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A few notes on the Borda and Condorcet methods

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Abstract

The present Technical Report (TR) contains a short analysis of both the Borda and the Condorcet methods. We start with a description of both methods in their twofold roles:

- of methods for defining a winning alternative from a set of available alternatives,
- of possible outranking methods over a set of alternatives.

We then discuss the properties of each method and analyze them with regard to the conditions posed by the Arrow's impossibility theorem.
The last step, that represents the core part of the TR, is represented by the search of possible relations between the two methods.
The TR closes with the proposal of a composed outranking method based on both the Borda and the Condorcet methods executed on the same profiles of preferences.
As usual, reports of errors and inaccuracies are gratefully appreciated.

Contents

1 Introduction

In this Technical report (TR) we present an analysis of two classical methods such as the Borda method and the Condorcet method.
We start with a description of both methods in their twofold roles:

- as methods for defining a winning alternative from a set of available alternatives,
- as possible outranking methods over a set of alternatives ([2], [3])

We then discuss the properties of each method and analyze them with regard to the conditions posed by the Arrow's impossibility theorem.
The last step, that represents the core part of the TR, is represented by the search of possible relations (to be defined) between the two methods.
The TR closes with the proposal of a composed outranking method based on both the Borda and the Condorcet methods executed on the same profiles of preferences.

2 The search of the relations

The search of possible relations between the mentioned methods occurs through a comparison that is made using the following parameters:

- the number n of the deciders[1] of the set N,
- the number m of the alternatives of the set A.

The two sets N and A are assumed to be independent one from the other so that there is no relation between their cardinalities. At this level we note only that if $m! < n$ (or the number of the possible transitive orderings[2] of the alternatives is lower than the number of the deciders) at least two deciders must have the same transitive ordering of the alternatives.
The set A contains the elements on which the deciders are going to act at two levels:

- with individual decisions,
- with a collective decision.

[1] We use the term decider as a synonym of the term decision maker.

[2] An ordering on a set A is termed transitive if for any triple of elements $a, b, c \in A$ we can say that:

- if a precedes b in the ordering
- and b precedes c in the ordering
- then a precedes c in the ordering.

More on transitivity in section 3.

The Borda and the Condorcet methods, as it will be evident from their descriptions, are based on very different principles and, as it will be shown in this TR, suffer from different drawbacks. Notwithstanding such differences, our aim in writing the present TR is to find possible relations between them in order to see if it is possible to derive certain consequences in one method if certain conditions are verified in the other and vice versa.
The Borda method is indeed based essentially on the assignment of numerical values and on the execution of sums on such values. On the other hand the Condorcet method is based on the execution of pairwise comparisons.
Such pairwise comparisons may be performed:

- by assuming a strict preference relation $\succ$

  or

- by assuming a weak preference relation $\succeq$

between pairs of alternatives ([1]).
In the former case we suppose that the deciders, whenever they compare the alternatives in pairs, consider one of the alternatives strictly better than the other so that no tie is possible.
In the latter case, on the other hand, we suppose that the deciders, whenever they compare the alternatives in pairs, may consider some of them as indifferent since, given a pair of alternatives, the former is weakly better than the latter and vice versa and so ties are possible.
More on relations in section 3.
At this level we note that in the former case a possible loss of transitivity may be seen as a sign of an irrational behavior of the deciders ([5], [6], [1]) whereas in the latter case such a loss can be solved through the definition of an indifference relation among the involved alternatives so to mean that they are indistinguishable from the point of view of the deciders.
The search of relations between the two methods aims at establishing possible relations between the following conditions:

(1) a condition of parity or of tie in the Borda method,

(2) a condition of parity or of tie in the Condorcet method,

(3) a condition of loss of transitivity in the Condorcet method with the presence of cycles involving subsets of the alternatives.

The condition (1) is assumed to be valid on the whole set A and expresses the fact that the alternatives of the set have received the same number of points from the execution of the method (to be defined shortly) and so can be seen as equivalent. We call such a condition a **full tie**.
This condition may hold also on subsets of A and in this TR we comment a little on this fact though we consider it less important than a parity on the whole set. The presence of subsets of equally ranked alternatives allows us to define a certain number of equivalence classes with the associated equivalence relation.

We call such a condition a **partial tie**.
The condition (2) expresses the fact that the number of the voters that rank two alternatives in a certain way coincides with the number of those who make the opposite ranking. In this way we express a condition of indifference between such alternatives.
The condition (3) is associated with the presence of cycles that involve either all the members of the set A or some of its proper subsets.
As we have already argued, if we use a weak preference relation, we can imagine to solve such problems by thinking the involved alternatives as equivalent. On the other hand, in the case we use a strict preference relation, both the presence of ties and the loss of transitivity are a problem that is generally said to derive from an irrational behavior of the deciders.
To see what all this means in practice we can consider the following set $A = \{a, b, c\}$.
If, in the Borda method, v_x denotes the number of votes received (from all the deciders) by alternative x the condition of full tie can be expressed as $v_a = v_b = v_c$ so that the three alternatives can be seen as equivalent. A partial tie, for instance, can be expressed as $v_a = v_b > v_c$ so that alternatives a and b can be seen as equivalent and both preferred to alternative c.
In the case of the Condorcet method, the condition (2) can be understood if we define, for instance, with v_{xy} the number of the deciders that weakly prefer alternative x to alternative y (more on preferences in the following sections) so that we have a tie between the alternatives x and y if $v_{xy} = v_{yx}$. A condition of parity in the Condorcet method involves all the possible pairs of alternatives but can be even partial if it involves a subset of the alternatives. In the case of A we have that if $v_{ab} = v_{ba}$, $v_{ac} = v_{ca}$ and $v_{cb} = v_{bc}$ the three alternatives can be seen as equivalent. A partial condition can be easily derived.
If, on the other hand, we use the strict preference relation[3] $\succ$ we can have also in this case $v_{ab} = v_{ba}$, $v_{ac} = v_{ca}$ and $v_{cb} = v_{bc}$ but the three alternatives cannot be seen as equivalent. The condition (3) will be fully examined only after the introduction of the preference relations.
For the moment we only note how, in the case of the set $A = \{a, b, c, d\}$, we can have $v_{ab} \geq v_{ba}$ so that a is weakly preferred to b, $v_{bc} \geq v_{cb}$ so that b is weakly preferred to c and $v_{ca} \geq v_{ac}$ so that c is weakly preferred to a. Together with these preferences we may also have $v_{ad} \geq v_{da}$ so that a is weakly preferred to d and similar cases can occur for b and c with regard to d so we get a cycle among a, b, c but all of them are weakly preferred to d.
It can be shown ([5]) that by choosing in a certain way the profiles of preferences of the various deciders we can always produce cyclical global (i. e. involving all the alternatives) rankings (see further on for details). This possibility cannot be ruled out since, according to the property of **universal** or **unrestricted domain** ([5], [6], [2]), all the possible profiles must be seen as legal profiles of the deciders. According to this condition if we have m alternatives all the $m!$

[3]To appreciate the meaning of $\succ$ we note that if we write $a \succ b$ we mean that a is strictly preferred to b.

transitive orderings of such alternatives must be allowed as well as the intransitive orderings though we tend to rule them out in order to avoid a "garbage in, garbage out" effect.

3 The preference relations

The starting point is, therefore, a set of alternatives $A = \{a_1, \ldots, a_m\}$ (with $m = |A|$) as the set of the elements that each decider $i \in N$ can order according to his individual preferences.
On the set A each decider can apply a binary relation $R \subseteq A \times A$ that allows him to define which is his preferred alternative of every possible pair of distinct alternatives[4]. Such binary relation is termed also a **preference relation** since it allows a decider to establish a preference order between any pair of alternatives (see also section 4).
Over every R that we think suitable for this purpose we pose the constraint that it is **complete** or **total**[5] so that for any pair of alternatives $a, b \in A$ at least one of the following conditions holds for sure[6]:

(1) aRb

or

(2) bRa

We note that both such conditions may hold at the same time depending on the nature of the relation R. This is true if R is $\succeq$ but cannot be true if R is $\succ$.

[4]We note that a binary relation can compare an alternative with itself. Though this comparison may allow the definition of some properties of the binary relation itself (such as reflexivity, irreflexivity and non reflexivity, [4]) it has no particular relevance and will be considered only marginally in what follows. We note how the property of reflexivity allows us to define equivalence relations with their equivalence classes. Also in this case we consider only marginally such concepts since an equivalence relation does not define any ordering on a set of alternatives but, on the contrary, allows to identify some alternative as equivalent among themselves.

[5]In [4] we can find the following definitions:

- of a **complete** relation,
- of a **strongly complete** relation.

A binary relation on the set A is **complete** if $\forall a, b \in A$ with $a \neq b$ we have aRb or bRa.
A binary relation on the set A is **strongly complete** if $\forall a, b \in A$ we have aRb or bRa. In this case we admit also $a = b$.
To appreciate the difference we note that on the set $\mathbb{R}$ of the real numbers we have that the relation $>$ is complete but not strongly complete whereas the relation $\geq$ is both strongly complete and complete.
We note that strong completeness implies completeness but the converse is not true. In this TR we use essentially only the former definition since we are not really interested in comparing an alternative with itself.

[6]We note that the set A if fixed in advance and then we define a relation R on it and discover if it is complete or not. If R is complete, for any pair of distinct alternatives a and b, we have that either aRb or bRa hold for sure. In this way if we discover that aRb if false then it must be bRa for completeness to hold so that if also bRa fails then R cannot be complete.

If we had a pair of alternatives $a, b \in A$ such that neither of the two condition hold we would term them as incomparable[7].
In this case the relation R would be termed as incomplete or non total meaning that the set A contains incomparable elements. Such possibility, however, is ruled out by the very definition of R.
Other properties that we may wish R to satisfy include the following[8] ([4]):

- transitivity,
- negative transitivity,
- antisymmetry,
- asymmetry,
- reflexivity.

Such properties on a binary relation ([4]) allow us to define (with a minimal set of requirements for each ordering) the following orderings[9] on the whole set A:

- a **pre order** if the relation R is reflexive and transitive;
- an **order** if it is an antisymmetric pre order;
- a **strict simple order** if the relation R is asymmetric and transitive;
- a **strict weak order** if it is asymmetric and negatively transitive.

The property of **transitivity** involves triples of alternatives $a, b, c \in A$ in the following relation[10]:

$$aRb \,\&\, bRc \;\Rightarrow aRc \tag{1}$$

If from the set A we derive that relation (1) holds for any triple of alternatives we may say that R is a transitive relation (descriptive approach) whereas if we suppose that R is a transitive relation from the validity of the antecedent of the relation (1) we can derive the validity of its consequent (normative approach).

[7] Depending on A and R we can have a complete binary relation or not. If we choose the relation $|$ (or "divide") we have that the relation is incomplete since we have $a|b$ if and only if we have $b = ka$ where $a, b, k \in A = \mathbb{N}$. In this case we have that $(2, 3) \notin |$ and $(3, 2) \notin |$ so they are incomparable through $|$. The same holds also for the set inclusion $\subseteq$ relation since, in general, two sets cannot be compared through such relation if one is not a subset of the other. If we have a set A and two subsets a and b we have that if $\subseteq$ were complete then we should have either $a \subseteq b$ or $b \subseteq a$ but since both may fail, and this can happen in many cases, then $\subseteq$ cannot be complete.

[8] We introduce only a small set of properties that we consider as essential within the current framework.

[9] We recall the requirement that the orderings we define are at least total or complete.

[10] In what follows with the symbol & we denote a logical binary **and** operator, with the symbol + we denote a logical binary **or** operator whereas with the symbol $\neg$ we denote a logical unary operator **not**. We recall that the relation $a \Rightarrow b$ is true in all the cases but where a is true and b is false when it is false.

Also the property of **negative transitivity** involves triples of alternatives $a, b, c \in A$ in the following relation:

$$\neg(aRb) \,\&\, \neg(bRc) \;\Rightarrow \neg(aRc) \tag{2}$$

that can be expressed also, in an equivalent way, as[11]:

$$aRc \;\Rightarrow aRb \,+\, bRc \tag{3}$$

If as R we use $\succ$ and consider completeness we have (see further on) that $\neg(a \succ b)$ can be expressed as $b \succeq a$ so that relation (2) can be expressed as:

$$b \succeq a \,\&\, c \succeq b \;\Rightarrow c \succeq a \tag{4}$$

In this way we can introduce a relation between negative transitivity and transitivity through the complementary relations $\succ$ and $\succeq$.
The following two properties, on the other hand, involve a generic pair of alternatives $a, b \in A$.
The property of **antisymmetry** can be expressed as follows:

$$aRb \,\&\, bRa \;\Rightarrow a = b \tag{5}$$

The property of **asymmetry** can be expressed as follows:

$$aRb \Rightarrow \neg(bRa) \tag{6}$$

We note that ([4]) if a binary relation is asymmetric it is also antisymmetric[12] whereas the converse is not true. As an example we have that on the set $\mathbb{R}$ the binary relation $\geq$ is antisymmetric but not asymmetric.
The property of **reflexivity**, on the other hand, involves a single element and can be expressed as follows:

$$\forall a \in A \; aRa \tag{7}$$

In what follows we consider reflexivity only as a property that distinguishes a weak preference relation (for which such property holds) from a strict preference relation (for which such property does not hold) since we usually have no interest in comparing an alternative with itself.
In what follows we consider the following binary relations as implementations of the relation R:

(r_1) a weak preference relation $\succeq$,

(r_2) a strict preference relation $\succ$,

(r_3) an indifference relation $\sim$.

[11]By using the fact that $a \Rightarrow b$ is equivalent to $\neg b \Rightarrow \neg a$ and that $\neg(\neg a \& \neg b) = a + b$.

[12]If a binary relation is asymmetric then the antecedent of the definition of antisymmetry is alway false so that the condition of antisymmetry is always true independently from the fact that we have $a = b$ or $a \neq b$.

We note that with the relation $\sim$ (that can be proved to be **reflexive**, **transitive** and **symmetric**[13]) it is possible to define an equivalence relation with the associated equivalence classes.
According to their basic definitions such relations are assumed to be transitive. As to the other properties we have ([4]):

$\succ$ is asymmetric and so defines a strict simple order;

$\succeq$ is reflexive and antisymmetric and so defines an order;

$\sim$ is reflexive, transitive and symmetric and so defines an equivalence relation but not an order.

The next step is to recognize that the three relations ($\succ$, $\succeq$ and $\sim$) are not independent one from the others.
If we indeed take the relation $\succeq$ as primitive we can:

- use it to define relation $\succ$,
- use it to define relation $\sim$.

We can, indeed, state the following definitions:

$$a \succ b \Leftrightarrow a \succeq b \,\&\, \neg(b \succeq a) \tag{8}$$

$$a \sim b \Leftrightarrow a \succeq b \,\&\, b \succeq a \tag{9}$$

If we consider the relation $\succ$ as primitive we can state the following definitions[14]:

$$a \succeq b \Leftrightarrow \neg(b \succ a) \tag{10}$$

$$a \sim b \Leftrightarrow \neg(a \succ b) \,\&\, \neg(b \succ a) \tag{11}$$

From relations (9) and (10) we derive:

$$a \sim b \Leftrightarrow a \succeq b \,\&\, b \succeq a \Leftrightarrow \neg(b \succ a) \& \neg(a \succ b) \tag{12}$$

and so relation (11).
It is easy to see, lastly, how it is impossible to define the other two relations by taking the relation $\sim$ as primitive[15]. This fact confirms that $\sim$ cannot be used to define an ordering on the set A.
In what follows we are going to use both relations $\succ$ and $\succeq$ depending on the context but with a slight preference for $\succeq$. As a closing comment we underline that:

[13]A binary relation R is said to be **symmetric** if for any $a, b \in A$ we have $aRb \Rightarrow bRa$. We note that from symmetry and strong completeness we derive reflexivity.

[14]For what concerns definition (10) we can justify it as follows. We start from $a \succeq b$ iff $(a \succ b) + (a \sim b)$ then we replace $(a \sim b)$ with relation (11). We note that we use *iff* as a shorthand for "if and only if". At this point we perform a transformation and note that $(a \succ b) + \neg(a \succ b)$ is a tautology so we can discard it. In this way with a chain of equivalences we arrive at $(a \succ b) \& \neg(b \succ a)$ whose truth value coincides with that of $\neg(b \succ a)$ so that the definition is justified.

[15]If we have $a \sim b$ and we evaluate $\neg(a \sim b)$ we have no information to define any further relation between a and b

- $a \succ b \Rightarrow a \succeq b$ but the converse is not true since we can have $a \sim b$,
- the Borda method can be described with both relations $\succ$ and $\succeq$ though the former relation reflects better its real nature,
- the Condorcet method can be described both with $\succ$ and $\succeq$ though the former gives rise to a loss of transitivity that can be resolved by introducing equivalences if we use the latter.

4 The ranking methods

Given the set of m alternatives A and the n deciders we have that each decider can produce an ordering of the alternatives that we call local ranking or local ordering[16] r_i, $i = 1, \ldots, n$.
By the very definition of the Borda method (see section 5) its rankings never suffer a loss of transitivity nor they contain ties. It can be said that such a method imposes to the decider a rational behavior ([5], [6]).
On the other hand such condition cannot be enforced in the case of the Condorcet method where a decider can have any ordering based on pairwise comparisons of alternatives. In order to avoid a "garbage in, garbage out" effect, however, we impose also on the single Condorcet rankings such condition ([5], [6]).
With this hypothesis we impose that the each of the deciders we deal with has a total ordering of the alternatives without ties since things cannot be different (if the each decider uses the Borda method) and things can only worsen (owing to the known "garbage in, garbage out" effect) if such constraint is relaxed and we allow each decider to have an arbitrary ordering (in the case of the Condorcet method).
Such local rankings r_i are then to be merged in a single global ranking r_X (with $X = B$ for the Borda method and $X = C$ for the Condorcet method) that the deciders can use to produce a global ordering of the alternatives.
From what we have said so far it should be clear that all the local rankings we take as the input of a merging method are total and without ties so there is no alternative that cannot be compared with some of the others.
With this we mean that any binary relation R on the set A allows us to state either aRb or bRa or both for any pair of alternatives $a, b \in A$ and that relation is a good relation that act on the individual preference profiles.
Any method that can be used to perform a global merging of the local rankings can be seen as an outranking method though the global ranking r_X may fail to satisfy the property of transitivity and so may contain one or more cycles among the alternatives and may also contain full or partial ties among the alternatives.
With this we mean that even if we start from good data (the local rankings) we have no guarantee that the global ranking can be good as well.
If this occurs and so if the outcome contains some garbage that it produced

[16] In many cases we can consider the two terms, ranking and ordering, as synonyms.

autonomously (since it does not belong to the input data) we have to devise a solution of some kind if we want to be able to produce a method that allows the selection of one of the available alternatives.
Once again we remark that in this TR we present both the Borda method and the Condorcet method and see if and how they can be used as outranking methods.

5 The Borda method

The main idea on which the Borda method is based is that of assigning a distinct numeric[17] value to each alternative. This value is related to the positioning of an alternative in a local ranking r_i. Such values are summed for each alternative over all the available local rankings so that each alternative receives a single global value. All these global values are then ordered so to produce a global ranking of the alternatives.
The method is really simple and allows us to define a global ranking of the alternatives with possible ties among them. It is possible to show how by using the Borda method we can have all the alternatives getting the same global value so that they can be seen as tied or equivalent or as belonging to the same equivalence class. In this case the alternatives are related through the $\sim$ binary relation (see section 3).
More formally we define the method as follows.
We suppose to have m alternatives so that each decider has a range of integer values from m (his most preferred alternative) to 1 (his less preferred alternative). The procedure is based on the following steps:

(1) each decider assigns a distinct numeric value to every alternative so to define a local profile of preferences without ties among the alternatives;

(2) the n profiles are merged in a global preference profile or ranking.

The merging is obtained simply by summing the various numbers assigned to each alternative so that at the end we have that every alternative is associated to a numeric quantifier.
We note that though the single profiles of preferences are strict and so without ties there is no guarantee (see further on) that this property holds also for the global preference profile.
The method has possible variants[18] but the essence is the same: the various

[17] In the case of the Borda method the values assigned to the alternatives are always numeric, as integer values from a well defined range, and distinct. Such features will not be always specified but must be implicitly assumed whenever we deal with this method.

[18] Among the possible variants we cite the following two variants:

- we can assign the values in the opposite way and so from 1 (the most preferred alternative) to m (the less preferred alternative) so that the lower the better;
- we can have the range of the values as running from 0 to $m-1$ so that, in the descending order case, every value counts the number of the alternatives less preferred to the current one.

numbers assigned to each alternative are summed up so that the alternatives can be compared among themselves within a global ranking.
The method is easy to understand and implement but has some drawbacks that we are going to examine in section 8.
In order to understand those drawbacks we note how the Borda method mixes qualitative and quantitative features.
If we have m alternatives every value $i \in [1, m]$ can be seen indeed as defining:

- a position within a profile (qualitative feature);
- a weight for each alternative (quantitative feature) to be summed with the other weights for the same alternative in order to obtain a global weight for that alternative.

According to the former feature the relative position of two alternatives does not vary qualitatively as a function of the number of the in between alternatives. From this we derive that if a occupies a given position i and b is in a position in the interval $[i-1, 1]$ then we have $a \succ b$. With this we therefore mean that their ordinal ordering does not vary as a function of the number of the in between alternatives.
On the other hand, according to the latter feature, the relative importance of two alternatives depends on such number so that if $a \in A$ has a weight equal to k another less preferred alternative $b \in A$ has a weight equal to $k - 1 - h$ if between a and b we have h alternatives.
This twofold feature has the following side effects:

(1) if an alternative is removed from the set A after the global ranking r_B has been determined the removal may modify the ranking of the remaining alternatives;

(2) if some deciders modify the ordering of a pair of alternatives in some rankings r_i this may affect the ranking of the other alternatives in the global ranking r_B though the relative positions of such alternatives in every local ranking r_i are unchanged.

If one alternative is removed from the set A after the global ranking has been evaluated all the alternatives that follow it are upgraded so that their global ranking may be modified.
To show such side effects in practice we are going to present in section 7 two toy examples. Before doing so we note that:

- the first side effect depends on the fact that the removal leaves untouched the qualitative feature but modifies the quantitative feature,
- the second side effect is usually seen as a violation of the principle of independence of irrelevant alternatives ([5], [6], [2]).

In order to deal with the first side effect we could keep the rankings r_i fixed but for the empty holes of the removed alternatives. In this way both the qualitative

feature and the quantitative feature would be unaffected by the removals.
In this way it is as if we use m holes numbered from 1 to m and each hole can contain 0 or 1 alternative. In order to get the global ranking of an alternative we sum the numbers corresponding to the holes occupied by that alternative. Empty holes correspond to removed alternatives.
The second side effect is an expression of strategic voting and depends on the definition of the concept of irrelevant alternative in the Borda method possibly to be compared with the same definition in the Condorcet method. Some more comments will be made in section 8. At this level we only note what follows.
If we consider the alternatives $a, b \in A$ we may have that[19]:

- some of the deciders rank $a \succeq b$,
- the rest of the n deciders rank $b \succeq a$

so that, for instance, in the global ranking we may get $b \succeq a$.
If there is some knowledge of this fact before the global ranking is calculated the deciders of the first group may worsen the ranking of b (by assigning it less points and so by putting some alternatives between a and b or by exchanging the rankings of b with respect to some other distinct alternative) so that in the global ranking we now have $a \succeq b$. This rank reversal occurs without any change in the relative ordering of the alternatives a and b but simply worsening the ranking of b relatively to a.
This fact is, in our opinion, a direct consequence of the twofold qualitative/quantitative nature of the Borda rankings.
To understand this fact we note that if we have $a \succeq b$ in our version of the Borda method a gets more points than b but how many more points it gets depends on how many alternatives are comprised between a and b. From this perspective such in between alternatives cannot be seen as irrelevant. If a gets k points b may get $k-1-h$ points where h is the number of the in between alternatives so that if h is high enough it can determine a rank reversal between a and b even if the relative ordering between a and b is unchanged for all the deciders.
It is therefore evident how such h alternatives are really not irrelevant with respect to a and b.

6 The Condorcet method

Given the set A of alternatives this method is based on the execution of pairwise comparisons among the m alternatives of the set A.
In this way we have to perform, in general, $m(m-1)/2$ comparisons for each decider so that we can evaluate his preferences for any alternative. In this way we have a profile of preferences for a decider.
Then we have to repeat such evaluation for all the n deciders in order to understand how many deciders [weakly] prefer one alternative to another, be it m,

[19] We recall that by its very definition in the Borda method the single rankings r_i cannot contain ties so we could use as R even the strict preference relation $\succ$.

in a given pair so to derive how many of them have the opposite preference as $n - m$.
In this way we start from the local rankings r_i for the single deciders (that we assume to be rational and so to have a total preferences without ties nor cycles) and derive the global ranking r_C.
As we have already seen the assumption on the local rankings derives from the will to avoid any "garbage in, garbage out" effect and to make easier a comparison with the Borda method that enforces such condition from its very definition.
If we have n deciders, for a pair (a, b) we can have[20]:

- n_1 deciders have the $a \succeq b$ ranking,
- n_2 deciders have the $b \succeq a$ ranking,

with $n = n_1 + n_2$. We can therefore have the following cases:

- if $n_1 = n_2$ we can say that $a \sim b$ is the social choice of the n deciders,
- if $n_1 > n_2$ we can say that $a \succeq b$ is the social choice of the n deciders,
- if $n_1 < n_2$ we can say that $b \succeq a$ is the social choice of the n deciders.

Of course the condition $n_1 = n_2$, from the constraint $n = n_1 + n_2$, is possible iff[21] n is even.
By using such method we can get the definition of ([2]):

- the Condorcet winner,
- the Condorcet loser.

A related concept is that of **Condorcet score** ([7]) that allows the assignment of a numerical value to each of the possible transitive orderings of the alternatives.
Given a transitive global ordering among the alternatives we consider all the relations between the possible pairs of alternatives (that respect the ordering). For each pair we count the number of deciders that have the corresponding ordering in their own ordering. By summing all such values we get the Condorcet score of an ordering.
If we have two distinct global ordering the one to be preferred is therefore the one with the highest Condorcet score.
The Condorcet score, therefore, refers to global rankings of the alternatives and allows their ordering according to easy to compute numerical values. On the other hand both the Condorcet winner and the Condorcet loser involve the single alternatives in comparison with all the others.
The **Condorcet winner** is an alternative that defeats, by a majority, every other alternative in pairwise comparisons. We can show ([2]) that if the Condorcet winner exists it is unique though we have no guarantee of its existence.

[20] In this case we use as R the weak preference relation $\succeq$ but similar considerations hold also if we consider a strict preference relation $\succ$.
[21] We note that we use *iff* as a shorthand for "if and only if".

On the other hand, a **Condorcet loser** is an alternative that is defeated by a majority by every other alternative in pairwise comparisons.
Summing up we can say that the Condorcet method works as follows[22] ([2]):

- given an alternative $a_i \in A$ we compare it with all the remaining $m-1$ alternatives a_j of A with $i \neq j$;

- we count the number n_1 of times where a_i is weakly preferred to a_j;

- we count the number n_2 of times where a_j is weakly preferred to a_i;

- alternative a_i is preferred to alternative a_j if $n_1 > n_2$, alternative a_j is preferred to alternative a_i if $n_2 > n_1$ otherwise (if $n_1 = n_2$) the two alternatives are indifferent.

The method can be used also to define a global ranking of the alternatives of the set A though such global ranking may suffer both the presence of ties and a loss of transitivity and so may contain cycles that involve either the whole set A or some of its subsets.
The method is based on qualitative comparisons of pair of alternatives and this lack of connection among larger sets of alternatives causes the arising of the loss of transitivity even if the single preference orderings of the deciders satisfy such a property ([5], [6]).
As a final remarks on both methods we recall that the Borda method imposes the rationality on the deciders as transitive individual rankings so, in order to make a comparison easier, we assume also for the Condorcet method rational individual rankings. Notwithstanding this both methods can introduce violations of rationality through ties and cycles.

7 Borda and Condorcet, some toy examples

We now give some toy examples of the two methods in order to show their working. The examples we present here can be used, in other dedicated sections, to understand their possible relations.

Example 7.1 *As a first example we consider the set $A = \{a, b, c\}$ and three deciders with the following profiles:*

r_1: $a \succeq c \succeq b$

r_2: $b \succeq a \succeq c$

r_3: $c \succeq a \succeq b$

If we use our version of the Borda method we assign to a $7 = 3 + 2 + 2$ points, to b $5 = 1 + 3 + 1$ points and to c $6 = 2 + 1 + 3$ points so that the global ranking is:

[22]In this case we adopt a weak preference relation.

r_B: $a \succ c \succ b$

If we use the Condorcet method we get:

a versus b wins a,

a versus c wins a,

c versus b wins c,

so we get the same global ranking as before[23]*:*

r_C: $a \succ c \succ b$

In this case we have $r_B = r_C$. This event can occur quite often and is the basis of the method we propose in section 14.
Such result continues to hold, for instance, if we have $n > 3$ deciders split in three groups of cardinalities n_1, n_2 and n_3 such that $n_1 \geq n_2 \geq n_3$.
We note that since n is odd we cannot have direct ties in the Condorcet method so we can have, at the most, a loss of transitivity. This happens if we have, for instance, the following profiles:

r_1: $a \succeq b \succeq c$

r_2: $c \succeq a \succeq b$

r_3: $b \succeq c \succeq a$

In this case it is easy to see how we have (if we use $\succeq$):

r_C: $a \succeq b \succeq c \succeq a$

and:

r_B: $a \sim b \sim c$

If, however, we have 3 deciders with a r_1 type profile, 2 deciders with a r_2 type profile and 2 deciders with a r_3 type profile we again have a cycle with the Condorcet method since we get:

a versus b wins a,

a versus c wins c,

c versus b wins b,

and so we get $a \succeq b \succeq c \succeq a$.
On the other hand we have not a tie, neither full nor partial, with the Borda method where we get[24] *$\{a, b, c\} = \{15, 14, 13\}$.*

[23] We use $\succ$ but similar considerations hold also if we use $\succeq$.

[24] The notation $\{a, b, c\} = \{15, 14, 13\}$ represents a concise way to say that a gets 15 points, b gets 14 point and c gets 13 points.

Example 7.2 *If we consider the following set $A = \{a, b, c\}$ with four deciders we can have the following preference profiles:*

r_1: $a \succeq b \succeq c$

r_2: $c \succeq b \succeq a$

r_3: $a \succeq c \succeq b$

r_4: $b \succeq c \succeq a$

If we use the Borda method in this case we have:

r_B: $a \sim b \sim c$

since every alternative gets 8 *points. In the case of the Condorcet method we have three direct ties among the alternatives and so:*

$a \sim b$ *since we have* $v_{ab} = v_{ba} = 2$,

$a \sim c$ *since we have* $v_{ac} = v_{ca} = 2$,

$b \sim c$ *since we have* $v_{bc} = v_{cb} = 2$.

This can occur since n is even. On the same set we can have:

- 3 *deciders with a profile* r_1,
- 3 *deciders with a profile* r_2,
- 2 *deciders with a profile* r_3,
- 2 *deciders with a profile* r_4.

In this way we have again:

r_C: $a \sim b \sim c$

owing to direct ties among the alternatives. If we use the Borda method we get:

r_B: $a \sim b \sim c$

since every alternative gets 20 *points. In this case we note that:*

- r_1 *and* r_4 *are easily obtained one from the other with a shift operation*[25],
- r_2 *and* r_3 *are easily obtained one from the other with a shift operation.*

If we have:

- 3 *deciders with a profile* r_1,
- 2 *deciders with a profile* r_2,

[25] With this term we denote the shifting of one alternative in the preference profile. This term will be defined more formally shortly.

- 3 *deciders with a profile* r_3,
- 2 *deciders with a profile* r_4.

we get:

r_B: $a \succ b \sim c$

and:

r_C: $a \succ b \sim c$

As a last case we consider to have:

- 3 *deciders with a profile* r_1,
- 1 *deciders with a profile* r_2,
- 1 *deciders with a profile* r_3,
- 3 *deciders with a profile* r_4.

so that we get:

r_B: $b \succ b \succ c$

but:

r_C: $a \sim b \sim c$

From these examples it seems possible to derive that:

- *a full tie in the Borda method is associated to a cyclical ranking in the Condorcet method with possible direct ties if* n *is even;*
- *a cyclical ranking in the Condorcet method or direct ties are not necessarily associated to a full tie in the Borda method;*
- *partial ties can relate each other in the two methods.*

Example 7.3 *In another case we can have the set* $A = \{a, b, c, d\}$ *with three deciders with the following profiles:*

r_1: $a \succeq b \succeq c \succeq d$

r_2: $d \succeq a \succeq b \succeq c$

r_3: $c \succeq d \succeq a \succeq b$

If we use the Borda method in this case we have:

r_B: $a \succeq d \succeq c \succeq b$

whereas if we use the Condorcet method we get:

r_C: $d \succeq a \succeq b \succeq c \succeq d$

ad so a cycle that in his case we can resolve with indifference conditions between pair of alternatives but not with direct ties since n is odd.

Example 7.4 *As another example we consider the set $A = \{a, b, c, d\}$ with three deciders with the following profiles:*

r_1: $a \succeq d \succeq c \succeq b$

r_2: $b \succeq a \succeq c \succeq d$

r_3: $b \succeq c \succeq a \succeq d$

If we use the Borda method in this case we have:

r_B: $a \sim b \succeq c \succeq d$

whereas if we use the Condorcet method we get:

r_C: $b \succeq a \succeq c \succeq d$

so partial ties in the Borda method do not have necessarily an equivalent condition in the Condorcet method.

Example 7.5 *Another example is the following. We can consider the set $A = \{a, b, c\}$ with five deciders and with the corresponding Borda rankings that form the following matrix*[26]*:*

$$R = \begin{pmatrix} 3 & 2 & 1 \\ 3 & 2 & 1 \\ 2 & 3 & 1 \\ 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \tag{13}$$

From R we can derive the following preference profiles:

r_1: $a \succeq b \succeq c$

r_2: $a \succeq b \succeq c$

r_3: $b \succeq a \succeq c$

r_4: $c \succeq b \succeq a$

r_5: $c \succeq a \succeq b$

If we use the Borda method in this case we have:

r_B: $a \succeq b \succeq c$

[26] The matrix R will be defined more precisely in section 11. For the moment we note that we have a row for each decider and a column for each alternative of the set A in the order in which they are listed in the set A. Every row defines the Borda scores for each decider and so the preference profile for that decider.

since we have $\{a, b, c\} = \{11, 10, 9\}$. *If we use the Condorcet method we get:*

r_C: $a \succeq b \succeq c$

In this case we have $r_B = r_C$. *If we modify the matrix* R *in a way that leaves unchanged the Borda scores we may get the following matrix:*

$$R = \begin{pmatrix} 3 & 1 & 2 \\ 3 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & 1 \\ 2 & 3 & 1 \end{pmatrix} \tag{14}$$

Such modified matrix has been obtained through operations that are in contrast with those we are going to define as correct in section 11.
The new matrix defines the following preference profiles:

r_1: $a \succeq c \succeq b$

r_2: $a \succeq c \succeq b$

r_3: $c \succeq b \succeq a$

r_4: $b \succeq a \succeq c$

r_5: $b \succeq a \succeq c$

In this case the Borda ranking r_B *is unchanged but for the Condorcet ranking we get:*

r_C: $c \succeq b \succeq a \succeq c$

and so a cycle involving all the alternatives or a full indirect tie (see further on). A full indirect tie among the alternatives can be resolved, by using the definitions of the relations $\succeq$ *and* $\sim$ *and the transitivity of the former, assuming that the alternatives are equivalent. In this way we reduce an indirect tie to a direct tie. We note that this example shows how a cycle in the Condorcet method has no necessary consequence on the Borda method in the sense that it does not impose a particular ordering (such as a full tie) on the alternatives according to that method.*

Before closing this section we make two examples to illustrate some problematic features of the Borda method.

Example 7.6 *We start with a set* $A = \{a, b, c, d\}$ *with seven deciders and with the Borda rankings that form the following matrix:*

$$R = \begin{pmatrix} 4 & 3 & 2 & 1 \\ 1 & 4 & 3 & 2 \\ 2 & 1 & 4 & 3 \\ 4 & 3 & 2 & 1 \\ 1 & 4 & 3 & 2 \\ 2 & 1 & 4 & 3 \\ 4 & 3 & 2 & 1 \end{pmatrix} \tag{15}$$

If we use the Borda method in this case we have:

r_B: $c \succeq b \succeq a \succeq d$

since we have $\{a, b, c, d\} = \{18, 19, 20, 13\}$. *If we use the Condorcet method we have:*

r_C: $c \succeq d \succeq a \succeq b \succeq c$

If, after the rankings have been defined, the alternative d is discarded for some reason, we get:

$$R = \begin{pmatrix} 3 & 2 & 1 \\ 1 & 3 & 2 \\ 2 & 1 & 3 \\ 3 & 2 & 1 \\ 1 & 3 & 2 \\ 2 & 1 & 3 \\ 3 & 2 & 1 \end{pmatrix} \tag{16}$$

If we use the Borda method in this case we have:

r_B: $a \succeq b \succeq c$

since we have $\{a, b, c\} = \{15, 14, 13\}$. *In this case the cancellation of an alternative has caused a cascaded modification of the rankings with a rank reversal of the alternatives. If we use the Condorcet method, on the other hand, we have:*

r_C: $a \succeq b \succeq c \succeq a$

so that the Condorcet method proves immune from the cancellation of one alternative.

Example 7.7 *As another example of the difficulties of the Borda method we can use the following case.*
We can consider the set $A = \{a, b, c\}$ *with* $n = 11$ *deciders where five deciders have the following preference profile:*

r_1: $a \succeq b \succeq c$

whereas the other six deciders have the following preference profile:

r_2: $b \succeq a \succeq c$

If we use the Borda method we get:

r_B: $b \succeq a \succeq c$

since we have $\{a, b, c\} = \{27, 28, 11\}$. *If we use the Condorcet method we have:*

r_C: $b \succeq a \succeq c$

so that we have $r_C = r_B$.
If the five deciders reverse their ranking of b and c in an attempt to favor their preferred choice "a" they succeed if they use the Borda method since with that method we get:

r_B*:* $a \succeq b \succeq c$

since we have $\{a, b, c\} = \{27, 23, 16\}$*.*
If, on the other hand, we use the Condorcet method we see that the ranking of the alternatives is unchanged since also in this case we have:

r_C*:* $b \succeq a \succeq c$

so that we have $r_C \neq r_B$. In this case we had a rank reversal of the alternatives "a" and "b" in the Borda method though their relative orderings did not change from one set of preference profiles to the other.
In this case the five deciders that modify their preference profiles are said to behave or vote strategically. Their move may be unsuccessful if also the other six deciders strategically modify their preference profile so we get, respectively:

r_1*:* $a \succeq c \succeq b$

and:

r_2*:* $b \succeq c \succeq a$

It is easy to see that in this case we get:

r_C*:* $b \succeq c \succeq a$

and:

r_B*:* $b \succeq c \succeq a$

and so again $r_B = r_C$.
We underline:

- *the obvious improvement of the ranking of the alternative "c" as a consequence of the actions of the deciders;*
- *that also in the modified profiles the relative ordering of the alternatives "a" and "b" is the same that we had in the original profiles.*

Further details will be given in the following sections.

8 The Arrow's theorem and our methods

In this section we firstly introduce some basic properties we wish are satisfied by our methods then we state that they are incompatible so that they cannot be all satisfied at the same time by any method. The next step is to see which of the introduced properties is violated by the Borda or by the Condorcet method.

One of the key points is the definition of irrelevant alternatives that comes naturally for the Condorcet method (where we can say that any alternative that is outside the current pair under examination is irrelevant) but must be reconsidered for the Borda method though such reconsideration is usually not carried out in the literature (see [5] and [6]).
The properties we introduce are a minimal set of properties so it is easy to see that things can only worsen if we take a wider set.
Such properties are the following ([5], [6], [2], [3]):

(1) transitivity,

(2) universal domain,

(3) Pareto condition,

(4) binary independence.

We say that a profile and a social welfare function satisfy **transitivity** if for any triple of alternatives a, b, c we have that from $a \succeq b$ and $b \succeq c$ we can derive $a \succeq c$.
We say that a social welfare function satisfy **universal domain** if all the profiles of preferences of the deciders are legal profiles for the function so we do not impose any restriction on the preference profiles. We note that imposing a transitivity is a consequence of the first requirement.
The **Pareto condition** says that if, for a pair of alternatives a and b, the all the deciders prefer a to b then the social welfare function must produce a ranking where a is preferred to b.
A social welfare function satisfies **binary independence** if, given a pair of alternatives a and b, the ranking of such alternatives in the global ranking depends only on that alternatives and not on any of the remaining alternatives.
In order to understand the theorem 8.1 we need to define:

- what we mean as a **social welfare function**,

- what is a **dictator**.

A **social welfare function** is a function that from a set of preference profiles of the members of a set of deciders on a set of alternatives defines a ranking of the alternatives.
A **dictator**, on the other hand, is a particular decider. We say that a social welfare function is equivalent to a **dictator** if there is a decider whose profile of preferences always coincides with the outcome of that function independently from the profiles of preferences of the other deciders. In this way we say that in any way we modify the profiles of $n-1$ deciders the outcome is defined by the profile of the remaining decider.
We now present the Arrow's impossibility theorem in the formulation of [5] and then we use it to check the goodness of both the Borda method and the Condorcet method.

Theorem 8.1 (Arrow's impossibility theorem) *A social welfare function with:*

- *three or more alternatives,*

- *two or more deciders,*

where the profiles of preferences:

- *satisfy transitivity,*

- *satisfy the universal domain condition,*

- *satisfy binary independence,*

- *satisfy the Pareto condition,*

is equivalent to a dictator.

We note that the theorem is in the following form:

$$if\ \cap_i P_i\ then\ Q \tag{17}$$

where the P_i are its premises and Q is the dictatorial conclusion.
Such formulation is equivalent to the following one:

$$if\ \neg Q\ then\ \cup_i \neg P_i \tag{18}$$

From this alternative formulation we derive that if we know that a given function is not equivalent to a dictator (so the premise $\neg Q$ in the alternative formulation is true) then one of its premises P_i must be false (since we require that also the consequence is true). We underline how we do not consider the number of the alternatives and the number of the deciders in this analysis. The reason is that if we have two alternatives only there is no problem at all and the same is true in presence of a single decider.
We apply this approach to the evaluation of our two methods to see, for each of them, which premise must be violated. We can use this approach since we know that neither the Borda method nor the Condorcet method (as social welfare functions) are equivalent to a dictator.
We start with the **Borda method** ([5] and [6]).
This method:

- satisfies transitivity from the association between each alternative and the corresponding Borda score so that a transitivity on the alternatives derives from a the transitivity of such scores;

- satisfies unrestricted domain since it can be used for any preference profile that is transitive from the definition of the method itself;

- satisfies the Pareto condition since if every decider prefers a to b then a gets more points than b and so is better ranked than b in the overall ranking;

- we underline how this method is not dictatorial since it satisfies anonymity so there is no way to identify a particular voter as the dictator.

From all this we derive that it must fail binary independence as it can be easily shown by providing simple examples (see section 7).
We have therefore seen how the **Borda method** satisfies all the required properties but the **binary independence** or **independence from irrelevant alternatives**.
An illustration of this fact has been given both in example 7.6 and in example 7.7.
The fact that the Borda method violates this property is usually seen as an expression of strategic voting from the deciders and as an expression of the manipulability of the method. With this we mean that its outcome may be altered through the introduction of straw alternatives (to understand what this means it is sufficient to consider the example 7.6 in the opposite direction or from the narrow set to the wider set with d as the straw alternative) or by the modification of the relative rankings of pair of alternatives (see example 7.7).
We note how the last feature, to be effective, requires the full knowledge of the preference profiles of all the deciders and is vulnerable to counter strategical behaviors, how it is shown by the example 7.7.
We now switch to the **Condorcet method**.
This method:

- satisfies binary independence from its definition since the alternatives are pairwise compared so that any other alternative has no influence in the ranking of a pair;

- satisfies unrestricted domain since it can be used for any preference profile of the deciders;

- satisfies the Pareto condition since if every decider prefers a to b then a is better ranked than b in the overall ranking;

- also in this case we underline how the method is not dictatorial since it satisfies anonymity so there is no way to identify a particular voter as the dictator.

From all this we derive that it must fail transitivity as it can be easily shown by providing simple examples (see section 7).
We have so shown how the **Condorcet method** satisfies all such properties but transitivity.
The method is said to fail transitivity since, as we have seen, it can give rise to cycles among the alternatives.
Examples of this fact can be found in section 7. As we have already seen the presence of cycles can be either local or global.
In the former case a cycle involves only a subset of the alternatives whereas in the latter it involves all the alternatives. In both case a cycle can be seen as a loss of transitivity, if we use the $\succ$ preference relation, or as a tie among the

involved alternatives, if we use the $\succeq$ preference relation.
The presence of cycles cannot be avoided in the Condorcet method where we can have also direct ties between pairs of alternatives if n (the number of the deciders) is even. This feature of the method ([5], [6]) depends on the fact that the alternatives are pairwise compared so that in this way we lose any information concerning the relations among all the alternatives and this can give rise to cycles.
We note that for any pair $x, y \in A$ all the other alternatives in A are naturally irrelevant in the comparison.
On th other hand the problems with the Borda method arise since it has a twofold qualitative and quantitative nature and since what is irrelevant for the Condorcet method is not irrelevant, at least from a quantitative point of view, for the Borda method.
We comment here a little on these features in order to clarify the method we are going to propose in section 14.
In the Borda method when we compare two alternatives a_i and a_j we say:

- if a_i is more preferred to a_j;

- how much it is more preferred.

In this case we consider the quantitative feature to evaluate the ranking so that if we have $a_i \succeq a_j$ (qualitative feature) we can assign (where h is the number of in between alternatives):

- $k \in [1, m]$ points to a_i,

- $k - h - 1 \in [1, m]$ points to a_j.

In this way the h in between alternatives cannot be said to be irrelevant since they cause a modification of the ratio of the evaluations between the two alternatives from:

$$\frac{k}{k-1} \tag{19}$$

(when $h = 0$) to:

$$\frac{k}{k-m+1} \tag{20}$$

(when $h = m - 2$). We note that in the general case the ratio is:

$$\frac{k}{k-h-1} \tag{21}$$

and that, if $h = m - 2$, we get $k - h - 1 = k - (m - 2) - 1 = k - m + 1$.
We can analyze the interaction between qualitative and quantitative features (that can be said to be coincident in the Condorcet method) if we consider the following approaches: descriptive and normative.
If we take a normative approach we state that in the comparison between a_i and a_j any other alternative should be irrelevant whereas if we take a descriptive approach we state that in the comparison between a_i and a_j any other alternative

is irrelevant. In the Condorcet method the two approaches coincide whereas in the Borda method they diverge.
In order to have them converge also in the case of the Borda method we can define only the alternatives but the in between alternatives as irrelevant for the Borda method.
If we define as A_{ij} the set of the alternatives between a_i and a_j in the ordering we have that the irrelevant alternatives are those of the set:

$$A_I = A \setminus (A_{ij} \cup \{a_i, a_i\}) \tag{22}$$

In the Condorcet method we can define, from the definition of the method itself, $A_{ij} = \emptyset$ for any a_i and a_j so that we have:

$$A_I = A \setminus \{a_i, a_i\} \tag{23}$$

The problem with the Borda method is that A_I depends on both a_i and a_j since there is no way to set permanently $A_{ij} = \emptyset$. In order to see why is this we note how a strategic behavior has the effect to widen or shrink the set A_{ij} (depending on how many alternatives are inserted in or extracted from it) so that the problem with the irrelevant alternatives is back again and there do not seem to be any easy solution. Both the insertion and the extraction depend on the individual will of the deciders and cannot be constrained in general cases. For some tentative solutions of this problem we refer to [5] and [6].

9 The parameters and the various conditions

The parameters we think are meaningful for both the Borda and the Condorcet method and for our purposes include:

- the number n of the deciders,
- the number m of the alternatives.

In the **Borda method** every decider i assigns to each alternative j a distinct value $r_{i,j}$ in the range $[1, m]$ for a total of:

$$\frac{m(m+1)}{2} \tag{24}$$

points so that the n deciders assign globally:

$$n\frac{m(m+1)}{2} \tag{25}$$

points to the set of the alternatives or, on the average,

$$n\frac{m+1}{2} \tag{26}$$

points to each of the m alternatives.
The condition that the values $r_{i,j}$ that are assigned to the alternatives $j \in [1,m]$ by each of the $i \in [1,n]$ deciders are distinct can be expressed, for any $i \in [1,n]$, as:

$$\prod_{j=1}^{m} r_{i,j} = m! \tag{27}$$

with $r_{i,j} \in [1,m]$.
Under the foregoing conditions we can have a full parity (or a full tie or the same amount of points) over all the elements of A, according to the Borda method, iff:

$$n(m+1) \tag{28}$$

is even. This can occur under the following conditions:

- if n is even,

- if m is odd so that $m+1$ is even.

If at least one of such conditions is verified the m alternatives of A can receive the same number of points equal to:

$$n\frac{m+1}{2} \tag{29}$$

In this way they are equivalent alternatives from the point of view of the deciders, at least according to the Borda method.
Besides the case of the parity over all the elements of A we can have a wide number of cases where k_i alternatives receive the same number of global points p_i under the following constraints:

$\sum_i k_i = m$

$\sum_i k_i p_i = n\frac{m(m+1)}{2}$

for each $j \in N$ we have $\sum_i r_{ij} = \frac{m(m+1)}{2}$

for each $j \in N$ we have $\prod_i r_{ij} = m!$

where $r_{ij} \in [1,m]$ is the weight that decider i assigns to the alternatives $j \in A$.
The former type of tie is termed a full tie whereas the latter type is termed a partial tie.

Example 9.1 (Simple example with the Borda count and the Condorcet method) *If we have the set $A = \{a,b,c\}$ and three deciders*[27] *with the following Borda rankings:*

- $\{a,b,c\} = \{3,1,2\}$

[27] Similar considerations hold also for n deicers subdivided in three groups of equal size and so if $n = 3k$ with $k \in \mathbb{N}$.

- $\{a, b, c\} = \{1, 2, 3\}$
- $\{a, b, c\} = \{2, 3, 1\}$

we get the following global ranking[28]*:*

- *a gets 6,*
- *b gets 6,*
- *c gets 6,*

and so $a \sim b \sim c$. In this case we have a full tie. We note that we have $n = 3$ odd but $m = 3$ is odd so $m + 1$ is even and so the necessary condition for a full tie is satisfied.
If we use the Condorcet method in this case it is easy to see that we get $b \succeq a \succeq c \succeq b$ and so, from the transitivity of $\succeq$ (and from the definition of $\sim$) a full tie among the alternatives.
If we have the following rankings:

- $\{a, b, c\} = \{3, 2, 1\}$
- $\{a, b, c\} = \{2, 1, 3\}$
- $\{a, b, c\} = \{3, 2, 1\}$

we get the following global ranking:

- *a gets 8,*
- *b gets 5,*
- *c gets 5,*

and so $a \succ b \sim c$ and therefore a partial tie[29].
If we use the Condorcet method it is easy to see that we get $a \succeq b \succeq c$ or a total transitive ordering among the alternatives.
If we have:

- $\{a, b, c\} = \{3, 2, 1\}$
- $\{a, b, c\} = \{2, 3, 1\}$
- $\{a, b, c\} = \{3, 1, 2\}$

we get the following global ranking:

- *a gets 8,*
- *b gets 6,*

[28] With the phrase "a gets 6" we summarize the phrase "a gets a number of points equal to 6". The same holds also for the other cases in this TR.

[29] In this case we have used $\succ$ as the binary relation but nothing changes if we use $\succeq$.

- *c gets* 4*,*

and so $a \succ b \succ c$ *and therefore no tie.*
If we use the Condorcet method also in this case it is easy to see that we get $a \succeq b \succeq c$ *ad so again a total transitive ordering among the alternatives that coincides with the ordering we got from the Borda method.*

For what concerns the **Condorcet method** we can consider, as the meaningful features, both the number n of the deciders and some particular types of profile.
For what concerns n we have two cases:

- n is even,

- n is odd.

If n is even we can have **direct ties** between alternatives so that we can have, for two alternatives a and b, that $v_{ab} = v_{ba}$ where v_{ab} is the number of deciders that have $a \succeq b$. More formally we have:

- $v_{ab} = v_{ba}$

- $v_{ab} + v_{ba} = n$

so we have $v_{ab} = v_{ba} = n/2$ and this condition can be verified only if n is even.
On the other hand if we have a succession of links among the alternatives from which we can derive that both $a \succeq b$ and $b \succeq a$ hold then we have $a \sim b$ though we do not necessarily have $v_{ab} = v_{ba}$. In this case we speak of **indirect ties**, see the examples we have made in section 7.
We note how direct ties can involve all the pairs of alternatives so that such alternatives can be said to belong to the same equivalence class and, so, to be equivalent to the other alternatives through a series of direct ties.
On the other hand if n is odd it is easy to see, from what we have said before, that we can have only indirect ties but no direct ties. This is the main reason why, in the literature, many of the examples of application of the Condorcet method are given with n odd.
For what concerns the profiles we can define the following types of profiles[30]:

- compensating profiles,

- shift profiles,

globally termed as **particular profiles**.
We define two profiles as **compensating profiles** if they define opposite orderings on the same alternatives.
If we consider the m alternatives $a_1, \ldots, a_m$ we have, for instance, that the two profiles[31]:

[30] In what follows we use the relation $\succeq$ but many of the considerations we make hold also if we use the relation $\succ$.

[31] It is obvious that if we have $a \succeq b$ then we have $b \preceq a$ and that if we have $a \succ b$ then we have $b \prec a$. If $\succeq$ means "is weakly preferred to" then $\preceq$ means "is weakly preferred by". Similarly if $\succ$ means "is strictly preferred to" then $\preceq$ means "is strictly preferred by".

$a_1 \succeq a_2 \cdots \succeq a_m$

and

$a_1 \preceq a_2 \cdots \preceq a_m$

define two opposite orderings on the set A and so two compensating profiles. If we have m alternatives we can easily see how they can be ordered in $m!$ different ways (without repetitions) so that we can have $m!/2$ profiles with the corresponding $m!/2$ opposing or compensating profiles[32].
It is easy to see that:

- given a generic profile it is always possible to derive the corresponding compensating profile by simply replacing $\succeq$ with $\preceq$ or vice versa;
- compensating profiles cancel out each other and have no effect on the global ordering.

From this we have that if n is even we can have pairs of compensating profiles that make the alternatives as equivalent. On the other hand if $n = 2k + 1$ (with $k \in \mathbb{N}$) is odd we may have $2k$ profiles that can cancel out as pairs of compensating profiles so that the final ranking of the alternatives coincides with the ranking of the remaining profile that is a sort of dictator.

Example 9.2 *If for instance we have the following preference profiles:*

$a \succeq b \succeq c \succeq d$

$d \succeq c \succeq b \succeq a$

$b \succeq a \succeq c \succeq d$

we have that the first two profiles are compensating profiles so that the total ranking coincides with the one determined by the last profile for both the Borda and the Condorcet method.

Compensating profiles have an influence also on the Borda method. It is easy to see how each pair of compensating profiles contribute with the same weight to each alternative in the global ranking so that such profiles can be safely discarded.
For what concerns the **shift profiles** ([5], [6]), if we imagine the m alternatives $a_1, \ldots, a_m$ as marks on the border of a wheel in a clockwise order we have that, starting from the basic profile $a_1 \succeq a_2 \cdots \succeq a_m$, we can get[33] the following types of shift profiles:

- **clockwise shift [one] profile**,
- **counterclockwise shift [one] profile**.

[32] We note that $m!$ is even for any $m \neq 1$ as it easily results from its definition.
[33] We can imagine more complex shift profiles ([5]) whose introduction and use are however unnecessary for our purposes.

In the foregoing case we get;

- $a_m \succeq a_1 \succeq a_2 \cdots \succeq a_{m-1}$ as a **clockwise shift one profile**,
- $a_2 \succeq a_3 \cdots \succeq a_{m-1} \succeq a_m \succeq a_1$ as a **counterclockwise shift one profile**.

For instance, if we have a preference profile of the form:

$$a \succeq b \succeq c \tag{30}$$

a **clockwise shift one profile** is:

$$c \succeq a \succeq b \tag{31}$$

whereas a **counterclockwise shift one profile** is:

$$b \succeq c \succeq a \tag{32}$$

From such three profiles we have a full tie with the Borda method where each alternative gets 6 points. If we use the Condorcet method we get:

a versus b wins a,

a versus c wins c,

b versus c wins b,

so we have $c \succeq a \succeq b \succeq c$ and therefore a relation of indifference among the alternatives or $a \sim b \sim c$ from the definition of $\sim$ and from the transitivity of $\succeq$.
In order to prove this fact we note that, for instance, from $c \succeq a$ and $a \succeq c$ (obtained through the transitivity of $\succeq$) we get $a \sim c$. The other cases follow in similar ways.
If, in the same case, we use relation $\succ$ we get a loss of transitivity that cannot be solved in an indifference among the alternatives but remains as a contradictory ranking from the deciders ([5], [6]).
We call the possible profiles homogeneous if they are of the same type and heterogeneous if they are of distinct types though for any clockwise shift we can define a counterclockwise shift that gives the same outcome. If we have m alternatives we can indeed have:

- m clockwise shift one profiles,
- m corresponding counterclockwise shift one profiles.

It is easy to see how using the right number of homogeneous shift one profiles we can obtain both a cyclical outcome in the Condorcet method and an equal ranking of the alternatives in the Borda method.

Example 9.3 *If for instance we have the following preference profiles:*

$a \succeq b \succeq c$

$b \succeq c \succeq a$

$c \succeq a \succeq b$

we have that the second profile is obtained with a shift operation[34] *from the first and in the same way we obtain the third from the second. In this way we get a full cycle with the Condorcet method and a full tie with the Borda method.*
This eventuality can occur in all the cases where $n = km$ with $k \in \mathbb{N}$ and where m denotes both the number of the alternatives and the number of distinct shift profiles. In more complex cases we can have a mixture of compensating profiles and shift profile that give an outcome with the same properties.

Last but not least if we consider the possible relations between the values of the parameters n and m we can have the following cases:

- $n < m$,
- $n = m$,
- $n > m$.

If $n = m$ or $n < m$ we can use distinct shift one profiles whereas if $n > m$ we are forced to use repeated shift one profiles.
If, for instance, we have $n = 5$ and $m = 3$ (so we can have 6 distinct transitive profiles but only 3 distinct shift profiles) we can start with the following profiles:

- $a \succ b \succ c$
- $c \succ a \succ b$
- $b \succ c \succ a$

but now we are forced to use either repeated shift profiles or different types of profiles such as $b \succ a \succ c$ and $c \succ b \succ a$ obtained through local permutations. We note, however, how the latter profile is obtained from a clockwise shift of the former.

10 Some examples of the particular profiles

In this section we give some toy examples of the particular profiles we have described in section 9 and precisely:

- compensating profiles,
- shift profiles.

[34] A shift operation, to be defined more formally shortly, can be seen as the moving of an alternative in the ordering in two ways: from the head of the ordering to the tail or vice versa. We call the former mode a counter clockwise shift and the latter a clockwise shift.

In all the examples of this section we assume to have the sets $A = \{a, b, c, d\}$ and $B = \{a, b, c\}$.
As a simple example of a pair of compensating profiles we can use the following profiles over the set A:

r_1: $a \succeq b \succeq c \succeq d$

r_2: $d \succeq c \succeq b \succeq a$

Such profiles contribute, in the Borda method, with a weight of 5 to every alternative so that these profiles can be safely discarded from the global ranking r_B made by more than two deciders. If we have only those two deciders then the four alternatives are seen as equivalent according to the Borda method.
If we apply the Condorcet method to the foregoing profiles we get[35]:

a versus b gives $a \sim b$

a versus c gives $a \sim c$

a versus d gives $a \sim d$

b versus c gives $b \sim c$

b versus d gives $b \sim d$

c versus d gives $c \sim d$

In this way the four alternatives are to be seen as equivalent according to such profiles so that they can be safely discarded from the global ranking r_C made by more than two deciders. If we have only those two deciders then the four alternatives are seen as equivalent according to the Condorcet method.
If we use the set B and use $\succ$ as the preference relation we can start from the profile:

r_1: $a \succ b \succ c$

and generate in succession the following clockwise shift one profiles:

r_2: $c \succ a \succ b$

r_3: $b \succ c \succ a$

If we apply the Borda method to such profiles we get:

- a gets 6 points,

- b gets 6 points,

[35]We recall how for 4 alternatives we have $4*3/2 = 6$ possible comparisons between distinct alternatives. In general if we have m alternatives we can compare the first alternative with $m-1$ different alternatives, the second one with $m-2$ different alternatives and so on for a total of $m*(m-1)/2$ alternatives.

- c gets 6 points.

In this case we have a parity over the set B. If we apply the Condorcet method we have:

a versus b gives $a \succ b$

b versus c gives $b \succ c$

c versus a gives $c \succ a$

so that we get a Condorcet cycle with a loss of transitivity $a \succ b \succ c \succ a$. If on the other hand we use the relation $\succeq$ we get:

$$a \succeq b \succeq c \succeq a \tag{33}$$

that, owing to the transitivity of $\succeq$ and the definition of the indifference relation $\sim$, gives rise to the following relations:

$a \sim b$

$b \sim c$

$c \sim a$

so that the three alternatives can be seen as equivalent.
If we use the set A we can have the following profiles:

r_1: $a \succeq c \succeq d \succeq b$

r_2: $b \succeq d \succeq c \succeq a$

r_3: $b \succeq d \succeq c \succeq a$

If we apply the Borda method to such profiles we get $b \succeq d \succeq c \succeq a$ and the same holds if we apply the Condorcet method to the same profiles since by pairwise comparisons we get:

a versus b gives $b \succeq a$

a versus c gives $c \succeq a$

a versus d gives $d \succeq a$

b versus c gives $b \succeq c$

b versus d gives $b \succeq d$

c versus d gives $d \succeq c$

From this we get that in simple cases the two method give the same rankings. On the same set we can have, however, the following profiles that produce a cycle if we apply the Condorcet method but do not produce an indifference condition among the alternatives if we apply the Borda method:

r_1: $a \succeq b \succeq c \succeq d$

r_2: $b \succeq c \succeq d \succeq a$

r_3: $c \succeq d \succeq a \succeq b$

If we apply the Borda method we indeed get:

- a gets 7,
- b gets 8,
- c gets 9,
- d gets 6,

and so we obtain $c \succeq b \succeq a \succeq d$ whereas if we apply the Condorcet method we get:

a versus b gives $a \succeq b$

b versus c gives $b \succeq c$

c versus d gives $c \succeq d$

d versus a gives $d \succeq a$

and so we get $a \succeq b \succeq c \succeq d \succeq a$.

11 The R matrix

We can denote[36] the matrix $n \times m$ of the $r_{i,j}$ as R.
Each $r_{i,j}$, with $i \in [1, n]$ and with $j \in [1, m]$, is the score that each decider i assigns to each alternative j. If we fix i the $r_{i,j}$ must be distinct and belong to the interval $[1, m]$.
We recall that, in our context, $r_{i,j} > r_{i,k}$ means that alternative j is [weakly] preferred to alternative K in both the Borda method and the Condorcet method.
The members of R satisfies the following constraints:

$$\sum_{j=1}^{m} r_{i,j} = \frac{m(m+1)}{2} \; \forall\, i \; \in [1, n] \tag{34}$$

$$\prod_{j=1}^{m} r_{i,j} = m! \; \forall\, i \; \in [1, n] \tag{35}$$

If they satisfy also the following constraint:

$$\sum_{i=1}^{n} r_{i,j} = \frac{n(m+1)}{2} \; \forall\, j \; \in [1, m] \tag{36}$$

[36] In this section we define more formally the matrix R that we have already used in section 7 in a rather informal way.

we say we have a full tie if we use the Borda method whereas, if we use the Condorcet method, we can have both direct and indirect ties among the alternatives. We call such a matrix as a tied R and denote it as R_t.
If on R_t we apply the following operations:

- swap of the contents of two columns,
- swap of the contents of two rows,

we get a matrix R'_t that satisfies the same properties as R_t and so has a full tie according to the Borda method and may have either direct or indirect ties if we use the Condorcet method.
We underline how such operations, together with those on some sub matrices that we are going to present shortly, are the only legal operations we can perform on a matrix R_t so to have it keep on satisfying the Borda full tie condition.
We note that if we swap the contents of two rows we swap the preference profiles of two deciders so that the global ranking cannot change.
To see why things do not change if we swap the content of two columns we can proceed as follows:

- we start from an R_t that defines a condition of Borda full tie;
- we use R_t to get an r_C (with full direct or indirect ties) by using the numerical columns identifiers;
- we swap the contents of two columns and define a new R'_t;
- we get the corresponding r'_C (with full direct or indirect ties) by using again the numerical columns identifiers;
- we replace in both cases the numerical column identifies with the names of the corresponding unchanged alternatives (since we swap the content and not the columns);
- we obtain two full cycles among the alternatives that produce among such alternatives a set of direct or indirect ties so that the two cycles can be seen as equivalent.

Similar considerations hold if we work on a sub matrix $\hat{R}_t$ of R_t so that we can swap the content of two rows i and j of $\hat{R}_t$ only if their elements give rise to the same sum and we can swap the content of two columns h and k of $\hat{R}$ only if their elements give rise to the same sum.

Example 11.1 *If we consider the set $A = \{a, b, c\}$ and $n = 5$ deciders we can start from the following ad hoc matrix that describes the profiles of preferences of the deciders:*

$$R_t = \begin{pmatrix} 3 & 2 & 1 \\ 2 & 1 & 3 \\ 1 & 3 & 2 \\ 2 & 3 & 1 \\ 2 & 1 & 3 \end{pmatrix} \tag{37}$$

If we apply the Borda method we get of course a full tie since we get:

r_B*:* $a \sim b \sim c$

since every alternative gets 10 *points. If, on the other hand, we apply the Condorcet method we easily get:*

r_C*:* $a \succeq b \succeq c \succeq a$

and so a cycle that gives indirect ties between pairs of alternatives. We cannot have direct ties since n *is odd.*
If we swap the contents of columns 2 *and* 3 *we get the following equivalent matrix:*

$$R'_t = \begin{pmatrix} 3 & 1 & 2 \\ 2 & 3 & 1 \\ 1 & 2 & 3 \\ 2 & 1 & 3 \\ 2 & 3 & 1 \end{pmatrix} \tag{38}$$

If we apply the Borda method we get in this case we get again:

r_B*:* $a \sim b \sim c$

since every alternative gets 10 *points. If, on the other hand, we apply the Condorcet method we easily get:*

r_C*:* $b \succeq a \succeq c \succeq b$

and so again a cycle that gives indirect ties between pairs of alternatives.
If we work on the following sub matrix of R_t*:*

$$\hat{R}_t = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix} \tag{39}$$

we can get:

$$\hat{R}'_t = \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix} \tag{40}$$

so we get:

$$R''_t = \begin{pmatrix} 3 & 2 & 1 \\ 2 & 1 & 3 \\ 1 & 3 & 2 \\ 2 & 1 & 3 \\ 2 & 3 & 1 \end{pmatrix} \tag{41}$$

where both r_B *and* r_C *are unchanged,* r_C *since it defines a cycle among the alternatives that produces a indirect ties between pairs of alternatives.*

In this way we have seen how we can start from R_t that gives:

- a Borda full tie,
- a Condorcet full cycle,

and modify it with admissible operations so to get equivalent matrices with the same outcomes.

12 Borda and Condorcet

In this and the following section we aim at understanding if given a relation among the alternatives of the set A in the Borda method we can state a relation among the same alternatives in the Condorcet method and vice versa (see also section 1) assuming to have the same voters with the same preference profiles. With this we mean that we want to check if the following implications (to be read as possibilities, see further on) hold or not:

(1) if $c_B \Rightarrow c_C$

(2) if $c_C \Rightarrow c_B$

where c_B denotes a condition of the Borda method and c_C a condition of the Condorcet method.
If condition (1) is satisfied we have that from a condition on the Borda method applied to a certain number of preference profiles it is possible to derive a certain condition on the Condorcet method applied to the same profiles.
On the other hand if condition (2) is satisfied we have that from a condition on the Condorcet method applied to a certain number of preference profiles it is possible to derive a certain condition on the Borda method applied to the same profiles of preferences.
Our aims, therefore, are:

(1) to verify if the presence of Borda ties has some consequence on the outcome of the Condorcet method,

(2) to verify if the presence of Condorcet direct and indirect ties has some consequence on the outcome of the Borda method,

(3) to verify if and when we can use the two methods in conjunction so to have one reinforce the outcome of the other (see section 14).

We recall that a **Borda tie** can be of two types (see Table 1):

- **full** if all the alternatives get the same number of points and so the same ranking,

- **partial** if the alternatives of a subset of the set A get the same ranking.

If the conditions for a full tie are satisfied we can have also some partial ties whereas if they are violated we can have only partial ties.
A **full tie** can be possible only if either n or $m+1$ or both are even[37] whereas a **partial tie** can be possible also when both are odd.
On the other hand for what concerns the **Condorcet method** we must differentiate between direct and indirect ties.
A **direct tie** defines a condition of parity between two alternatives a and b so that we say that they are tied if $v_{ab} = v_{ba}$ or the number of the deciders who

[37] We recall that if $m+1$ is even then m is odd and vice versa.

n \ m+1	even	odd
even	full/partial	full/partial
odd	full/partial	only partial

Table 1: *Possible types of tie in the Borda method*

weakly prefer a to b is equal to the number of the deciders who weakly prefer b to a.
A direct tie is said to be **full** if it involves all the alternatives of the set A other wise it is termed **partial**.
Indirect ties, on the other hand, derive from the presence of cycles, from the transitivity of the $\succeq$ preference relation and from the definition of the $\sim$ relation.
More concretely if we have the set $A = \{a, b, c, d\}$ we can have $a \succeq b \succeq c \succeq a$ so we have a tie among the alternatives a, b and c and we call such a tie as partial since is derives from a partial cycle or a cycle not involving all the alternatives.
The possible cycles can indeed be seen as:

- **full**,

- **partial**.

A **full cycle** is a cycle that involves all the m alternatives $a_i \in A$ and can be expressed as:

$$a_1 \succeq a_2 \succeq \cdots \succeq a_m \succeq a_1 \tag{42}$$

A **partial cycle**, on the other hand, involves only a subset of the alternatives.
If, for instance, we have $A = \{a, b, c\}$ we can have $a \succeq b \succeq a$ as well as $a \succeq c$ and $b \succeq c$.
Given a set of the m alternatives A and n preference profiles, one for each decider, we note that:

(c1) we can obtain a full cycle through the composition of $2h$ different compensating profiles,

(c2) we can obtain a full cycle through the composition of km different shift one profiles,

(c3) we can obtain a full cycle through the composition of km different shift one profiles and $2h$ different compensating profiles.

13 The conditions of interest

In this section we define so called conditions of interest or the conditions we think are meaningful, then we give the conditions under which any of such conditions can occur and, last but not least, we provide the links between such conditions. To each condition we assign a label that will be used to identify it in the rest of the section.

B_1 : Borda full ties, n even;

B_2 : Borda full ties, n odd & m odd;

B_3 : Borda partial ties;

C_1 : Condorcet full direct ties;

C_2 : Condorcet full indirect ties;

C_3 : Condorcet partial direct ties;

C_4 : Condorcet partial indirect ties.

We note that B_1 and B_2 are equivalent conditions but for what concerns their link with the Condorcet related conditions.
From what we have already seen we can state that also C_1 and C_2 are equivalent conditions but for the conditions under which they can occur and their relations with the Borda related conditions.
As an example of B_3 we can consider the set $A = \{a, b, c\}$ with five deciders with the following preference profiles:

r_1: $a \succeq c \succeq b$

r_2: $b \succeq a \succeq c$

r_3: $a \succeq b \succeq c$

r_4: $b \succeq a \succeq c$

r_5: $c \succeq b \succeq a$

In this case if we apply the Borda method we get $\{a, b, c\} = \{11, 11, 8\}$ or:

r_B: $a \sim b \succeq c$

whereas if we use the Condorcet method we get:

r_C: $b \succeq a \succeq c$

As an example of C_3 we can use the following global ranking:

r_C: $a \sim b \succeq c \sim d$

whereas as an example of C_4 we can use the following global rankings among the alternatives of the set $A = \{a, b, c, d\}$:

r_C: $a \succeq b \succeq c \succeq a$

r_C: $a \succeq d$

r_C: $c \succeq d$

r_C: $d \succeq b$

We recall that a **Borda full tie** can be defined as $\forall\, x, y \in A\ v_x = v_y$. It can occur in two cases:

B_1 : if n is even and for any m;

B_2 : if n is odd and m is odd.

A **Borda partial tie** (B_3) is defined through the following conditions:

- $\exists A_i$ such that $\cup_i A_i = A$ and $A_i \cap A_j = \emptyset$ for any $i \neq j$;
- $\forall\, x, y \in A_i\ v_x = v_y$

A **Borda partial tie** can occur for any values of n and m.
A **Condorcet direct tie** can occur if $\exists x, y \in A$ such that $v_{xy} = v_{yx}$.
We define a **Condorcet full direct tie** (C_1) if $\forall x, y \in A$ we have $v_{xy} = v_{yx}$ whereas we have a **Condorcet partial direct tie** (C_3) if such condition holds for a proper subset of the set A.
On the other hand we define a **Condorcet full indirect tie** (C_2) if we have a cycle among all the alternatives whereas we have a **Condorcet partial indirect tie** (C_4) if such condition holds for a proper subset of the set A.
We note that C_1 can occur if n is even and for any m. If we can express n as $km + 2h$, so that we compose k distinct shift profiles and h compensating profiles, we have that C_2 is verified though the converse does not necessarily hold. It is easy to see this if we consider that we can get the same result if we compose k distinct shift profiles and a certain number of profiles that are in agreement with the cycle that is defined by the shift profiles.
Last but not least C_3 can occur only if n is even and for any value of m whereas C_4 can occur for any values of n and m.
We note how the conditions on the Condorcet method do not depend in any case from the value of m.
As an example we can consider the set $A = \{a, b, c, d\}$ with four deciders and the following preference profiles:

r_1: $a \succeq b \succeq c \succeq d$

r_2: $d \succeq a \succeq b \succeq c$

r_3: $c \succeq d \succeq a \succeq b$

r_4: $b \succeq c \succeq d \succeq a$

In this case if we apply the Borda method we get:

r_B: $a \sim b \sim c \sim d$

(since every alternative gets 10 points) whereas if we use the Condorcet method we get:

r_C: $a \succeq b \succeq c \succeq d \succeq a$

If we consider the set $A = \{a, b, c\}$ with three deciders and the following preference profiles:

r_1: $a \succeq b \succeq c$

r_2: $c \succeq a \succeq b$

r_3: $b \succeq c \succeq a$

if we apply the Borda method we get:

r_B: $a \sim b \sim c$

(since every alternative gets 60 points) whereas if we use the Condorcet method we get:

r_C: $a \succeq b \succeq c \succeq a$

We now try to detail the conditions under which C_1 can occur. We have already seen that a sufficient condition is that n is even since we have that if n is even then C_1 can occur whereas if n is odd then we cannot have a full direct tie among the alternatives.
We can state that:

- if we compose pairs of compensating profiles we surely have that the condition C_1 is satisfied.

To see this we note that, in each pair of compensating profiles, for each pair of alternatives $x, y \in A$ if we have $x \succeq y$ in one profile we have $y \succeq x$ in the other so that we necessarily have $v_{xy} = v_{yx}$ in the global profile.
On the other hand we wish to verify that:

- if the condition C_1 is satisfied then we have $n/2$ pairs of compensating profiles.

We can argue as follows. If C_1 is satisfied then we have that for any $x, y \in A$ we have $v_{xy} = v_{yx}$ so that for any profile where we have $x \succeq y$ we must have a profile where $y \succeq x$. Since this condition must hold for any pair of alternatives it turns in the definition of pairs of compensating profiles.
For what concerns the condition C_2 we have that if we define:

$$n_g = km + 2h \tag{43}$$

(with $k \neq 0$, $k, h \in \mathbb{N}$) then if we have $n = n_g$ then the condition C_2 can be satisfied though the converse is false.
This can be easily seen if we consider that we can have a full indirect tie if we use, for instance, m shift profiles (that produce a cycle) and a certain number of profiles that are in accordance with the cycle but do not form pairs of compensating profiles.
For instance if we have the set $A = \{a, b, c\}$ we can use the three shift profiles $a \succeq b \succeq c$, $c \succeq a \succeq b$ and $b \succeq c \succeq a$ (that form the cycle $a \succeq b \succeq c \succeq a$) together

with two profiles of the form $a \succeq b \succeq c$ and one of the form $b \succeq c \succeq a$ so without using any compensating profiles.
We recall that k counts the number of groups of distinct shift profiles (that are m) whereas h denotes the number of pairs of compensating profiles.
For what concerns B_3, on one side, and C_3 and C_4 on the other we cannot draw any general conclusion since we can find examples where we have partial ties according to the Borda method but a transitive ordering according to the Condorcet method as well as cases where the presence of partial ties in one method turns into similar partial ties in the other.
Anyway, in order to compare B_3 with C_3 we must consider n even otherwise C_3 cannot occur.
As an example we can consider the set $A = \{a, b, c, d\}$ with four deciders and the following preference profiles:

r_1: $a \succeq b \succeq c \succeq d$

r_2: $b \succeq a \succeq c \succeq d$

r_3: $d \succeq a \succeq b \succeq c$

r_4: $c \succeq b \succeq a \succeq d$

In this case if we apply the Borda method we get:

r_B: $a \sim b \succeq c \succeq d$

(since we get $\{a, b, c, d\} = \{12, 12, 9, 7\}$) and if we use the Condorcet method we get:

r_C: $a \sim b \succeq c \succeq d$

At this point we can introduce some relations between the Borda related conditions and the Condorcet related conditions, relations that can be justified, if not proved, by the considerations we have made so far. We argue that:

(A_1) $B_1 \Rightarrow C_1$ or C_2 depending on both n and m;

(A_2) $B_2 \Rightarrow C_2$;

(A_3) $C_1 \Rightarrow B_1$;

(A_4) $C_2 \not\Rightarrow B_1$;

(A_5) $C_2 \not\Rightarrow B_2$.

We note how we disregard relations between partial conditions since there is no general rule we can derive and, moreover, such conditions are less harmful and less significant in the context of the method we present in section 14.
We moreover note how we do not aim at establishing conditions of necessity or of sufficiency but rather conditions of possibility. To understand what this means we make an example. In the case (A_1), for instance, we aim at verifying if,

from a condition B_1, we can derive either a condition C_1 or a condition C_2 (but not both, obviously) and not that we necessarily derive one of such conditions (anyway which?).
Of course the cases (A_4) and (A_5) are a little bit different since in such cases we show how a relation does not necessarily hold by means of counter examples.
We now start with the various conditions.
To show (A_4) we use the following example. We consider the set $A = \{a, b, c\}$ with eight deciders with the following preference profiles:

r_1: $a \succeq b \succeq c$

r_2: $c \succeq a \succeq b$

r_3: $b \succeq c \succeq a$

where three deciders have the profile r_1, three deciders have the profile r_2 and two deciders have the profile r_3. In this case we have that n is even and so we are in a condition where B_1 can occur.
In this case if we apply the Borda method we get:

r_B: $a \succeq c \succeq b$

(since we get $\{a, b, c\} = \{17, 15, 16\}$) but if we use the Condorcet method we get:

r_C: $a \sim b \sim c \sim a$

On the other hand, to show (A_5) we use the following example. We consider the set $A = \{a, b, c\}$ with seven deciders with the following preference profiles:

r_1: $a \succeq b \succeq c$

r_2: $c \succeq a \succeq b$

r_3: $b \succeq c \succeq a$

where three deciders have the profile r_1, two deciders have the profile r_2 and two deciders have the profile r_3. In this case we have that n and m are odd and so we are in a condition where B_2 can occur.
In this case if we apply the Borda method we get:

r_B: $a \succeq b \succeq c$

(since we get $\{a, b, c\} = \{15, 14, 13\}$) but if we use the Condorcet method we get:

r_C: $a \sim b \sim c \sim a$

We have so shown with two examples that the presence of a Condorcet full indirect tie (and so C_2) does not produce necessarily a Borda full tie both in the case of n even (B_1) and in the case of n odd and m odd (B_2). We note that the difference between B_1 and B_2 is meaningful only in the cases (A_1) and (A_2).

We now want to argue about A_3 ($C_1 \Rightarrow B_1$) so that a full direct tie in Condorcet implies a Borda full tie with n even.
We note that C_1 is defined as: for any $x, y \in A$ we have $v_{xy} = v_{yx}$. If we have a decider who has $x \succeq y$ we need a decider with $y \succeq x$ and this must hold for any pair of alternatives. From this we derive that n must be even.
Since the condition $v_{xy} = v_{yx}$ must hold for any pair of alternatives and n is even necessarily we must have pairs of compensating profiles so that B_1 holds directly from the fact that two compensating profiles assign to each alternative the same sum of Borda scores.
We note indeed that we have $x \sim y$ iff $v_{xy} = v_{yx}$ and such relations hold for any pair $x, y \in A$ and must also satisfy transitivity (since both $\sim$ and $=$ are transitive relations) so that given a profile we necessarily get the corresponding compensating profile.
For instance if we have $a \sim b \sim c$ we obtain $a \sim b$ (or $v_{ab} = v_{ba}$), $b \sim c$ (or $v_{bc} = v_{cb}$) and $a \sim c$ (or $v_{ac} = v_{ca}$). In this case for any $a \succeq b$ condition we need a $b \succeq a$ condition for any pair of alternatives. The only possibility is through a compensating profile.
We now examine the case (A_1). In this case we have that n is even so that we can have a Borda full tie for any m.
In this case if we obtain B_1 from the combination of h pairs of compensating profiles so that $n = 2h$ we easily obtain C_1.
If, on the other hand, we obtain B_1 from the composition of:

- k groups of m shift profiles,
- h pairs of compensating profiles,

(so that we have $n = km + 2h$) we easily obtain C_2 and so a full indirect tie among the alternatives.
We note that the fact that n must be even imposes that km must be even so that we have the following cases:

- k even and any m,
- m even and any k.

As an example we can consider the set $A = \{a, b, c, d\}$ with six deciders and the following preference profiles (where we have $k = 1$ and $h = 1$):

r_1: $a \succeq b \succeq c \succeq d$

r_2: $d \succeq a \succeq b \succeq c$

r_3: $c \succeq d \succeq a \succeq b$

r_4: $b \succeq c \succeq d \succeq a$

r_5: $a \succeq b \succeq c \succeq d$

r_6: $d \succeq c \succeq b \succeq a$

We note that the first four profiles are shift profiles whereas the last two profiles are compensating profiles.
In this case if we apply the Borda method we get:

r_B: $a \sim b \sim c \sim d$

(since we get $\{a, b, c, d\} = \{15, 15, 15, 15\}$) and if we use the Condorcet method we get:

r_C: $a \succeq b \succeq c \succeq d \succeq a$

We now examine the case (A_2). In this case we have that n is odd so that we can have a Borda full tie only if m is odd.
In this case we can obtain B_1 from the composition of:

- k groups of m shift profiles,

- h pairs of compensating profiles,

so that we have $n = km + 2h$ and we easily obtain C_2 and so a full indirect tie among the alternatives.
Since n is odd and m is odd we have that that km must be odd so also k must be odd.
As a closing remark we note what follows.
In the case (A_1) we can define the following conditions:

- we can express n as $2h$,

- we can express n as $km + 2h$

If n is even such conditions imply B_1 and also imply either C_1 (the former) or C_2 (the latter) but from B_1 we do not derive either of such conditions. The idea is simple: we produce admissible profiles that give us a desired condition from which we derive a wished condition. The remaining step, or that B_1 implies either the former or the latter of such conditions, is still to be done though, probably, it cannot be done.
We note, for instance, that if we have $n = km + 2h$ (with $k \neq 0$ and $h \neq 0$ or $k \neq 0$ and $h = 0$) we have C_2 but the converse is not necessarily true. We can show this with the following example.
We can consider the set $A = \{a, b, c\}$ with seven deciders and the following preference profiles:

r_1: $a \succeq b \succeq c$

r_2: $c \succeq b \succeq a$

r_3: $b \succeq c \succeq a$

where three deciders have the profile r_1, two deciders have the profile r_2 and two deciders have the profile r_3. In this case we have C_2 since we get:

r_C: $a \succeq b \succeq c \succeq a$

but n cannot be expressed as $km+2h$ since we have two groups of 3 shift profiles (so $k=2$) and an added profile ($a \succeq b \succeq c$) in accordance with the cycle but not any pair of compensating profiles.
As another case we have that if we can express n as $2h$ or if we get a global profile with a full tie by composing pairs of compensating profiles we get both B_1 and C_1. On the other hand B_1 does not imply $n=2h$ since we can have at least also $n=km+2h$ (with $k \neq 0$ and $h \neq 0$) whereas, how we have already seen, from C_1 we get $n=2h$.
We close this section with the following example.
We can consider the set $A=\{a,b,c,d\}$ with four deciders and the following preference profiles:

r_1: $a \succeq c \succeq d \succeq b$

r_2: $b \succeq a \succeq c \succeq d$

r_3: $d \succeq b \succeq a \succeq c$

r_4: $c \succeq d \succeq b \succeq a$

In this case if we apply the Borda method we get:

r_B: $a \sim b \sim c \sim d$

(since we get $\{a,b,c,d\}=\{10,10,10,10\}$) but if we use the Condorcet method we get:

r_C: $a \succeq b \succeq c \succeq d \succeq a$

together with $a \sim d$ and $b \sim c$ that derive from the former profile.

14 A proposal: a compound method

Now that we have examined in some detail both the Borda and the Condorcet methods we can try to make a merge of such methods so to define a new composed method that can get the best features from both methods and, possibly, avoid (this fact must be proved yet) the drawbacks of the Arrow's impossibility theorem.
Given a set of n total and without ties preference profiles $p_1,\dots,p_n$ the proposed method is based on the following steps:

(1) we apply the Borda method and get a global ranking r_B;

(2) we apply the Condorcet method and get a global ranking r_C;

(3) if the two rankings coincide (so $r_B=r_C$) the procedure is over;

(4) if the two rankings differ (so $r_B \neq r_C$) we firstly consider r_C since we assume that r_B may be vitiated by strategic voting;

(5) if the Condorcet method succeeds then the procedure is over and r_C is the final global ranking;

(6) if the Condorcet method fails we use r_B though it can contain ties (but no cycles).

We say that, in the present context, the Condorcet method **succeeds** if it does not fail and **fails** if r_C contains indirect ties. In this case we abandon it since we want to avoid any "garbage out" effect and aim at getting a total order though possibly with ties but without cycles.
From what we have seen in the previous sections we can argue that the presences of a full cycle in the Condorcet method does not necessarily imply the presence of ties in the Borda method.
We underline that also r_B can contain ties as either a full tie or a partial tie but such ties are assumed to be of less harm in this case and at this level of decision.
If we have a full tie we can choose one of the alternatives at random since they are equivalent, at least according to the given preference profiles.
If, on the other hand, we have partial ties they are even less harmful since they, in the worse case, involve top ranked alternatives as in the full tie case. If, on the other hand, partial ties involve low ranked alternatives they can be discarded with no harm since they affect alternatives that the deciders are not going to select.
In this way we can define a compound voting method that can gather the best features from both the Borda and the Condorcet methods without possibly (this fact has not been proved in this TR) suffering the drawbacks of Arrow's impossibility theorem.

15 Conclusions

The present TR has presented a detailed analysis of both the Borda method and the Condorcet method so to understand their properties, their relations and their weaknesses. The examples we have presented are mainly toy examples but can be easily generalized. Future plans include a stronger and more accurate formalization of the two methods and their relations as well as a formal analysis of the proposed composed method to analyze its properties and to see if it can represent a loophole from the limitations imposed by the Arrow's impossibility theorem.

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