1. the very first time when a double refusal occurs, to split I in labelled lots so to make clear to B which are the subsets of goods the he is disposed to barter;

2. on successive double refusals, to rearrange his lots as a reply to unfavourable (for him) partitioning from B of B's pool of goods.

In this case one possible strategy for the players involve subsets and not single goods. Except for this we again refer to what we have noticed in section 6.1.

7 The uses of the models or disclosing the metaphor

In this section we briefly list the basic assumptions that drove us to the formulation of the models we introduced in the previous sections and present some of their possible applications.
As to the first point we already noted how the basic idea is avoiding any use of common scales for the evaluation of the goods. In this way we have that both actors perform their evaluations one independently from the other and only accept or refuse a barter and their acceptances and refusals define the effective possibility of having the barter done.
As to the applications we can devise a "positive", a "negative" and a "mixed" framework[34].
Of course equations such as (17) and (18) must be adapted case by case, since they have been devised to deal with the barter of goods, whereas rules such as (19) remains almost unchanged and can be used to drive A's and B's behaviour.

1. In the "positive" framework we have that both A and B offer goods or positive externalities. In this case both A and B propose what they are almost sure the other will be willing to accept. We note here that what A thinks is a good for B may be a good or have no value or even be a bad for A himself and the same holds also for B.

2. In the "negative" framework we have that both A and B present bads or chores. In this case we have that A asks B to accept some bads or to carry out some chores in exchange for other bads or chores that B asks A to accept or to carry out. We note here that what A thinks is a bad/chore for B usually is a bad/chore for A himself and the same holds also for B.

3. In the "mixed" framework we have that goods and bads/chores can be mixed in any proportion. To make things simpler and tractable we imagine the following cases:

   (a) A offers a prevalence[35] goods but B offers a prevalence bads/chores,

   (b) both A and B offer a balanced mixture of goods and bads/chores.

[34] It is obvious how asymmetric cases (such as A offers only goods to B and B only bads/chores that A must take or execute) cannot give rise to any barter.

[35] We avoid any quantification since, from our descriptive perspective, we think that this task of quantifying is up to the players in a pre barter phase.

In these cases we have an exchange of items where each actors tend to maximize the goods and minimize the bads/chores he/she obtains.
In practice there can be two solutions:

(a) both A and B splits their pools in two subsets, each containing only goods or bads/chores and negotiate separately on them as in the "pure" frameworks;

(b) A and B agree on a many-to-many barter so to be able to obtain more or less balanced subsets of goods and bads/chores.

8 Fairness of the proposed solutions

We now try to verify if the solutions we have proposed in the previous sections satisfy the criteria (envy-freeness, equitability and efficiency) we stated in section 2 so that we can say whether they produce fair barters or not.
As we have seen in the sections so far, if a barter occurs this means that both players think each of them gets more than one looses (as it results[36], in case of A, from relations (17) and (19)) so the barter is **envy-free**. This holds in all the models we have seen so far.
From our definitions, this is equivalent at saying that it is also proportional.
We note that in every case where a set of goods is involved we can evaluate its worth by using the additivity hypothesis.
As to **equitability** (see relations (7) and (8)) and **efficiency** (see relations (9) and (10)) each must e verified for each barter since there is no a-priori guarantee that either of them holds for a particular case.
In conclusion, we can say that, in all the cases, fairness is a by-product of the barter process and is not a-priori guaranteed by its structure.

9 Hidden goods: alternating requests

9.1 Introduction

All the models we have seen so far are based on the following common structure:

1. both players show each other the goods they want to barter;

2. both agree on the type of barter they are going to have;

[36] Similar considerations hold also for player B.

3. both start the process that can end either with or without an exchange of goods.

In this section we very briefly present two more models.
In the first model we drop the hypothesis that the two actors show each other their goods before the barter process starts. We call it the "pure model" where none of the players shows anything to the other. So to compare this model with those we have seen so far we note how it is a one shot, one-to-one barter model with successive requests where two actors aim at bartering one good for one good[37].
In the second model (we call it the "mixed" model) we have a mixed situation where:

1. only one of the two players, say A, shows his goods;
2. the other, B, proposes a barter that A can either accept or refuse;
3. if A accepts we have an agreement and the process ends whereas if A refuses he can make a counterproposal[38] so it is again B's turn;
4. things go on until both reach an agreement and a barter occurs or they decide to give up and no barter occurs.

9.2 Pure model: nobody shows, hidden items

The situation we are interested in can be described in the following terms. One of the two players is interested in giving a good or a service (we may call it a "bad") to the other player so to get back a good or a bad (gods and bads collectively may be called items).
Such an exchange may be carried out with a barter where the players in turn propose a pair[39] (i, j) that can be either accepted or refused. Things go on until:

1. both agree on a proposal and the barter occurs,

[37]It is possible to devise barter processes where one or both players ask for one basket of goods but, in the present paper, we do not deal with such cases.
[38]With this term we denote a proposal made as a response to a proposal that is being judged as unsatisfactory. It is possible (and in many cases advisable so to avoid infinite bartering) to design the rules so that a proposal cannot be used more than once during the whole process.
[39]Such a pair may be read in two ways depending on who is the player who proposes it. If A is the proposer, A gives i to player B so to get j that is supposed to be in B's availability. If B is the proposer, B gives j to player A so to get i that is supposed to be in A's availability. The right meaning will be clarified form the context.

2. one of the two refuses without a counterproposal so that the barter closes with a failure.

During the process, the two players reveal each other the items they are willing to barter and this revelation process (Myerson (1991)) allows the definition of some sets that we denote as[40] I_i for player A and J_i for players B. We call such sets **revelation set**s. Such sets, indeed, reveal the barter sets of the two players and are common knowledge between them.
At the very start of the barter process we have the two sets $I_0 = \emptyset$ and $J_0 = \emptyset$ since none of the players has revealed anything. After each move of the active player[41] his proposal is added to his current set so that it can be used by the other player to frame his successive proposals.
In our case we think that A moves first and B follows[42]. We note that:

1. when A moves for the first time we have $I_0 = \emptyset$ and $J_0 = \emptyset$,
2. A proposes $\{(i_0, j_0)\}$,
3. then we have $I_1 = \{(i_0, j_0)\}$ and $J_1 = \emptyset$ so that B can use I_1 to frame his counterproposal.

To describe this kind of barter we can use a decision tree (see Figure 1). In such a tree inner nodes are represented with white dots whereas black dots denote leaves.
The labels near to each inner node denote both the player that has the right to move at that node and the composition of the revelation sets at that node. On the other hand, the labels on the outgoing arcs denote the acceptance (**a**) or the refusal (**r**) of the proposal made at the previous step or the proposed barters[43] of that player to the other player. In our case we supposed A moved first so that the root is labelled as A. From this it follows that the nodes where A has to move have even depth[44] whereas those where B has to move have odd depth.

[40]In I_i and J_i the term i identifies the step of the process and also the depth of the decision nodes in the decision tree, see Figure 1.

[41]With this term we denote the player who has to move at a given point in the barter process.

[42]The situation where B moves first is symmetrical and will not be examined here.

[43]We note that such barters are not known in advance so the tree cannot be seen as a representation of a game in extensive form since it is dynamically built up level by level so that no fine grained strategy is possible. We note moreover how the process has no predefined maximum duration. We only know that such duration is finite if the number of possible proposals is finite.

[44]In a tree the **depth** of a node in the number of arcs between that node and the root of the tree.

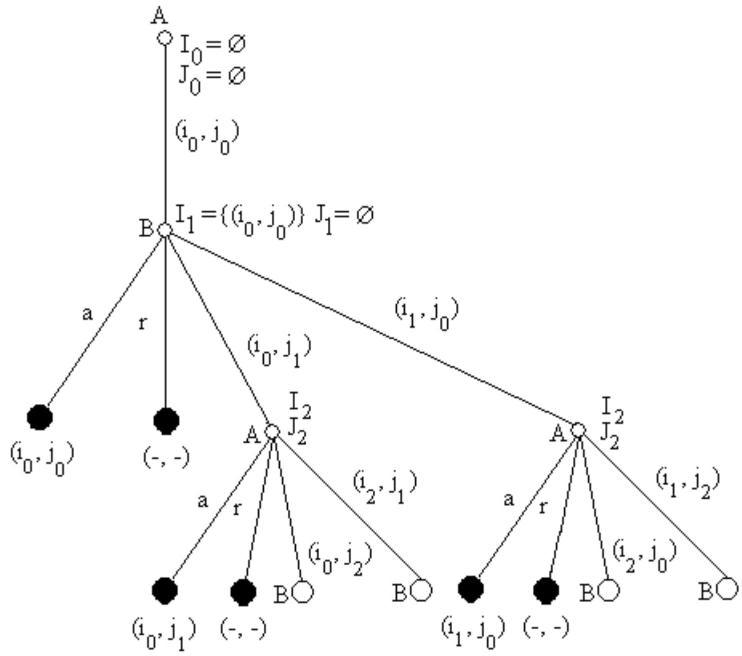

Figure 1: *A part of the barter tree*

After all these premises we can describe the portion of the barter portrayed in Figure 1.

1. A proposes a barter of (i_0, j_0);

2. in this way A reveals to B his set $I_1 = \{(i_0, j_0)\}$;

3. B has the following possibilities:

   (a) accepts,

   (b) refuses,

   (c) proposes a different item j_1 instead of j_0 so that she proposes (i_0, j_1),

   (d) asks for an item other than i_0 so that she proposes (i_1, j_0).

We have to specify what is i_1 since $J_1 = \emptyset$ and $i_1 \notin I_1$. The basic idea is that $i_1 = \alpha i_0$ with $\alpha > 1$ if the barter concerns a good and $0 < \alpha < 1$ if the barter concerns a bad, the effective value being fixed by B.
If B accepts, the barter of (i_0, j_0) occurs. This is seldom the case, however,

because it is in A's interest to ask for the most by giving the less. The acceptance reveals that $(i_0, j_0) \in J_2$ but this revelation has no further consequence since the process ends.
If B refuses the process ends and no barter occurs. Both players suffer a loss but there is no possibility either of compensation or of penalties. B's refusal, on the other hand, means that A had insufficient knowledge of B so that the barter was badly planned and no agreement was possible. It again reveals that $(i_0, j_0) \notin J_2$ but this revelation has no further consequence since the process ends.
Before stepping to the last two cases we must state on which basis players accept or refuse the proposals of barter or make a counterproposal. To do so they use the functions:

1. $eval_A(i, j)$
2. $eval_B(i, j)$

(where i and j denote the items to be bartered) that return a value ≥ 0 if a player thinks he is getting a gain from the barter and a value < 0 otherwise. Such functions can be used both in rules such as the following:

$$\textbf{if}(eval_A(i, j) \geq 0) \textbf{ then } accept_A \textbf{ else } refuse_A \tag{22}$$

and to establish a strict preference ordering $\succ$ on the proposals. We can indeed say[45]:

$$(i, j) \succ_A (i', j') \Leftrightarrow eval_A(i, j) > eval_A(i', j') \tag{23}$$

and the same holds also for B.
If B neither accepts nor refuses she can make one of the two counterproposals (i_0, j_1) and (i_1, j_0). We note that j_1 is known to B since it belongs to the hidden set of her items whereas i_1 is a B's guess, as we have already seen.
In the first case we have $(i_0, j_1) \succ_B (i_0, j_0)$, $J_2 = \{(i_0, j_1)\}$ and A has to move, in the second case we have $(i_1, j_0) \succ_B (i_0, j_0)$, $J_2 = \{(i_1, j_0)\}$ and again A has to move.
In both cases A has now six possibilities for the two nodes at level $l = 2$ (see Figure 1):

1. accept,
2. refuse,

[45] It is obvious that with $\succ_A$ and $\succ_B$ we denote the strict preference relation of player A and B respectively.

3. propose (i_0, j_2),
4. propose (i_2, j_1),
5. propose (i_2, j_0),
6. propose (i_1, j_2),

that can be analysed as those of B at the previous step. At each step the[46] $A-$side can be freely defined by player A and the $B-$side from player B whereas the other side of each proposal can be a guess based on the current revealed set of the other player.
We have therefore identified the following strategies (each of them defining a thread):

1. **A-conservative** where i_0 is kept whereas the B-side of the barter changes at each step,
2. **B-conservative** where j_0 is kept whereas the A-side of the barter changes at each step,
3. **mixed** where at each step both components of a proposal can change starting from $depth = 2$,

and such threads can, at least theoretically, last forever.
As a closing comment of this model, that deserves further and deeper investigations, we note how each (but not necessarily every) refusal move can be replaced with a completely new barter process where one player implicitly refuses and closes one barter but both players can open a new one by giving a new proposal to the other player (see Figure 2 where triangles represent subtrees). In this way the two players that cannot agree on a line of bartering can change line so to try to reach an agreement starting with a completely different barter proposal. This case cannot, however, be seen as a case of consecutive barters since, also in this case, at the most we can have one successful barter.

9.3 Mixed model: shown goods, hidden goods

In the mixed model we have a barter process where A shows his goods and B tries to get one or more of them by giving one of her goods to A.
This model can be seen as either a one-to-one or a one-to-many barter model with successive requests with the first turn to move for B (the player with

[46] Given a barter proposal (i, j) we say i the A-side and j the B-side of the barter.

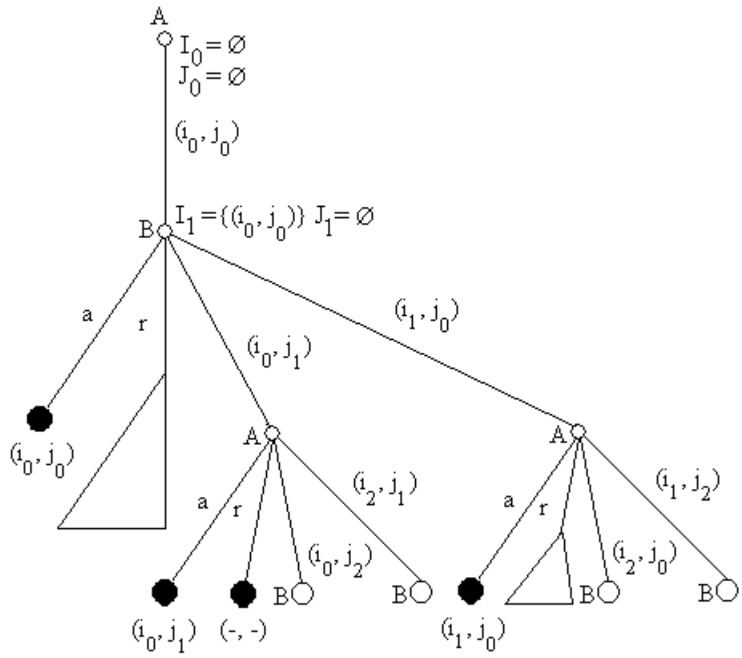

Figure 2: *Grafting new barters*

hidden goods). In the present section we present only the one-to-one version, further investigations in a forthcoming paper. The goods of A are common knowledge between the two players and we have:

1. A assigns to each of the n goods of his set $I = \{i_1, \ldots, i_n\}$ a value $v_A(i)$;
2. B assigns to each of the n goods of this set I a value $s_B(i)$;
3. B knows the value of all her (hidden to A) goods $j \in J$, $v_B(j)$;
4. A can evaluate (as $s_A(j)$) the single goods of B only after she has made one of her proposals.

At the very start of the algorithm we have that:

1. A knows his set of goods, I;
2. A has no idea of the set of goods of B, $J = \emptyset$;
3. B knows her set of goods, hidden to A, J;
4. B knows the set of goods of A, I.

We note that A has no possibility of revelation since his goods are common knowledge whereas B undergoes a process of revelation since every time she makes a proposal may reveal to A something about her goods. From this we have that I is fixed whereas as to B we start with the set J_1 that is enriched during the barter process.
The main steps of the algorithm are the followings:

1. A shows his goods I;

2. B propose a barter (i_0, j_0) with $i_0 \in I$ and with $j_0 \in J_1$ where J_1 is the currently revealed set of B;

3. A has the following possibilities:

   (a) accept so that the barter occurs,

   (b) refuse,

   (c) propose a barter (i_1, j_0),

   (d) if $J_0 \setminus \{j_0\} \neq \emptyset$ propose (i_0, j_1) with $j_1 \in J_0$.

4. B can either accept one of A's proposals, refuse or make a counterproposal using one of the not yet proposed goods of A or revealing one more of her hidden goods.

In this way (but for some details) the evolution of the model is very similar to that of the pure model. Acceptance and refusal are decided by both actors independently and using relations we have seen in section 9.2. The main difference between the two models is in the use from A of the set of B's revealed proposals to A. We note indeed that, through the bartering with B, A can create an history of proposals through which he can reply to a proposal of B that is judged unacceptable. In this way B, making her proposals, allows A to build up the set J_0 so to carry out the barter as in the case where both show each other their goods but for the fact that A is "many steps back" since can update the set J_0 only after B has made his proposal and adding one good at a time.
Again a refusal may represent for both players an opportunity to start a new barter process with a new proposal that can be built using past proposals of both players.

10 Extensions

The basic extensions of the proposed models involve essentially:

1. the possibility of repeated barters between two actors;
2. the possibility that more than two actors are involved in the barter;
3. both possibilities;
4. the relaxing of the additivity hypothesis.

If we allow the execution of repeated barters we must introduce and manage the possibility of retaliations between the players from one barter session to the following and how are defined and/or modified the pool of goods between consecutive bartering sessions.
If, on the other hand, we allow the presence of more than two actors we must introduce mechanisms for the execution of parallel negotiations.
If, for instance, we have three actors A, B and C we can have (in the case of one-to-one barter with simultaneous requests).

1. Circular one-to-one requests where, for instance, A makes a request to B, B to C and C to A.
2. One-to-many requests so that A makes a request to B and C, B makes a request to A and C and C makes a request to B and A.

In the former case there can be no conflict whereas in the latter it can occur that two actors ask the same good to the third causing a conflict that must be resolved some way.
In both cases we have:

1. the barter occurs if and only if all the actors accept what is proposed by the others;
2. if all actors refuse the others' proposals a rearrangement of the respective pools occurs followed by a repetition of the barter;
3. in all the other cases the procedure must allow the refusing actors (two at the most) to repeat their request.

Obviously in all the other cases the interactions tend to be more and more complex. Analysis of such extensions can be carried out using the tools suggested in Myerson (1991), section 9.5 where *graphical cooperation structures* are introduced and used.
As a last extension we mention the relaxing of additivity. Additivity is undoubtedly a simplifying assumption and is based on the hypothesis of relative independence of the goods that the actors want to barter among themselves.

This hypothesis in many cases is not justified since functional links, for instance, make goods acquire a value when and only when are properly combined. In such cases such goods must be bartered as a whole and cannot enter properly in a one-to-one barter. The issue is very complex (so complex that Brams and Taylor (1996) and Brams and Taylor (1999) deal with it only marginally) and here we only note how relaxing additivity can bring us to the adoption of either superadditivity or subadditivity.
As to player A (the situation with B is fully symmetrical), under additivity (see equations (15) and (16)) we saw that what A loses is:

$$v_A(H) = \sum_{h \in H} v_A(i_h) \quad \forall H \subseteq I \tag{24}$$

and what A gets is:

$$s_A(K) = \sum_{k \in K} s_A(j_k) \quad \forall K \subseteq J \tag{25}$$

We can relax additivity on either only one of equations (24) and (25) or on both. In this section we concentrate only on equation (25) and so the attitude of A towards the barter. In this case we think subadditivity is really not interesting[47] since A would be better off by asking one single good from B and so by entering in either a one-to-one or a one-to-many barter. On the other hand superadditivity makes A better off if he can pick subsets of the goods of B and so can be involved in a many-to-one or a many-to-many barter.
If we allow this superadditivity this means that:

$$s_A(K) \geq \sum_{k \in K} s_A(j_k) \quad \forall K \subseteq J \tag{26}$$

A is of course more interested in subsets K of J such that:

$$s_A(K) > \sum_{k \in K} s_A(j_k) \tag{27}$$

We call such subsets **superadditive subsets** of J and in this way we formalize both the fact that for A not every subset of the goods of B has a value greater than the sum of the values of its elements and the functional links among the goods of B.
Under this premise, the many-to-many barter case is really interesting so we make some more comments on it. It requires that also for B there is the same

[47] We note how subadditivity could be interesting for A if occurred on equation (24).

sort of superadditivity on the corresponding equation so that both players can see the goods of the other as composed of superadditive subsets that have nothing to do with a partitioning.
If we have:

1. $I = \{i_1, i_2, i_3, i_4, i_5\}$

2. $J = \{j_1, j_2, j_3, j_4, j_5, j_6\}$

then a many-to-many barter of this kind may involve not all the possible subsets of I and J but only some of them so that:

1. B can see I as made of the following set of superadditive subsets $\{\{i_1, i_2\}, \{i_1, i_3, i_4\}, \{i_4, i_5\}, \{i_2, i_3, i_5\}\}$;

2. A can see J as made of the following set of superadditive subsets $\{\{j_1, j_2, j_3\}, \{j_1, j_3, j_4\}, \{j_4, j_5\}, \{j_3, j_5, j_6\}\}, \{j_2, j_3, j_5, j_6\}\}$.

In this case the many to many barter is reduced to a one-to-one barter where the goods to be bartered are the superadditive subsets of goods and not the single goods.

11 Concluding remarks and future plans

In this paper we have introduced some barter models between two actors that executes a one shot barter through which they exchange, according to various mechanisms, the goods of two separate and privately owned pools.
This is an introductory paper so a lot of formalization is still to be done for what concerns both the basic versions of the model and its extensions.
More precisely we need:

1. to examine more formally the basic model of one shot barter with all its variants;

2. to improve the algorithms of the various proposed solutions;

3. to examine the properties of such solutions and their plausibility;

4. to develop more thoroughly the model we introduced in section 9,

5. to analyse and formalize the extensions we listed in section 10.

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