On the product of balanced sequences

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Balanced sequences

A infinite sequence v is *balanced* if for each letter a of the alphabet A and for all factors u and u' of v s.t. $|u| = |u'|$ we have that

$$||u|_a - |u'|_a| \leq 1$$

Example

- $w = abcadbcadbacbdacbd\cdots$ is a balanced sequence.
- $v = \mathbf{abcb}dbcad\mathbf{bacbdacbd}\cdots$ is not a balanced sequence.

Remark

For a two-letter alphabet, being balanced is equivalent to being balanced with respect to one letter.

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Binary case

- An infinite aperiodic sequence v is balanced if and only if v is a sturmian sequence.
- Sturmian sequences are defined as the infinite sequences having exactly $n+1$ distinct factors of length n.
- An infinite periodic sequence v^ω is balanced if and only if v is a conjugate of a standard word.

Example

Fibonacci words

$f_0 = b$

$f_1 = a$

$f_2 = ab$

$f_3 = aba$

$$f_0 = b \qquad f_1 = a$$
$$f_{n+1} = f_n f_{n-1} \; (n \geq 1)$$

The infinite Fibonacci word is the limit of the sequence of Fibonacci words.

Balanced words on larger alphabets

- If $|A| > 2$, the general structure of balanced words is not known.
- As a direct consequence of a result of Graham, one has that balanced sequences on a set of letters having different frequencies must be periodic.

Fraenkel's conjecture

Let $A_k = \{a_1, a_2, \ldots, a_k\}$. For each $k > 2$, there is only one circularly balanced word $F_k \in A_k^*$, having different frequencies. It is defined recursively as follow $F_1 = a_1$ and $F_k = F_{k-1} a_k F_{k-1}$ for all $k \geq 2$.

Direct product

Let us define a *direct product* of two infinite sequences $u = u_0 u_1 \cdots$ and $v = v_0 v_1 \cdots$ on $\Sigma = \{a, b\}$ as the sequence

$$u \otimes v =< u_0, v_0 >< u_1, v_1 > \cdots .$$

on $\Sigma \times \Sigma$.

u :	0	0	1	1
v :	0	1	0	1
w :	a	b	c	d

We define the *degree* of product, $deg(w)$, as the cardinality of the alphabet of the product itself.

The notion of product of two sequences has been introduced in [P. Salimov. On uniform recurrence of a direct product. In AutoMathA 2009], where the author studies the class of uniformly recurrent sequences such that the product of any of its members and each uniformly recurrent sequence is also uniformly recurrent.

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Question

We ask us: when the product of two balanced sequences is balanced too?

Example

Consider the Fibonacci sequence f and the sturmian sequence s:

$$\begin{array}{r|cccccccccccccccccccccc} f: & 0&1&0&0&1&0&1&0&0&1&0&0&1&0&1&0&0&1&0&1&0&\cdots \\ s: & 0&0&1&0&0&0&1&0&0&1&0&0&0&1&0&0&0&1&0&0&1&\cdots \\ \hline w: & a&c&b&a&c&a&d&a&a&d&a&a&c&b&c&a&a&d&a&c&b&\cdots \end{array}$$

w is not a balanced sequence, because w has factors $u = aa$ and $v = cb$, for which $||u|_a - |v|_a| = 2$.

Example

Consider the two following sturmian sequences:

$$\begin{array}{r|cccccccccccccccccccccc} r: & 0&1&0&0&1&0&0&0&1&0&0&0&1&0&0&1&0&0&0&1&0&\cdots \\ z: & 0&1&0&1&0&0&1&0&1&0&1&0&0&1&0&1&0&1&0&0&1&\cdots \\ \hline t: & a&d&a&b&c&a&b&a&d&a&b&a&c&b&a&d&a&b&a&c&b&\cdots \end{array}$$

t is a balanced sequence.

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f :	0	1	0	0	1	0	1	0	0	1	0	0	1	0	1	0	0	1	0	1	0	$\cdots$
s :	0	0	1	0	0	0	1	0	0	1	0	0	0	1	0	0	0	1	0	0	1	$\cdots$
w :	a	c	b	a	c	a	d	a	a	d	a	a	c	b	c	a	a	d	a	c	b	$\cdots$

w is not a balanced sequence, because w has factors $u = aa$ and $v = cb$, for which $||u|_a - |v|_a| = 2$.

Example

Consider the two following sturmian sequences:

r :	0	1	0	0	1	0	0	0	1	0	0	0	1	0	0	1	0	0	0	1	0	$\cdots$
z :	0	1	0	1	0	0	1	0	1	0	1	0	0	1	0	1	0	1	0	0	1	$\cdots$
t :	a	d	a	b	c	a	b	a	d	a	b	a	c	b	a	d	a	b	a	c	b	$\cdots$

t is a balanced sequence.

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w :	a	c	b	a	c	a	d	a	a	d	a	a	c	b	c	a	a	d	a	c	b	$\cdots$

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t :	a	d	a	b	c	a	b	a	d	a	b	a	c	b	a	d	a	b	a	c	b	$\cdots$

t is a balanced sequence.

On four-letters alphabets

Theorem

Let u, v be two binary balanced sequences. If $w = u \otimes v$ is balanced and $deg(w) = 4$ then w is (ultimately) periodic and is a suffix of one of the following sequences:

i) $(adacb)^t(adabc)^\omega$
ii) $(adabc)^t(adacb)^\omega$
iii) $(adabacb)^t(adabcab)^\omega$
iv) $(adabcab)^t(adabacb)^\omega$

where $t \in \mathbb{N}$.

On three-letters alphabets

Theorem

Any balanced sequence w on three letters can be obtained as the product of two binary balanced sequences u and v.

Example

The balanced sequence $w = abaadaabadaabaada\cdots$ is the product of two balanced sequences $u = 00001000010000010\cdots$ and $v = 01001001010010010\cdots$.

$$\begin{array}{cccccccccccccccccc} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & \cdots \\ \hline\hline a & b & a & a & d & a & a & b & a & d & a & a & b & a & a & d & a & \cdots \end{array}$$

On three-letters alphabets

Theorem

For any binary balanced sequence v, one can construct a binary balanced sequence u such that $w = u \otimes v$ is balanced and $deg(w) = 3$.

Example

If $v = 0010010010001001001 0\cdots$ then $u = 000001000000100000 10\cdots$.

0	0	0	0	0	1	0	0	0	0	0	0	1	0	0	0	0	0	1	0	$\cdots$
0	0	1	0	0	1	0	0	1	0	0	0	1	0	0	1	0	0	1	0	$\cdots$
a	a	b	a	a	d	a	a	b	a	a	a	d	a	a	b	a	a	d	a	$\cdots$

And $w = u \otimes v = aabaadaabaaadaabaada\cdots$ is balanced.

Conclusions and further works

- We have proved that:
  - All balanced (periodic or aperiodic) sequences on an alphabet with three letters are obtained by the product of two binary balanced sequences.
  - There exist only finitely many balanced sequences on four letters that can be obtained as product of two binary balanced sequences. Moreover they are ultimately periodic.

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  - All balanced (periodic or aperiodic) sequences on an alphabet with three letters are obtained by the product of two binary balanced sequences.
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Conclusions and further works

Given two integer k and h, one could determine the maximum degree of the product $w = u \otimes v$, such that u, v are balanced sequences, $deg(u) = k$ and $deg(v) = h$:

$$m(k,h) = max\{deg(w) \text{ s.t. } w = u \otimes v, u, v \in \mathcal{B}, deg(u) = k, deg(v) = h\}$$

where $\mathcal{B}$ denotes the set of the balanced sequences.

Example

$$\begin{array}{lcccccccccccccccccccl} u: & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & \cdots & deg(u) = 2 \\ v: & 0 & 2 & 1 & 2 & 0 & 2 & 1 & 2 & 2 & 0 & 2 & 1 & 2 & 0 & 2 & 1 & 2 & 2 & \cdots & deg(v) = 3 \\ \hline \hline w: & a & d & b & c & a & d & b & c & d & a & c & b & d & a & c & b & d & c & \cdots & deg(w) = 4 \end{array}$$

Several experiments suggest that it is not possible to obtain a balanced sequence w with $deg(w) = 5$ or $deg(w) = 6$ as product of two balanced sequences u and v, where $deg(u) = 2$ and $deg(v) = 3$.

Conclusions and further works

Example

u :	0	0	1	0	0	0	0	1	0	0	0	0	0	1	0	0	$\cdots$	$deg(u) = 2$
v :	0	1	2	0	1	2	0	2	1	0	2	1	0	2	1	2	$\cdots$	$deg(v) = 3$
w :	a	b	d	a	b	c	a	d	b	a	c	b	a	d	b	c	$\cdots$	$deg(w) = 4$

u, v, and w are balanced sequences.

Example

u' :	0	0	1	0	0	0	0	1	0	0	0	0	0	1	0	0	$\cdots$	$deg(u') = 2$
v' :	0	1	2	0	2	1	0	2	1	0	2	1	0	2	1	2	$\cdots$	$deg(v') = 3$
w' :	a	b	d	a	c	b	a	d	b	a	c	b	a	d	b	c	$\cdots$	$deg(w') = 4$

u', w' are two balanced sequences, but v' is not balanced sequence.

Conclusions and further works

Example

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v :	0	1	2	0	1	2	0	2	1	0	2	1	0	2	1	2	$\cdots$	$deg(v) = 3$
w :	a	b	d	a	b	c	a	d	b	a	c	b	a	d	b	c	$\cdots$	$deg(w) = 4$

u, v, and w are balanced sequences.

Example

u' :	0	0	1	0	0	0	0	1	0	0	0	0	0	1	0	0	$\cdots$	$deg(u') = 2$
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w' :	a	b	d	a	c	b	a	d	b	a	c	b	a	d	b	c	$\cdots$	$deg(w') = 4$

u', w' are two balanced sequences, but v' is not balanced sequence.

Conclusions and further works

And on five letters alphabets ...

Example

$u:$	0	0	0	0	1	0	0	0	2	0	0	0	1	0	0	0	0	2	0	0	0	1	0	0	0	2 $\cdots$
$v:$	0	1	2	0	2	1	2	0	2	1	2	0	2	1	0	2	1	2	0	2	1	2	0	2	1	2 $\cdots$
$w:$	a	b	c	a	d	b	c	a	e	b	c	a	d	b	a	c	b	e	a	c	b	d	a	c	b	e $\cdots$

u, v, and w are balanced sequences, where $deg(u) = 3, deg(v) = 3, deg(w) = 5$.

Example

$u':$	0	1	2	0	2	1	2	0	2	1	2	0	1	2	0	2	1	2	0	2	1	2	0	2	1	2 $\cdots$
$v':$	0	0	0	0	1	0	0	0	2	0	0	0	0	1	0	0	0	2	0	0	0	1	0	0	0	2 $\cdots$
$w':$	a	b	c	a	d	b	c	a	e	b	c	a	b	d	a	c	b	e	a	c	b	d	a	c	b	e $\cdots$

$deg(u) = 3, deg(v) = 3, deg(w) = 5$. w' is a balanced sequence.

But u' and v' are not balanced sequences.

Conclusions and further works

And on five letters alphabets . . .

Example

u :	0	0	0	0	1	0	0	0	2	0	0	0	1	0	0	0	0	2	0	0	0	1	0	0	0	2	···
v :	0	1	2	0	2	1	2	0	2	1	2	0	2	1	0	2	1	2	0	2	1	2	0	2	1	2	···
w :	a	b	c	a	d	b	c	a	e	b	c	a	d	b	a	c	b	e	a	c	b	d	a	c	b	e	···

u, v, and w are balanced sequences, where $deg(u) = 3, deg(v) = 3, deg(w) = 5$.

Example

u' :	0	1	2	0	2	1	2	0	2	1	2	0	1	2	0	2	1	2	0	2	1	2	0	2	1	2	···
v' :	0	0	0	0	1	0	0	0	2	0	0	0	0	1	0	0	0	2	0	0	0	1	0	0	0	2	···
w' :	a	b	c	a	d	b	c	a	e	b	c	a	b	d	a	c	b	e	a	c	b	d	a	c	b	e	···

$deg(u) = 3, deg(v) = 3, deg(w) = 5$. w' is a balanced sequence.

But u' and v' are not balanced sequences.

Conclusions and further works

Given k, is it possible to classify the balanced sequences $w = u \otimes v$, with $degree(w) = k$ according to $deg(u)$ and $deg(v)$?

Example

On a four-letter alphabet:

- There exist only finitely many balanced sequences on four letters that can be obtained as product of two binary balanced sequences. Moreover they are ultimately periodic.
- The balanced sequence $w = u \otimes v = adbcadbcdacbdacbdc \cdots$ is obtained as product of two balanced sequences u and v, where $deg(u) = 2$ and $deg(v) = 3$ (the previous example).
- Can all remaining balanced sequences w on four letters be obtained as product $u \otimes v$, where $deg(u) = 2$ and $deg(v) = 3$?

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- Can all remaining balanced sequences w on four letters be obtained as product $u \otimes v$, where $deg(u) = 2$ and $deg(v) = 3$?

Conclusions and further works

- Clearly, a balanced sequence over k letters can always be obtained by the product of $k-1$ sequences.

Example

1	0	0	0	1	0	0	0	1	0	0	0	1	0	0	0	0	⋯
0	0	0	1	0	0	0	0	0	1	0	0	0	0	0	1	0	⋯
0	1	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	⋯
0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	0	1	⋯
0	0	1	0	0	0	1	0	0	0	1	0	0	0	1	0	0	⋯
a	d	b	e	a	c	b	d	a	e	b	c	a	d	b	e	c	⋯

0	0	0	1	0	0	0	0	0	1	0	0	0	0	0	1	0	⋯
0	1	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	⋯
0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	0	1	⋯
0	0	1	0	0	0	1	0	0	0	1	0	0	0	1	0	0	⋯
a	d	b	e	a	c	b	d	a	e	b	c	a	d	b	e	c	⋯

Is it possible to obtain the sequence as product of 3 binary balanced sequences?

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1	0	0	0	1	0	0	0	1	0	0	0	1	0	0	0	0	$\cdots$
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0	1	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	$\cdots$
0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	0	1	$\cdots$
0	0	1	0	0	0	1	0	0	0	1	0	0	0	1	0	0	$\cdots$
a	d	b	e	a	c	b	d	a	e	b	c	a	d	b	e	c	$\cdots$

0	0	0	1	0	0	0	0	0	1	0	0	0	0	0	1	0	$\cdots$
0	1	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	$\cdots$
0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	0	1	$\cdots$
0	0	1	0	0	0	1	0	0	0	1	0	0	0	1	0	0	$\cdots$
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0	1	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	⋯
0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	0	1	⋯
0	0	1	0	0	0	1	0	0	0	1	0	0	0	1	0	0	⋯
a	d	b	e	a	c	b	d	a	e	b	c	a	d	b	e	c	⋯

0	0	0	1	0	0	0	0	0	1	0	0	0	0	0	1	0	⋯
0	1	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	⋯
0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	0	1	⋯
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a	d	b	e	a	c	b	d	a	e	b	c	a	d	b	e	c	⋯

Is it possible to obtain the sequence as product of 3 binary balanced sequences?

Conclusions and further works

- To determine the smallest value of h such that a balanced sequence over a k-letters alphabet is obtained as product of h binary balanced sequences.

$$g(k) = min\{h \text{ s.t. } w = u_1 \otimes u_2 \otimes \cdots \otimes u_h, deg(w) = k, u_i \in \mathcal{B}, deg(u_i) = 2, \text{ for each } i\}$$

- Is it possible to classify the balanced sequences according to the different value of h?

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Thank you
for your attention!