Brane Calculi

Introduction

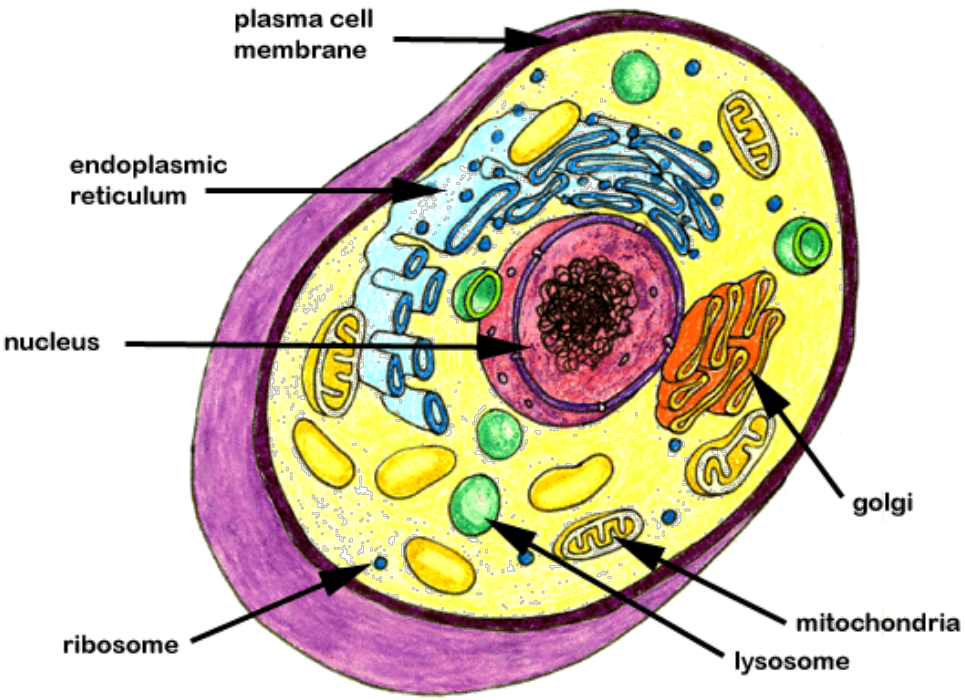
- A biological cellular membrane is an closed surface that can perform various molecular functions.
- Membranes are not just containers: they are coordinators and active sites of major activity.

  > *"For a cell to function properly, each of its numerous proteins must be localized to the correct cellular membrane or aqueous compartment."*

- Proteins may be embedded in membranes, and can act on both sides of the membrane simultaneously.

Introduction

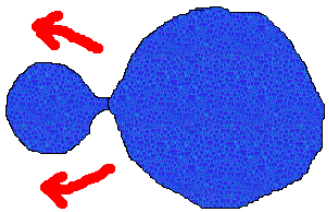
Cells contain a huge number of membranes...



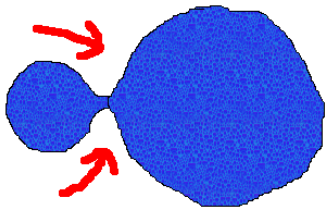
...and they are not static!

Introduction

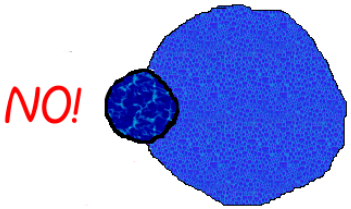
For example, it is possible for a membrane to gradually create a bubble that then detaches,



or for a bubble to merge with a membrane,



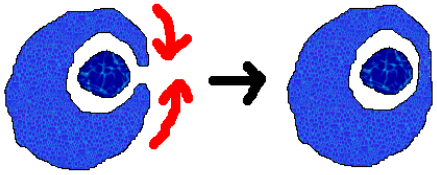
but it is not possible for a bubble to "jump across" a membrane (only small molecules can do that).



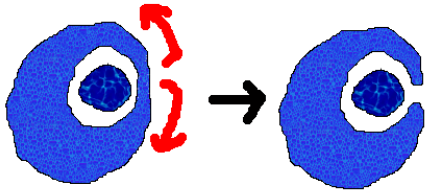
Introduction

New bubbles can appear autonomously (usually emtpy) or dissolve.

Finally, a membrane can engulf a bubble (this is called phagocytosis)



and a bubble can be exected (exocytosis)



Brane Calculi

- Brane Calculi are a family of process calculi for describing membrane interactions.
  - PEP Calculus
  - MBD Calculus
  - Other calculi including molecules
  - ...
- The idea is to have terms describing the membrane structures, and to have a *process* on each membrane describing what the membrane could do
- Processes are based on *actions*
- Two membranes can interact (e.g. merge) if they have two corresponding actions on their processes.

The Syntax of Brane Calculi

Syntax

Systems	$P,Q ::= \diamond \mid P \circ Q \mid !P \mid \sigma(\!\mid P \mid\!)$	nests of membranes
Branes	$\sigma,\tau ::= 0 \mid \sigma \mid \tau \mid !\sigma \mid a.\sigma$	combinations of actions
Actions	$a,b ::= \ldots$	(detailed later)

Systems $P, Q, \ldots$ describe the membrane structures

- $\diamond$ is the empty membrane
- $\circ$ is the juxtaposition (parallel composition) of membranes
- ! represents moltitudes of (equivalent) membranes
- $\sigma(\!\mid P \mid\!)$ denotes a single membrane, whose associated process (brane) is σ and which contains system P

The Syntax of Brane Calculi

Syntax

Systems	$P,Q ::= \diamond \mid P \circ Q \mid !P \mid \sigma(\!\mid P \mid\!)$	nests of membranes
Branes	$\sigma,\tau ::= 0 \mid \sigma \mid \tau \mid !\sigma \mid a.\sigma$	combinations of actions
Actions	$a,b ::= \ldots$	(detailed later)

Branes $\sigma, \tau, \ldots$ describe a process of a membrane

- **0** is the empty brane (does nothing)
- $|$ is the parallel composition of branes
- ! is process replication (describes recursion)
- $a.\sigma$ is sequential composition of actions (action prefixing)

A membrane $\sigma(\!\mid \mathbf{P} \mid\!)$
with actions σ and contents **P**

The Structural Congruence Relation

Structural Congruence

$P \circ Q \equiv Q \circ P$	$\sigma \mid \tau \equiv \tau \mid \sigma$
$P \circ (Q \circ R) \equiv (P \circ Q) \circ R$	$\sigma \mid (\tau \mid \rho) \equiv (\sigma \mid \tau) \mid \rho$
$P \circ \diamond \equiv P$	$\sigma \mid 0 \equiv \sigma$
$! \diamond \equiv \diamond$	$!0 \equiv 0$
$!(P \circ Q) \equiv !P \circ !Q$	$!(\sigma \mid \tau) \equiv !\sigma \mid !\tau$
$!!P \equiv !P$	$!!\sigma \equiv !\sigma$
$!P \equiv P \circ !P$	$!\sigma \equiv \sigma \mid !\sigma$
$0 \llparenthesis \diamond \rrparenthesis \equiv \diamond$	
$P \equiv Q \Rightarrow P \circ R \equiv Q \circ R$	$\sigma \equiv \tau \Rightarrow \sigma \mid \rho \equiv \tau \mid \rho$
$P \equiv Q \Rightarrow !P \equiv !Q$	$\sigma \equiv \tau \Rightarrow !\sigma \equiv !\tau$
$P \equiv Q \wedge \sigma \equiv \tau \Rightarrow \sigma \llparenthesis P \rrparenthesis \equiv \tau \llparenthesis Q \rrparenthesis$	$\sigma \equiv \tau \Rightarrow a.\sigma \equiv a.\tau$

The PEP Calculus

Different calculi of the Brane Calculi family can be obtained by considering different sets of actions.

The first calculus we consider has **P**hagocytosis, **E**xocytosis and **P**inocytosis as possible actions

- It is called PEP Calculus

Phagocytosis: a membrane engulfs another one
Exocytosis: a membrane expels another one
Pinocytosis: a membrane engulfs nothing (it creates a bubble inside itself)

In the first two there are two membranes which interact, while in the last one there is one only membrane performing the action

The PEP Calculus: Phagocytosis

Bitonal Actions

Actions	$a ::= \ldots \mid \circlearrowleft_n \mid \circlearrowleft^{\perp}_n(\sigma) \mid \circlearrowleft_n \mid \circlearrowleft^{\perp}_n \mid \circledcirc(\sigma)$	phago $\circlearrowleft$, exo $\circlearrowleft$, pino $\circledcirc$

Phagocytosis:

- $\circlearrowleft_n$ is the action executed by the membrane which is engulfed
- $\circlearrowleft^{\perp}_n(\sigma)$ is the corresponding action (co–action) executed by the membrane which engulfs the other one
- n is the "communication channel", the two membrane can perform the two actions if they use the same n
- σ is the process that will be associated with the bubble surrounding the engulfed membrane

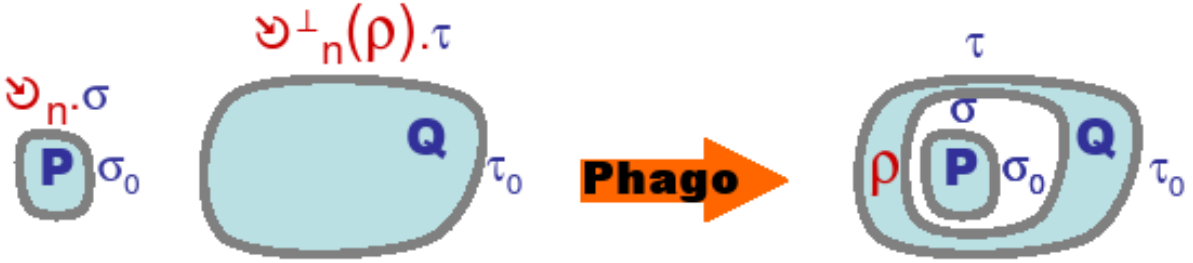
The PEP Calculus: Phagocytosis

Bitonal Actions

Actions	$a ::= \ldots \mid \mathfrak{Ꝑ}_n \mid \mathfrak{Ꝑ}^{\perp}_n(\sigma) \mid \mathfrak{Ꝑ}_n \mid \mathfrak{Ꝑ}^{\perp}_n \mid \circledcirc(\sigma)$	phago $\mathfrak{Ꝑ}$, exo $\mathfrak{Ꝑ}$, pino $\circledcirc$

Phagocytosis, semantics:

$$\mathfrak{Ꝑ}_n.\sigma|\sigma_0(\!|P|\!) \circ \mathfrak{Ꝑ}^{\perp}_n(\rho).\tau|\tau_0(\!|Q|\!) \rightarrow \tau|\tau_0(\!|\rho(\!|\sigma|\sigma_0(\!|P|\!)|\!) \circ Q|\!)$$



The PEP Calculus: Exocytosis

Bitonal Actions

Actions	$a ::= \ldots \mid \mathfrak{O}_n \mid \mathfrak{O}^{\perp}{}_n(\sigma) \mid \mathfrak{O}_n \mid \mathfrak{O}^{\perp}{}_n \mid \circledcirc(\sigma)$	phago $\mathfrak{O}$, exo $\mathfrak{O}$, pino $\circledcirc$

Exocytosis:

- $\mathfrak{O}_n$ is the action executed by the membrane which is ejected
- $\mathfrak{O}_n^{\perp}$ is the corresponding action (co–action) executed by the membrane which ejects the other one
- n is the "communication channel", the two membrane can perform the two actions if they use the same n

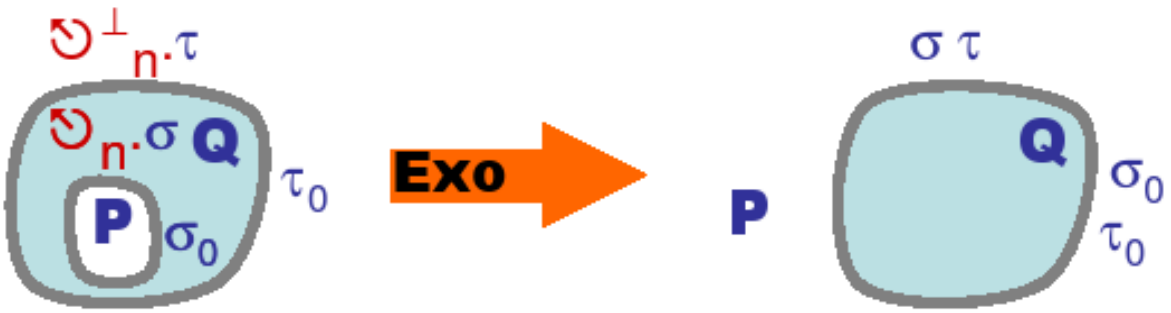
The PEP Calculus: Exocytosis

Bitonal Actions

Actions	$a ::= \ldots \mid \circlearrowleft_n \mid \circlearrowleft^{\perp}_n(\sigma) \mid \circlearrowleft_n \mid \circlearrowleft^{\perp}_n \mid \circledcirc(\sigma)$	phago $\circlearrowleft$, exo $\circlearrowleft$, pino $\circledcirc$

Exocytosis, semantics:

$$\circlearrowleft^{\perp}_n.\tau|\tau_0(|\circlearrowleft_n.\sigma|\sigma_0(|P|) \circ Q|) \rightarrow P \circ \sigma|\sigma_0|\tau|\tau_0(|Q|)$$



The PEP Calculus: Pinocytosis

Bitonal Actions

Actions	$a ::= \ldots \mid \circlearrowleft_n \mid \circlearrowleft^{\perp}_n(\sigma) \mid \circlearrowleft_n \mid \circlearrowleft^{\perp}_n \mid \circledcirc(\sigma)$	phago $\circlearrowleft$, exo $\circlearrowleft$, pino $\circledcirc$

Pinocytosis:

- $\circledcirc(\sigma)$ is the action executed by the membrane which is create a bubble
- σ is the process that will be associated with the new bubble

Semantics:

$$\circledcirc(\rho).\sigma|\sigma_0(\!|P|\!) \rightarrow \sigma|\sigma_0(\!|\rho(\!|\diamond|\!) \circ P|\!)$$

The PEP Calculus

Bitonal Actions

Actions	$a ::= \ldots \mid \circlearrowright_n \mid \circlearrowright^{\perp}{}_n(\sigma) \mid \circlearrowleft_n \mid \circlearrowleft^{\perp}{}_n \mid \circledcirc(\sigma)$	phago $\circlearrowright$, exo $\circlearrowleft$, pino $\circledcirc$

Actions of the PEP Calculus are called *bitonal actions* because they preserve bitonality of membrane contents

- assume that nested membranes are colored by alternating two colors,
- after phagocytosis, exocytosis and pinocytosis, colors are still alternating

The PEP Calculus is Turing Complete!

The PEP Calculus: Semantics

Summing up, the semantics of the PEP Calculus is the least transition relation satisfying the following axioms

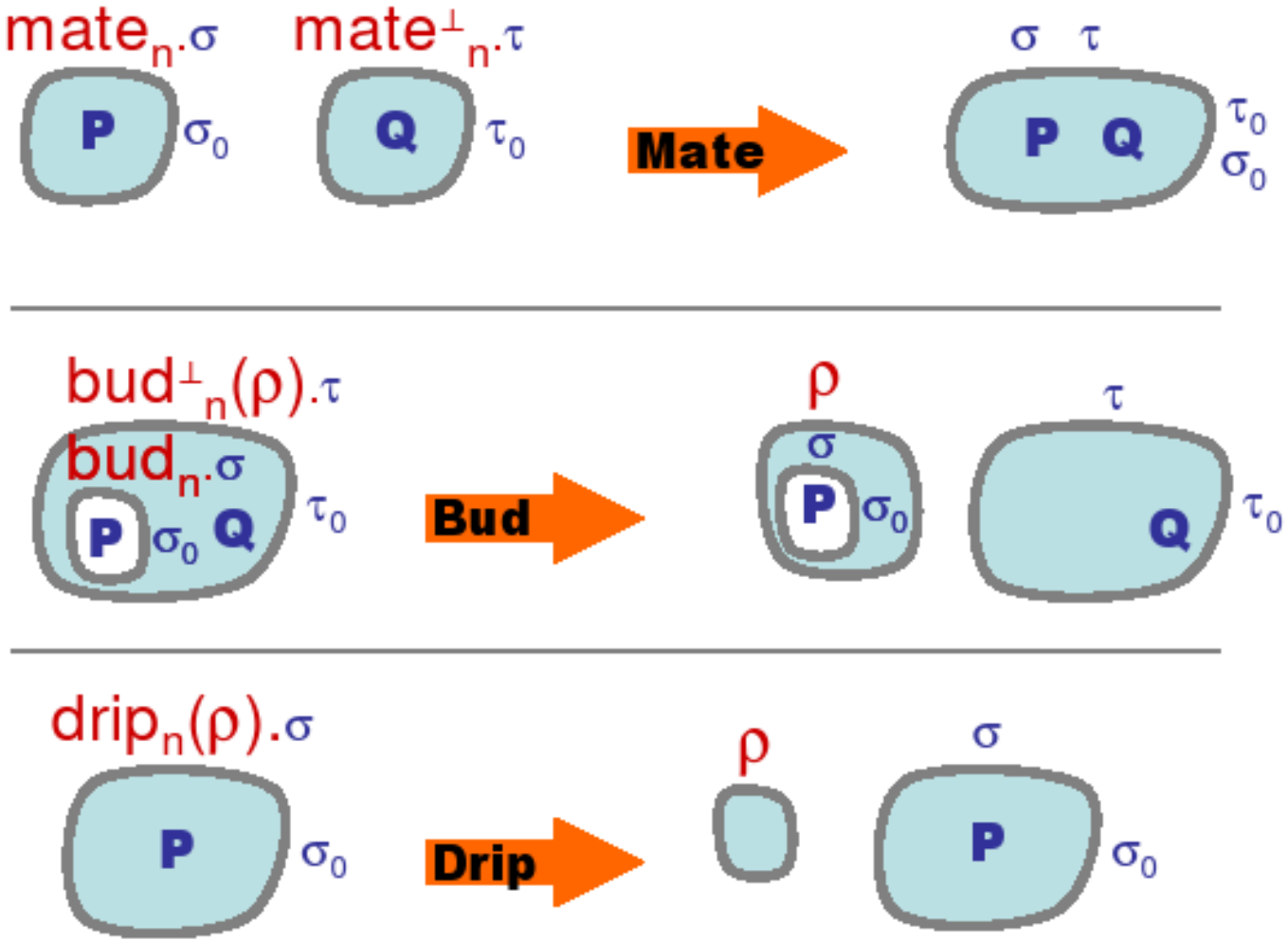
$$\text{(phago)} \quad \circlearrowleft_n.\sigma|\sigma_0(\!|P|\!) \circ \circlearrowleft_n^{\perp}(\rho).\tau|\tau_0(\!|Q|\!) \rightarrow \tau|\tau_0(\!|\rho(\!|\sigma|\sigma_0(\!|P|\!)|\!) \circ Q|\!)$$

$$\text{(exo)} \quad \circlearrowleft_n^{\perp}.\tau|\tau_0(\!|\circlearrowleft_n.\sigma|\sigma_0(\!|P|\!) \circ Q|\!) \rightarrow P \circ \sigma|\sigma_0|\tau|\tau_0(\!|Q|\!)$$

$$\text{(pino)} \quad \circledcirc\,(\rho).\sigma|\sigma_0(\!|P|\!) \rightarrow \sigma|\sigma_0(\!|\rho(\!|\diamond|\!) \circ P|\!)$$

and closed wrt $\_ \circ P$, $\sigma(\!|\_|\!)$ and $\equiv$

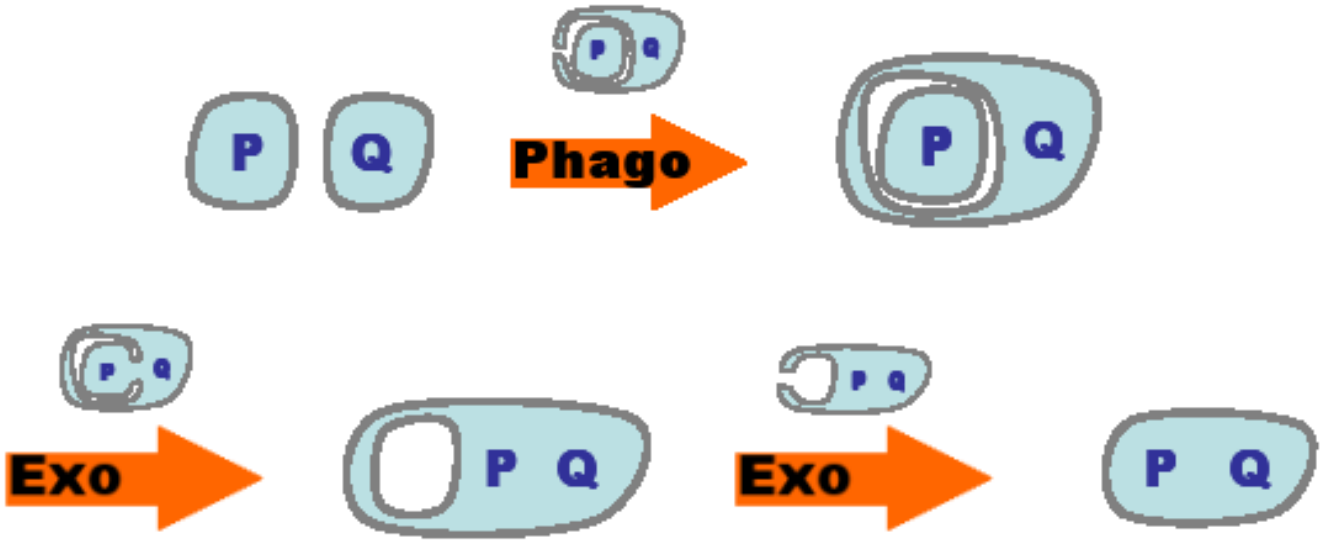
The MBD Calculus



The MBD Calculus

Actions of the MBD Calculus can be translated into sequences of actions of the PEP Calculus

Example: the mate action:

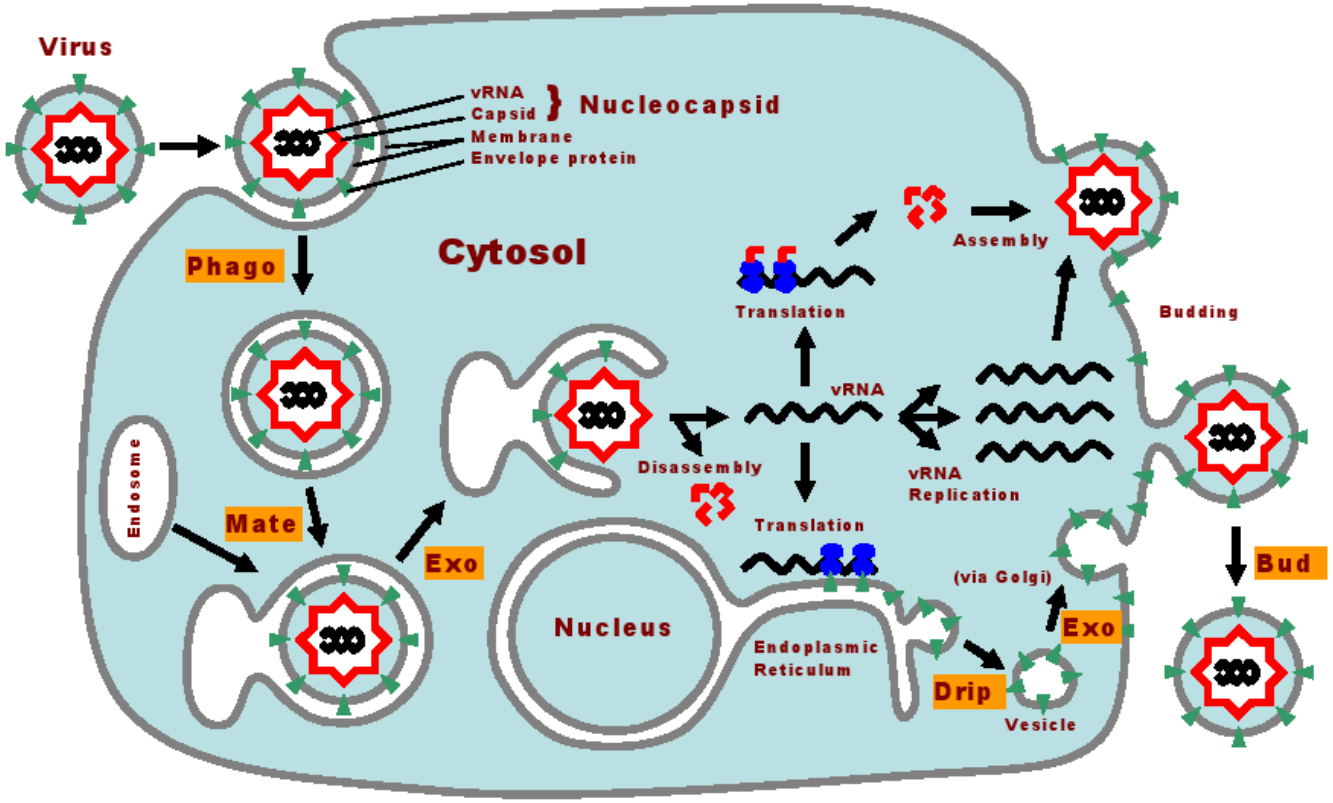


$$\mathrm{mate}_n.\sigma \triangleq \mathfrak{o}_n.\mathfrak{o}_{n'}.\sigma$$

$$\mathrm{mate}^{\perp}{}_n.\tau \triangleq \mathfrak{o}^{\perp}{}_n(\mathfrak{o}^{\perp}{}_{n'}.\mathfrak{o}_{n''}).\mathfrak{o}^{\perp}{}_{n''}.\tau$$

Example: Virus Infection

By using Brane Calculi we can describe a virus infection process.



Some steps of the process (e.g. vRNA replication) require actions we have not seen.

Translation of Brane Calculi into CLS

Brane Calculi can be translated into CLS

- this shows the expressiveness of CLS
- a simulator based on CLS can be used to simulate Brane Calculi

We define a bisimulation relation for Brane Calculi

- bisimilarity is preserved by the translation

We consider only the PEP Calculus, but the translation of other Brane Calculi is similar.

The Calculus of Looping Sequences (CLS)

We assume an alphabet $\mathcal{E}$. **Terms** T and **Sequences** S of CLS are given by the following grammar:

$$\begin{array}{lcl} T & ::= & S \quad | \quad (S)^L \rfloor T \quad | \quad T \mid T \\ S & ::= & \epsilon \quad | \quad a \quad | \quad S \cdot S \end{array}$$

where a is a generic element of $\mathcal{E}$, and ϵ is the empty sequence.

The operators are:

$S \cdot S$	:	Sequencing
$(S)^L$	:	Looping (S is closed and it can rotate)
$T_1 \rfloor T_2$	:	Containment (T_1 contains T_2)
$T \mid T$	:	Parallel composition (juxtaposition)

Actually, looping and containment form a single binary operator $(S)^L \rfloor T$.

Structural Congruence

The **Structural Congruence** relations $\equiv_S$ and $\equiv_T$ are the least congruence relations on sequences and on terms, respectively, satisfying the following rules:

$$S_1 \cdot (S_2 \cdot S_3) \equiv_S (S_1 \cdot S_2) \cdot S_3 \qquad S \cdot \epsilon \equiv_S \epsilon \cdot S \equiv_S S$$

$$T_1 \mid T_2 \equiv_T T_2 \mid T_1 \qquad T_1 \mid (T_2 \mid T_3) \equiv_T (T_1 \mid T_2) \mid T_3$$

$$T \mid \epsilon \equiv_T T \quad (\epsilon)^L \rfloor \epsilon \equiv_T \epsilon \quad (S_1 \cdot S_2)^L \rfloor T \equiv_T (S_2 \cdot S_1)^L \rfloor T$$

We write $\equiv$ for $\equiv_T$.

Dinamics of the Calculus (1)

Let $\mathcal{T}_{\mathcal{V}}$ be the set of terms which may contain variables of three kinds:

- term variables $(X, Y, Z, \ldots)$
- sequence variables $(\widetilde{x}, \widetilde{y}, \widetilde{z}, \ldots)$
- element variables $(x, y, z, \ldots)$

$T\sigma$ denotes the term obtained by replacing any variable in T with the corresponding term, sequence or element.

A **Rewrite Rule** is a pair (T, T'), denoted $T \mapsto T'$, where:

- $T, T' \in \mathcal{T}_{\mathcal{V}}$
- variables in T' are a subset of those in T

A rule $T \mapsto T'$ can be applied to all terms $T\sigma$.

Example: $a \cdot x \cdot a \mapsto b \cdot x \cdot b$

- can be applied to $a \cdot c \cdot a$ (producing $b \cdot c \cdot b$)
- cannot be applied to $a \cdot c \cdot c \cdot a$

The Encoding Function

We translate systems of the PEP calculus into CLS terms.

- We define an encoding function $\{\![\_]\!\}$ that takes a PEP system and results in a pair of a CLS sequence and a set of alphabet symbols.
- $\{\![\_]\!\} : PEP \rightarrow \mathcal{T} \times \mathcal{P}(\mathcal{E})$

Operators and actions of the encoded system are translated into elements of the sequence.

- as regards systems: $\diamond$,$\_ \circ \_$, $!\_$ and $\_(|\_|)$ are translated into $\mathbf{0}$,*circ*,*bangS* and *brane*, respectively
- as regards branes: $\mathbf{0}$,$\_|\_$, and $!\_$ are translated into $\mathbf{0}$,*par* and *bangB*, respectively
- Phagocytosis and exocytosis actions are translated into sequences of two elements, namely $\mathfrak{D}_n$, $\mathfrak{D}_n^{\perp}$, $\mathfrak{S}_n$ and $\mathfrak{S}_n^{\perp}$ are translated into $\mathfrak{D} \cdot n$, $\mathfrak{D}^{\perp} \cdot n$, $\mathfrak{S} \cdot n$ and $\mathfrak{S}^{\perp} \cdot n$, respectively
- pinocytosis $\circledcirc$ is translated into $\circledcirc$
- $\mathbf{0}$, *circ*, *bangS*, *brane*, *par*, *bangB*, $\mathfrak{D}$, $\mathfrak{D}^{\perp}$, $\mathfrak{S}$, $\circledcirc$, $n \in \mathcal{E}$

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The encodings of the operands and of the action parameters follow in the sequence the encodings of the corresponding operators and actions, respectively, and are delimited by symbols acting as separators.

- the set of symbols returned by the encoding contains all these separators.

The alphabet symbol *act* is used in the result of the encoding as a program counter:

- during the evolution of the term it preceeds every element which encodes a currently active action.

Example: consider the simple PEP system $\diamond \circ \diamond$

- $\{\![\diamond \circ \diamond]\!\} = (\quad act \cdot circ \cdot a \cdot \mathbf{0} \cdot a \cdot \mathbf{0} \quad , \quad \{a\} \quad)$

The Encoding Function

Definition (Encoding) The translation of a system P of the PEP calculus into CLS is the term $T \in \mathcal{T}$ such that, for some (finite) $E \subset \mathcal{E}$, it holds $\{\![P]\!\} = (T, E)$, where $\{\![\cdot]\!\} : PEP \to \mathcal{T} \times \mathcal{P}(\mathcal{E})$ is given by the following recursive definition:

$$\{\![\diamond]\!\} = \big(act \cdot \mathbf{0}, \varnothing\big)$$

$$\{\![P_1 \circ P_2]\!\} = \big(act \cdot circ \cdot a \cdot P_1'\{^{\epsilon}/act\} \cdot a \cdot P_2'\{^{\epsilon}/act\}, \{a\} \cup E_1 \cup E_2\big)$$
$$\text{where } \{\![P_i]\!\} = (P_i', E_i), E_1 \cap E_2 = \varnothing \text{ and } a \in \mathcal{E} \setminus (E_1 \cup E_2)$$

$$\{\![!P]\!\} = \big(act \cdot bangS \cdot P'\{^{\epsilon}/act\}, E\big) \qquad \text{where } \{\![P]\!\} = (P', E)$$

$$\{\![\sigma(\!| P |\!)]\!\} = \big(act \cdot brane \cdot a \cdot \sigma'\{^{\epsilon}/act\} \cdot a \cdot P'\{^{\epsilon}/act\}, \{a\} \cup E_P \cup E_\sigma\big)$$
$$\text{where } \{\![P]\!\} = (P', E_P), [\![\sigma]\!] = (\sigma', E_\sigma),$$
$$a \in \mathcal{E} \setminus (E_P \cap E_\sigma) \text{ and } E_P \cap E_\sigma = \varnothing$$

The Encoding Function

where $[\![\cdot]\!] : \mathit{Branes} \to \mathcal{T} \times \mathcal{P}(\mathcal{E})$ is given by the following recursive definition:

$$[\![\mathbf{0}]\!] = \big(\mathit{act} \cdot \mathbf{0}, \varnothing\big)$$

$$[\![\sigma_1 | \sigma_2]\!] = \big(\mathit{act} \cdot \mathit{par} \cdot a \cdot \sigma_1'\{^{\epsilon}/\mathit{act}\} \cdot a \cdot \sigma_2'\{^{\epsilon}/\mathit{act}\} \cdot a, E_1 \cup E_2 \cup \{a\}\big)$$
$$\text{where } [\![\sigma_i]\!] = (\sigma_i', E_i), E_1 \cap E_2 = \varnothing \text{ and } a \in \mathcal{E} \setminus (E_1 \cup E_2)$$

$$[\![!\sigma]\!] = \big(\mathit{act} \cdot \mathit{bangB} \cdot a \cdot \sigma'\{^{\epsilon}/\mathit{act}\} \cdot a, E \cup \{a\}\big)$$
$$\text{where } [\![\sigma]\!] = (\sigma', E) \text{ and } a \in \mathcal{E} \setminus E$$

$$[\![\circlearrowleft_n.\sigma]\!] = \big(\mathit{act} \cdot \circlearrowleft \cdot n \cdot a \cdot \sigma'\{^{\epsilon}/\mathit{act}\} \cdot a, E \cup \{a\}\big)$$
$$\text{where } [\![\sigma]\!] = (\sigma', E) \text{ and } a \in \mathcal{E} \setminus E$$

$$[\![\circlearrowleft_n^{\perp}(\rho).\sigma]\!] = \big(\mathit{act} \cdot \circlearrowleft^{\perp} \cdot n \cdot a \cdot \rho'\{^{\epsilon}/\mathit{act}\} \cdot a \cdot \sigma'\{^{\epsilon}/\mathit{act}\} \cdot a, E_\rho \cup E_\sigma \cup \{a\}\big)$$
$$\text{where } [\![\rho]\!] = (\rho', E_\rho), [\![\sigma]\!] = (\sigma', E_\sigma) \text{ and } a \in \mathcal{E} \setminus (E_\rho \cup E_\sigma)$$

The Encoding Function

$$[\![\circlearrowleft_n.\sigma]\!] = \big(act \cdot \circlearrowleft \cdot n \cdot a \cdot \sigma'\{^{\epsilon}/act\} \cdot a, E \cup \{a\}\big)$$
$$\text{where } [\![\sigma]\!] = (\sigma', E) \text{ and } a \in \mathcal{E} \setminus E$$

$$[\![\circlearrowleft_n^{\perp}.\sigma]\!] = \big(act \cdot \circlearrowleft^{\perp} \cdot n \cdot a \cdot \sigma'\{^{\epsilon}/act\} \cdot a, E \cup \{a\}\big)$$
$$\text{where } [\![\sigma]\!] = (\sigma', E) \text{ and } a \in \mathcal{E} \setminus E$$

$$[\![\circledcirc(\rho).\sigma]\!] = \big(act \cdot \circledcirc \cdot a \cdot \rho'\{^{\epsilon}/act\} \cdot a \cdot \sigma'\{^{\epsilon}/act\} \cdot a, E_\rho \cup E_\sigma \cup \{a\}\big)$$
$$\text{where } [\![\rho]\!] = (\rho', E_\rho), [\![\sigma]\!] = (\sigma', E_\sigma) \text{ and } a \in \mathcal{E} \setminus (E_\rho \cup E_\sigma)$$

Example of Translation

Let us consider the PEP system $!P$ where $P = \circlearrowright_n(\!| \diamond |\!) \circ \circlearrowright_n^{\perp}(\mathbf{0})(\!| \diamond |\!)$.

According to the semantics of the PEP calculus the system may evolve as follows:

$$!P \;\equiv\; !P \circ \circlearrowright_n(\!| \diamond |\!) \circ \circlearrowright_n^{\perp}(\mathbf{0})(\!| \diamond |\!) \;\rightarrow\; !P \circ \mathbf{0}(\!|\mathbf{0}(\!|\mathbf{0}(\!| \diamond |\!)|\!) \circ \diamond|\!) \;\equiv\; !P$$

By applying the encoding to the system we obtain the following CLS term:

$$act \cdot bangS \cdot circ \cdot e \cdot brane \cdot b \cdot \circlearrowright \cdot n \cdot a \cdot \mathbf{0} \cdot a \cdot b \cdot \mathbf{0} \cdot e \cdot brane \cdot d \cdot \circlearrowright^{\perp} \cdot n \cdot c \cdot \mathbf{0} \cdot c \cdot \mathbf{0} \cdot c \cdot d \cdot \mathbf{0}$$

How it evolves? We need rewrite rules!

Rewrite Rules for the Translation

We give a set of rewrite rules to be applied to terms obtained by the translation of PEP systems:

- the set of the rewrite rules is the same for any PEP systems
- and it is inspired by the semantics of the PEP Calculus

Rewrite Rules for the Translation

$$\left(act \cdot par \cdot x \cdot \widetilde{y} \cdot x \cdot \widetilde{z} \cdot x \cdot \widetilde{w}\right)^L \rfloor X \;\mapsto\; \left(act \cdot \widetilde{y} \cdot act \cdot \widetilde{z} \cdot \widetilde{w}\right)^L \rfloor X \quad \text{(par)}$$

$$act \cdot circ \cdot x \cdot \widetilde{y} \cdot x \cdot \widetilde{z} \;\mapsto\; act \cdot \widetilde{y} \mid act \cdot \widetilde{z} \quad \text{(circ)}$$

$$act \cdot brane \cdot x \cdot \widetilde{y} \cdot x \cdot \widetilde{z} \;\mapsto\; \left(act \cdot \widetilde{y}\right)^L \rfloor act \cdot \widetilde{z} \quad \text{(brane)}$$

$$x \cdot \widetilde{w} \mid act \cdot \mathbf{0} \;\mapsto\; x \cdot \widetilde{w} \qquad \left(x \cdot \widetilde{w}\right)^L \rfloor act \cdot \mathbf{0} \;\mapsto\; \left(x \cdot \widetilde{w}\right)^L \quad \text{(sc1,2)}$$

$$act \cdot bangS \cdot \mathbf{0} \;\mapsto\; act \cdot \mathbf{0} \qquad \left(act \cdot \mathbf{0}\right)^L \;\mapsto\; act \cdot \mathbf{0} \quad \text{(sc3,4)}$$

$$\left(act \cdot \mathbf{0} \cdot x \cdot \widetilde{w}\right)^L \rfloor X \;\mapsto\; \left(x \cdot \widetilde{w}\right)^L \rfloor X \quad \text{(sc5)}$$

$$\left(act \cdot bangB \cdot \mathbf{0} \cdot \widetilde{w}\right)^L \rfloor X \;\mapsto\; \left(act \cdot \mathbf{0} \cdot \widetilde{w}\right)^L \rfloor X \quad \text{(sc6)}$$

Rewrite Rules for the Translation

$$\left(act \cdot \circlearrowright^{\perp} \cdot x_n \cdot x \cdot \widetilde{y} \cdot x \cdot \widetilde{z} \cdot x \cdot \widetilde{w}\right)^L \rfloor X \mid \left(act \cdot \circlearrowright \cdot x_n \cdot x' \cdot \widetilde{y}' \cdot x' \cdot \widetilde{z}'\right)^L \rfloor Y$$
$$\mapsto \quad \left(act \cdot \widetilde{z} \cdot \widetilde{w}\right)^L \rfloor \left(X \mid \left(act \cdot \widetilde{y}\right)^L \rfloor \left(act \cdot \widetilde{y}' \cdot \widetilde{z}'\right)^L \rfloor Y\right) \qquad \text{(phago)}$$

$$\left(act \cdot \circlearrowleft^{\perp} \cdot x_n \cdot x \cdot \widetilde{y} \cdot x \cdot \widetilde{z}\right)^L \rfloor \left(X \mid \left(act \cdot \circlearrowleft \cdot x_n \cdot x' \cdot \widetilde{y}' \cdot x' \cdot \widetilde{z}'\right)^L \rfloor Y\right)$$
$$\mapsto \quad Y \mid \left(act \cdot \widetilde{y} \cdot \widetilde{z} \cdot act \cdot \widetilde{y}' \cdot \widetilde{z}'\right)^L \rfloor X \qquad \text{(exo)}$$

$$\left(act \cdot \circledcirc \cdot x \cdot \widetilde{y} \cdot x \cdot \widetilde{z} \cdot x \cdot \widetilde{w}\right)^L \rfloor X \quad \mapsto \quad \left(act \cdot \widetilde{z} \cdot \widetilde{w}\right)^L \rfloor \left(X \mid \left(act \cdot \widetilde{y}\right)^L\right) \qquad \text{(pino)}$$

$$act \cdot bangS \cdot \widetilde{x} \quad \mapsto \quad act \cdot bangS \cdot \widetilde{x} \mid act \cdot \widetilde{x} \qquad \text{(bangS)}$$

$$\left(act \cdot bangB \cdot x \cdot \widetilde{y} \cdot x \cdot \widetilde{w}\right)^L \rfloor X$$
$$\mapsto \quad \left(act \cdot bangB \cdot x \cdot \widetilde{y} \cdot x \cdot act \cdot \widetilde{y} \cdot \widetilde{w}\right)^L \rfloor X \qquad \text{(bangB)}$$

Example of Translation

Let's come back to the example: $!P$ where $P = \circlearrowleft_n(\!| \diamond |\!) \circ \circlearrowleft_n^{\perp}(\mathbf{0})(\!| \diamond |\!)$.

According to the semantics of the PEP calculus the system may evolve as follows:

$$!P \;\equiv\; !P \circ \circlearrowleft_n(\!| \diamond |\!) \circ \circlearrowleft_n^{\perp}(\mathbf{0})(\!| \diamond |\!) \;\rightarrow\; !P \circ \mathbf{0}(\!|\mathbf{0}(\!|\mathbf{0}(\!| \diamond |\!)|\!) \circ \diamond|\!) \;\equiv\; !P$$

By applying the encoding to the system we obtain the following CLS term:

$$act \cdot bangS \cdot circ \cdot e \cdot brane \cdot b \cdot \circlearrowleft \cdot n \cdot a \cdot \mathbf{0} \cdot a \cdot b \cdot \mathbf{0} \cdot e \cdot brane \cdot d \cdot \circlearrowleft^{\perp} \cdot n \cdot c \cdot \mathbf{0} \cdot c \cdot \mathbf{0} \cdot c \cdot d \cdot \mathbf{0}$$

which may evolve as follows (let us call it T)....

Example of Translation

$$
\begin{aligned}
T \;\rightarrow\;& T \mid act \cdot circ \cdot e \cdot brane \cdot b \cdot \circlearrowleft \cdot n \cdot a \cdot \mathbf{0} \cdot a \cdot b \cdot \mathbf{0} \cdot e \\
& \cdot brane \cdot d \cdot \circlearrowleft^{\perp} \cdot n \cdot c \cdot \mathbf{0} \cdot c \cdot \mathbf{0} \cdot c \cdot d \cdot \mathbf{0} && (bangS) \\
\rightarrow\;& T \mid act \cdot brane \cdot b \cdot \circlearrowleft \cdot n \cdot a \cdot \mathbf{0} \cdot a \cdot b \cdot \mathbf{0} \\
& \mid act \cdot brane \cdot d \cdot \circlearrowleft^{\perp} \cdot n \cdot c \cdot \mathbf{0} \cdot c \cdot \mathbf{0} \cdot c \cdot d && (circ) \\
\rightarrow^{*}\;& T \mid \left(act \cdot \circlearrowleft \cdot n \cdot a \cdot \mathbf{0} \cdot a\right)^{L} \rfloor act \cdot \mathbf{0} \\
& \left(act \cdot \circlearrowleft^{\perp} \cdot n \cdot c \cdot \mathbf{0} \cdot c \cdot \mathbf{0} \cdot c\right)^{L} \rfloor act \cdot \mathbf{0} && 2 \times (brane) \\
\rightarrow\;& T \mid \left(act \cdot \mathbf{0}\right)^{L} \rfloor \left(act \cdot \mathbf{0} \mid \left(act \cdot \mathbf{0}\right)^{L} \rfloor \left(act \cdot \mathbf{0}\right)^{L} \rfloor act \cdot \mathbf{0}\right) && (phago) \\
\rightarrow^{*}\;& T
\end{aligned}
$$

Correctness of the Translation

The translation of the PEP Calculus is correct:

- it is *sound* and *complete*

Soundness: for each step performed by a PEP system P there exists a corresponding sequence of steps performed by its translation

- Technical detail: we need a normal form $\langle T \rangle$ for a term T

Theorem (Soundness) Given a system P of the PEP calculus:

$$P \rightarrow P' \implies \exists T.\exists P''. \text{ s.t. } \{\!\![P]\!\!\} \rightarrow^* T, \langle T \rangle \equiv \langle \{\!\![P'']\!\!\} \rangle \text{ and } P'' \equiv P' \ .$$

Correctness of the Translation

The translation of the PEP Calculus is correct:

- it is *sound* and *complete*

Completenss: the translation of a PEP system P does not perform executions which does not correspond to an execution of P

- Technical detail: we need again a normal form $\langle T \rangle$ for a term T

Theorem (Completeness) Given a system P of the PEP calculus:

$$\{\![P]\!\} \rightarrow^* T \implies \exists P' \text{ s.t. } \langle T \rangle \equiv \langle \{\![P']\!\} \rangle \text{ and either } P \equiv P' \text{ or } P \rightarrow^* P' \ .$$

Towards Bisimulations for Brane Calculi

In order to define a notion of bisimulation for the PEP calculus, we introduce a labeled semantics.

- $\_$ denotes the silent (internal) action.
- We assume a set $\mathcal{L} = \{(\circlearrowleft_n, \sigma(\!|R|\!)), (\circlearrowleft_n^{\perp}, \sigma(\!|R|\!)), (\circlearrowleft_n, \sigma(\!|R|\!)), \_ \mid n \in \mathcal{N}, R \in PEP, \sigma \in Branes\}$ of transition labels.
- ℓ denotes a generic label.

A transition with a label in ℓ represents the potential action a system can perform when inserted in a suitable context.

- $P \xrightarrow{\circlearrowleft_n, \sigma(\!|R|\!)} P'$, means that P has a component $\sigma(\!|R|\!)$ that may enter another membrane
- $Q \xrightarrow{\circlearrowleft_n^{\perp}, \sigma(\!|R|\!)} Q'$, means that Q can engulf another system $\sigma(\!|R|\!)$
- When P and Q are composed by $\circ$, they can evolve together, with a silent action, to a new system in which $\sigma(\!|R|\!)$ is inside Q'

A Labeled Semantics for the PEP Calculus

Definition (Labeled Semantics) The labeled semantics of the PEP calculus is given by the labeled transition system generated by the following inference rules:

$$\circlearrowleft_n.\sigma|\sigma_0(\!|P|\!) \xrightarrow{\circlearrowleft_n,\sigma|\sigma_0(\!|P|\!)} \diamond \tag{Ph1}$$

$$\circlearrowleft_n^{\perp}(\rho).\tau|\tau_0(\!|Q|\!) \xrightarrow{\circlearrowleft_n^{\perp},\sigma(\!|R|\!)} \tau|\tau_0(\!|\rho(\!|\sigma(\!|R|\!)|\!) \circ Q|\!) \tag{Ph2}$$

$$\frac{P \xrightarrow{\circlearrowleft_n,\sigma(\!|R|\!)} P' \qquad Q \xrightarrow{\circlearrowleft_n^{\perp},\sigma(\!|R|\!)} Q'}{P \circ Q \xrightarrow{\tau} P' \circ Q'} \tag{Ph3}$$

To be continued....

A Labeled Semantics for the PEP Calculus

Definition (Labeled Semantics) more rules.....

$$\circlearrowleft_n.\sigma|\sigma_0(\!|P|\!) \xrightarrow{\circlearrowleft_n,\sigma|\sigma_0(\!|P|\!)} \diamond \tag{E1}$$

$$\frac{P \xrightarrow{\circlearrowleft_n,\sigma(\!|R|\!)} P'}{\circlearrowleft_n^{\perp}.\tau|\tau_0(\!|P \circ Q|\!) \xrightarrow{-} R \circ \sigma|\tau|\tau_0(\!|Q \circ P'|\!)} \tag{E2}$$

$$\circledcirc(\rho).\sigma|\sigma_0(\!|P|\!) \xrightarrow{-} \sigma|\sigma_0(\!|P \circ \rho(\!|\diamond|\!)|\!) \tag{Pi1}$$

$$\frac{P \xrightarrow{\ell} P'}{P \circ Q \xrightarrow{\ell} P' \circ Q} \qquad \frac{Q \xrightarrow{\ell} Q'}{P \circ Q \xrightarrow{\ell} P \circ Q'} \tag{Par1,Par2}$$

$$\frac{P \xrightarrow{-} P'}{\sigma(\!|P|\!) \xrightarrow{-} \sigma(\!|P'|\!)} \tag{Br1}$$

Strong Bisimulation for the PEP Calculus

Definition (PEP Strong Bisimulation) A binary relation κ on PEP terms is a *strong bisimulation* if, given P and Q such that $P\kappa Q$, the following conditions hold:

- $P \xrightarrow{-} P' \implies \exists Q'$ such that $Q \xrightarrow{-} Q'$ and $P'\kappa Q'$
- $Q \xrightarrow{-} Q' \implies \exists P'$ such that $P \xrightarrow{-} P'$ and $Q'\kappa P'$
- $P \xrightarrow{a,R} P' \implies \exists Q'$ such that $Q \xrightarrow{a,R'} Q', R\kappa R'$ and $P'\kappa Q'$
- $Q \xrightarrow{a,R} Q' \implies \exists P'$ such that $P \xrightarrow{a,R'} P', R\kappa R'$ and $Q'\kappa P'$

The *strong bisimilarity* $\simeq$ is the largest of such relations.

PEP Bisimilarity and Translation into CLS

We have seen that the PEP Calculus can be translated into CLS.

In CLS we have bisimulations.

The translations of two bisimilar PEP systems are bisimilar CLS terms?

YES!

Theorem (Full Abstraction) Given two systems P, Q of the PEP calculus, the following holds:

$$P \doteq Q \iff \{\![P]\!\} \approx \{\![Q]\!\} \ .$$