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# On Quantified Linear Implications

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**Abstract** A Quantified Linear Implication (QLI) is an inclusion query over two polyhedral sets, with a quantifier string that specifies which variables are existentially quantified and which are universally quantified. Equivalently, it can be viewed as a quantified implication of two systems of linear inequalities. In this paper, we provide a 2-person game semantics for the QLI problem, which allows us to explore the computational complexities of several of its classes. More specifically, we prove that the decision problem for QLIs with an arbitrary number of quantifier alternations is **PSPACE-hard**. Furthermore, we explore the computational complexities of several classes of 0, 1, and 2-quantifier alternation QLIs. We observed that some classes are decidable in polynomial time, some are **NP-complete**, some are **coNP-hard** and some are $\mathbf{\Pi_2^P}$**-hard**. We also establish the hardness of QLIs with 2 or more quantifier alternations with respect to the first quantifier in the quantifier string and the number of quantifier alternations. All the proofs that we provide for polynomially solvable problems are constructive, i.e., polynomial-time decision algorithms are devised that utilize well-known procedures. QLIs can be utilized as powerful modelling tools for real-life applications. Such applications include reactive systems, real-time schedulers, and static program analyzers.

**Keywords** Quantified Linear Implication · Linear constraints · Inclusion query over polyhedral sets · Computational complexity · Polynomial Hierarchy

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## 1 Introduction

*Quantified linear programming* is the problem of checking whether a set of linear inequalities over the reals, i.e., a linear system, is satisfiable with respect to a given quantifier string. In quantified linear programming all variables are either existentially or universally quantified. Hence, it represents a generalization of linear programming, where all variables are existentially quantified. The alternation of quantifiers in the quantifier string makes deciding a *Quantified Linear Program* (QLP) a non-trivial problem. In this paper, we extend the quantification of variables to *implications* over two linear systems of inequalities by exploring *Quantified Linear Implications* (QLIs). That is, QLIs correspond to inclusion queries of polyhedral solution sets of two linear systems with respect to a given quantifier string.

In order to facilitate the classification and analysis of QLIs, we provide a nomenclature, which succinctly specifies (a hierarchy of) QLI classes. Furthermore, we propose semantics for interpreting any QLI as a 2-person game. Using these semantics, we prove that the decision problem for QLIs with an arbitrary number of quantifier alternations is **PSPACE-hard**. Moreover, we present several classes of 0, 1, and 2-quantifier alternation QLIs and study their computational complexities. For all polynomially solvable problems, we provide proofs that are constructive, i.e., we design polynomial-time algorithms that are based on well-known procedures. Moreover, we establish the hardness of QLIs with 2 or more quantifier alternations with respect to the first quantifier in the quantifier string and the number of quantifier alternations. Finally, we characterize two interesting special cases of 2-quantifier alternation QLIs with respect to the structure of the linear system of the left-hand side of the implication.

The rest of this paper is organized as follows. Section 2 presents basic notions that are used throughout the paper. Section 3 formally introduces quantified linear implications and their corresponding nomenclature. Section 4 details the contributions of the paper. Motivations and related work are presented in Section 5 and 6 respectively. Section 7 presents a 2-person game semantics for QLIs. In Section 8, we explore the case of arbitrary quantifier alternations. Sections 9 and 10 cover the case of 0-quantifier and 1-quantifier alternations respectively, while Section 11 explores classes with 2 or more quantifier alternations. Section 12 concludes the paper, by summarizing our contributions and discussing avenues for future research.

## 2 Background

We use the standard notation of linear algebra (Schrijver, 1987) to formally present the basic notions of this paper. $\Re$ is the set of real numbers. Let small bold letters $(\mathbf{a}, \mathbf{b}, \ldots)$ denote column vectors, while capital bold letters $(\mathbf{A}, \mathbf{B}, \ldots)$ denote matrices. Let $\mathbf{0}$ and $\mathbf{1}$ be the column vectors with all elements equal to 0 and 1 respectively. Furthermore, let $a_i$ denote the $i^{th}$ element in $\mathbf{a}$, while $\mathbf{a}^T$ is the transposed vector of $\mathbf{a}$. Also, let $\mathbf{a}^T \cdot \mathbf{b}$ denote the inner product of $\mathbf{a}^T$ and $\mathbf{b}$ (it is used accordingly with matrices). Finally, we assume that the dimensions of vectors and matrices in inner products are of the appropriate size.

A *primitive linear constraint* (or *inequality*) is an expression of the form: $a_1 \cdot x_1 + \ldots + a_n \cdot x_n \leq b$, where $a_1, \ldots, a_n$ and $b$ are constants in $\Re$ and $x_1, \ldots, x_n$ are variables. The inner product form $\mathbf{a}^T \cdot \mathbf{x} \leq b$ can also be used to express a primitive linear constraint. Special cases include: (a) *difference constraints*, i.e., linear constraints of the form $x_i - x_j \leq b_{ij}$; and (b) *interval constraints*, i.e., linear constraints of the form $l \leq x \leq u$, where $l$ and $u$ are constants in $\Re$ representing a lower bound and an upper bound for variable $x$ respectively.

A *linear constraint* is a conjunction of primitive linear constraints. Conjunction is represented either by "$\wedge$" or by a comma. In terms of linear systems of inequalities, a linear constraint can also be denoted as $\mathbf{A} \cdot \mathbf{x} \leq \mathbf{b}$, so we will use the notions of linear constraints and *linear systems* interchangeably. A *polyhedron* is the set of solution points of a linear system: $Sol(\mathbf{A} \cdot \mathbf{x} \leq \mathbf{b}) = \{\mathbf{x}_0 \in \Re^{|\mathbf{x}|} \mid \mathbf{A} \cdot \mathbf{x}_0 \leq \mathbf{b}\}$. Polyhedra are convex sets (Schrijver, 1987). Deciding an (unquantified) *Linear Program* (LP) consists of checking satisfiability of a linear system in the domain of the reals:

$$\exists \mathbf{x}\ \mathbf{A} \cdot \mathbf{x} \leq \mathbf{b}$$

or, equivalently, checking whether $Sol(\mathbf{A} \cdot \mathbf{x} \leq \mathbf{b}) \neq \emptyset$.

Quantified Linear Programming was introduced in (Subramani, 2007) as an extension of linear programming with arbitrary variable quantifications. In a Quantified Linear Program, variables of a linear system are either existentially or universally (with bounds) quantified:

$$\exists \mathbf{x}_1\ \forall \mathbf{y}_1 \in [\mathbf{l}_1, \mathbf{u}_1]\ \ldots \exists \mathbf{x}_n\ \forall \mathbf{y}_n \in [\mathbf{l}_n, \mathbf{u}_n]\ \ \mathbf{A} \cdot \mathbf{x} + \mathbf{N} \cdot \mathbf{y} \leq \mathbf{b} \tag{1}$$

where $\mathbf{x}_1 \ldots \mathbf{x}_n$ is a partition of $\mathbf{x}$ with, possibly, $\mathbf{x}_1$ empty; $\mathbf{y}_1 \ldots \mathbf{y}_n$ is a partition of $\mathbf{y}$ with, possibly, $\mathbf{y}_n$ empty; and $\mathbf{l}_i$, $\mathbf{u}_i$ are lower and upper bounds in $\Re$ for $\mathbf{y}_i$, for $i = 1, \ldots, n$.

An example of a QLP with three quantifier alternations is given below:

$$\forall y_1 \in [0,1]\ \exists x_2\ \forall y_2 \in [1,3]\ \exists x_3 \qquad \begin{array}{l} 2y_1 - 2x_2 - y_2 + x_3 \geq 4 \\ 3x_2 - 5y_1 + y_2 - x_3 \leq 1 \end{array}$$

## 3 Problem statement

Consider two linear systems $P_1 : \mathbf{A} \cdot \mathbf{x} \leq \mathbf{b}$ and $P_2 : \mathbf{C} \cdot \mathbf{x} \leq \mathbf{d}$. We say that $P_1$ *is included* in $P_2$ if every solution of $P_1$ is also a solution of $P_2$. This holds if and only if the logic formula $\forall \mathbf{x}\ [\mathbf{A} \cdot \mathbf{x} \leq \mathbf{b} \rightarrow \mathbf{C} \cdot \mathbf{x} \leq \mathbf{d}]$ is true in the domain of the reals. We extend the notion of inclusion to arbitrary quantifiers by introducing Quantified Linear Implications of two linear systems:

$$\exists \mathbf{x}_1\ \forall \mathbf{y}_1\ \ldots \exists \mathbf{x}_n\ \forall \mathbf{y}_n\ \ [\mathbf{A} \cdot \mathbf{x} + \mathbf{N} \cdot \mathbf{y} \leq \mathbf{b} \rightarrow \mathbf{C} \cdot \mathbf{x} + \mathbf{M} \cdot \mathbf{y} \leq \mathbf{d}] \tag{2}$$

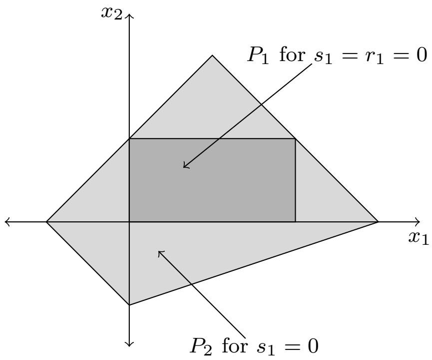


**Fig. 1** $P_1$ is included in $P_2$ for $s_1 = r_1 = 0$ (Example 1).

where $\mathbf{x}_1 \ldots \mathbf{x}_n$ and $\mathbf{y}_1 \ldots \mathbf{y}_n$ are partitions of $\mathbf{x}$ and $\mathbf{y}$ respectively, and where $\mathbf{x}_1$ and/or $\mathbf{y}_n$ may be empty. We say that a QLI *holds* if it is true as a first-order formula over the domain of the reals. The *decision problem for a QLI* consists of checking whether System (2) holds or not.

Let $\mathbf{Q}(\mathbf{x}, \mathbf{y})$ denote the quantifier string, namely $\exists \mathbf{x}_1 \ \forall \mathbf{y}_1 \ \ldots \exists \mathbf{x}_n \ \forall \mathbf{y}_n$ in System (2). We introduce a nomenclature to represent the different classes of QLIs that we will be examining. Consider a triple $\langle A, Q, R\rangle$. Let $A$ denote the number of quantifier alternations in the quantifier string $\mathbf{Q}(\mathbf{x}, \mathbf{y})$ and $Q$ the first quantifier of $\mathbf{Q}(\mathbf{x}, \mathbf{y})$. Also, let $R$ be an $(A + 1)$-character string, specifying for each quantified set of variables in $\mathbf{Q}(\mathbf{x}, \mathbf{y})$ whether they appear on the **L**eft, on the **R**ight, or on **B**oth sides of the implication. For instance, $\langle 1, \exists, \mathbf{LB}\rangle$ indicates a problem described by:

$$\exists \mathbf{x} \ \forall \mathbf{y} \ \ [\mathbf{A} \cdot \mathbf{x} + \mathbf{N} \cdot \mathbf{y} \leq \mathbf{b} \rightarrow \mathbf{M} \cdot \mathbf{y} \leq \mathbf{d}]$$

Note that this nomenclature was originally introduced in Eirinakis et al (2012).

*Example 1* Consider the following QLI of the class $\langle 2, \forall, \mathbf{RLB}\rangle$:

$$\forall s_1 \ \exists r_1 \ \forall x_1 \ \forall x_2$$
$$\left.\begin{array}{l} x_1 \geq 0 - 3r_1 \\ x_1 \leq 2 - 5r_1 \\ x_2 \geq 0 + 2r_1 \\ x_2 \leq 1 + 3r_1 \end{array}\right\} \rightarrow \left\{\begin{array}{l} x_1 + x_2 \geq -1 + 3s_1 \\ x_1 + x_2 \leq 1 + 4s_1 \\ x_1 - x_2 \geq -1 + 5s_1 \\ x_1 - 3x_2 \leq 3 + 7s_1 \end{array}\right.$$

Let $P_1$ denote the left-hand side and $P_2$ the right-hand side linear system of the implication. Figure 1 presents $P_1$ and $P_2$ for specific values of $s_1$ and $r_1$, i.e., $s_1 = r_1 = 0$. Note that for these values $P_1$ is included in $P_2$ (i.e., if both $s_1$ and $r_1$ were existentially quantified). However, in order for the above QLI to hold, for all values of $s_1$ there must exist a value of $r_1$ such that every solution $(x_1, x_2)$ of $P_1$ is also a solution of $P_2$.

## 4 Contributions

The principal contributions of the paper are as follows:

1. A 2-person game semantics for QLI is introduced. (Section 7).
2. The decision problem for QLIs with arbitrary number of quantifier alternations is **PSPACE-hard** (Theorem 5).
3. The complexity of 0-quantifier alternation class $\langle 0, \exists, \mathbf{B}\rangle$ is shown to be in **P** (Theorem 6).
4. The complexities of all classes with 1-quantifier alternation starting with $\exists$ are analyzed:
   a. Problems $\langle 1, \exists, \mathbf{BB}\rangle$ and $\langle 1, \exists, \mathbf{LB}\rangle$ are **NP-complete** (Theorem 8).
   b. Problems $\langle 1, \exists, \mathbf{RB}\rangle$ and $\langle 1, \exists, \mathbf{BR}\rangle$ (and their sub-classes) are in **P** (Theorems 9 and 10).
   c. Problems $\langle 1, \exists, \mathbf{BL}\rangle$ and $\langle 1, \exists, \mathbf{LL}\rangle$ are in **NP** (end of Section 10.1), but their hardness is unknown.
5. The complexities of all classes with 1-quantifier alternation starting with $\forall$ are analyzed:
   a. Problems $\langle 1, \forall, \mathbf{BB}\rangle$ and $\langle 1, \forall, \mathbf{BR}\rangle$ are **coNP-hard** (Theorem 11).
   b. Problems $\langle 1, \forall, \mathbf{LB}\rangle$ and $\langle 1, \forall, \mathbf{BL}\rangle$ (and their sub-classes) are in **P** (Theorems 12 and 13).
   c. Problems $\langle 1, \forall, \mathbf{RB}\rangle$ and $\langle 1, \forall, \mathbf{RR}\rangle$ are in **coNP** (end of Section 10.2), but their hardness is unknown.
6. Problems $\langle k, \exists, \mathbf{B}^{k+1}\rangle$ with $k$ odd are shown to be $\mathbf{\Sigma_k^P}$**-hard** (Theorem 14).
7. Problems $\langle k, \forall, \mathbf{B}^{k+1}\rangle$ with $k$ even are shown to be $\mathbf{\Pi_k^P}$**-hard** (Theorem 15).
8. The complexities of the following representative classes with 2-quantifier alternations starting with $\forall$ are analyzed:
   a. Problems $\langle 2, \forall, \mathbf{BBB}\rangle$ and $\langle 2, \forall, \mathbf{LLB}\rangle$ are polynomial time reducible to each other (Theorem 16), and both are $\mathbf{\Pi_2^P}$**-hard** (Corollary 3).
   b. Problem $\langle 2, \forall, \mathbf{LRB}\rangle$ is **coNP-hard** (Theorem 17).
   c. Problem $\langle 2, \forall, \mathbf{LRB}\rangle$ remains **coNP-hard** even if the left-hand side of the implication is restricted to difference constraints (Lemma 2).
   d. Problem $\langle 2, \forall, \mathbf{LRB}\rangle$ is in **P** if the left-hand side of the implication is restricted to interval constraints (Lemma 3).
   e. Problems $\langle 2, \forall, \mathbf{LRL}\rangle$ and $\langle 2, \forall, \mathbf{LRR}\rangle$ are in **P** (Theorems 18 and 19).

## 5 Motivation

We mention three application areas of QLIs that motivate their study: reactive systems, real-time scheduling, and type systems.

In many real-world applications, the input, which a system has to analyze and respond to, is massive and unpredictable. The domain is not fixed but keeps changing as a result of events that are initiated by the environment and given as input to the system. Likewise, the responses of the system may result in a change of the domain. Such a system is called *reactive* and its role is to maintain an ongoing interaction with its environment. In other words, a reactive system alters its actions and outputs and conditions its response to the actions of the environment. Reactive systems are used in several real-world important applications and in various fields (see e.g., Koo et al, 1999; Pfitzmann and Waidner, 2000, 2001; Kam et al, 2001; Hall, 2002; Harel, 2003). QLIs can be used to model reactive systems, where the values of the universally quantified variables represent the environmental input, while the values of the existentially quantified variables represent the system's response. QLIs offer an important modeling tool for the design and implementation of reactive systems.

In *real-time scheduling*, a dispatcher tries to determine whether a set of ordered, non-preemptive jobs can be scheduled within given time frames. Each job has a variable start time as well as a variable execution time. QLPs have been adopted to model such problems, representing the start times as existentially quantified variables and the execution times as universally quantified variables. Linear constraints are used to express the relation between the starting and ending times of different jobs (see e.g., Subramani, 2003, 2005). Now consider a real-life scenario where the dispatcher has already obtained a schedule (solution) but then some constraints are slightly altered. The question that arises is whether the dispatcher needs to recompute a solution or whether it can still use the current one. This problem can be formulated as a QLI with the old set of constraints in the left-hand side and the new set of constraints in the right-hand side of the implication.

*Types in programming languages* allow for program abstraction, static analysis, high-performance compilation, security enforcement, and many other applications (Pierce, 2002). Constraint logic programming provides an elegant scheme for dynamically building complex constraints by exploiting recursion, non-determinism and intertwined constraint generation and solving (Jaffar and Maher, 1994; Jaffar et al, 1998). A type system for constraint logic programming over the reals (Jaffar et al, 1992) has been proposed in (Ruggieri and Mesnard, 2010) by relying on QLIs of the class $\langle 2, \forall, \mathbf{LRB}\rangle$ for proving assertions over the types of input-output variables.

## 6 Related work

In this section, we provide a brief overview of related work in the literature. Consider two linear systems $P_1 : \mathbf{A} \cdot \mathbf{x} \leq \mathbf{b}$ and $P_2 : \mathbf{C} \cdot \mathbf{x} \leq \mathbf{d}$. The problem of checking whether $P_1$ is included in $P_2$ can also be expressed as a first-order sentence. Hence, it is equivalent to deciding the problem $\forall \mathbf{x}\ [P_1 \rightarrow P_2]$. This decision problem can be solved in polynomial time (Subramani, 2009). It suffices to show for every primitive constraint $\mathbf{c}_i^T \cdot \mathbf{x} \leq d_i$ in $P_2$ (where $\mathbf{c}_i^T$ is the vector corresponding to the $i^{th}$ row of $\mathbf{C}$), the following linear programming problem is either infeasible or its solution is bounded by $d_i$:

$$\max \mathbf{c}_i^T \cdot \mathbf{x}$$
$$\mathbf{A} \cdot \mathbf{x} \leq \mathbf{b}$$

The conclusion follows from the fact that linear programming problems are solvable in polynomial time (Khachiyan, 1979) and there is a finite number of primitive constraints in $P_2$. More formally, the following result holds.

**Theorem 1** *The decision problem for a QLI of the class* $\langle 0, \forall, \mathbf{B} \rangle$*, i.e., for* $\forall \mathbf{x}\ [\mathbf{A} \cdot \mathbf{x} \leq \mathbf{b} \rightarrow \mathbf{C} \cdot \mathbf{x} \leq \mathbf{d}]$*, is in* $\mathbf{P}$.

One could argue that the above approach can be utilized to decide any QLI efficiently. However, this is not the case. Consider, for example, a problem of the class $\langle 1, \exists, \mathbf{LB} \rangle$, i.e., an instance described by:

$$\exists \mathbf{y}\ \forall \mathbf{x}\ \ [\mathbf{A} \cdot \mathbf{x} + \mathbf{N} \cdot \mathbf{y} \leq \mathbf{b} \rightarrow \mathbf{C} \cdot \mathbf{x} \leq \mathbf{d}]$$

Here, $P_1$ is $\mathbf{A} \cdot \mathbf{x} + \mathbf{N} \cdot \mathbf{y} \leq \mathbf{b}$ and $P_2$ is $\mathbf{C} \cdot \mathbf{x} \leq \mathbf{d}$. Thus, the corresponding decision problem can be expressed as $\exists \mathbf{y}\ \forall \mathbf{x}\ [P_1 \rightarrow P_2]$. The QLI holds if and only if there exists a value $\mathbf{y}_0$ such that $\mathbf{A} \cdot \mathbf{x} + \mathbf{N} \cdot \mathbf{y}_0 \leq \mathbf{b}$ (i.e., $P_1$ for $\mathbf{y} = \mathbf{y}_0$) is included in $\mathbf{C} \cdot \mathbf{x} \leq \mathbf{d}$ (i.e., in $P_2$). Applying the above procedure requires solving the following problem:

$$\begin{gathered} \max \mathbf{c}_i^T \cdot \mathbf{x} \\ \mathbf{A} \cdot \mathbf{x} + \mathbf{N} \cdot \mathbf{y} \leq \mathbf{b} \end{gathered}$$

The solution of this problem is parametric in $\mathbf{y}$. Hence, this problem belongs to the class of *parametric linear programming* (Murty, 1980; Gal, 1995; Pistikopoulos et al, 2007). Its solution is a piecewise affine function in $\mathbf{y}$ defined over a polyhedral partition of the feasible parameters (Borrelli et al, 2003). Such a function can be exponentially large in the size of the problem, even if there is only one parameter (Murty, 1980). Hence, it cannot be adopted as a building block for deciding a QLI in polynomial time.

In a landmark thesis, Tarski analyzed a number of properties of real fields that make them amenable to the existence of decision procedures (Tarski, 1951). He showed, using quantifier elimination, that the language of reals with universal quantifiers is decidable (Van Den Vries, 1988). The complexity of quantifier elimination, in the worst case, is doubly exponential in the number of quantifier alternation and exponential in the number of variables (Davenport and Heintz, 1988; Weispfenning, 1988). Nevertheless, some approaches have been proved efficient in practice, e.g., (Brown, 2003; Dolzmann and Sturm, 1997; Dolzmann et al, 1998; Ratschan, 2006). The Fourier-Motzkin existential quantifier elimination method and a universal quantifier elimination method have also been employed in the context of QLPs (Subramani, 2007) to establish that deciding a QLP problem is in **PSPACE**. Also, they have been used to establish that the special case of **E**-QLP problems, which are of the form $\exists \mathbf{y}\ \forall \mathbf{x} \in [\mathbf{l}, \mathbf{u}]\ \mathbf{A} \cdot \mathbf{x} + \mathbf{N} \cdot \mathbf{y} \leq \mathbf{b}$, are solvable in polynomial time. The following result was obtained.

**Theorem 2** *The decision problem for a QLP of the form* $\exists \mathbf{y}\ \forall \mathbf{x} \in [\mathbf{l}, \mathbf{u}]\ \mathbf{A} \cdot \mathbf{x} + \mathbf{N} \cdot \mathbf{y} \leq \mathbf{b}$ *is in* $\mathbf{P}$.

Another special case that was characterized in (Subramani, 2007) is the **F-QLP** problem. The corresponding result is presented next.

**Theorem 3** *The decision problem for a QLP of the form* $\forall \mathbf{y} \in [\mathbf{l}, \mathbf{u}] \ \exists \mathbf{x} \ \mathbf{A} \cdot \mathbf{x} + \mathbf{N} \cdot \mathbf{y} \leq \mathbf{b}$ *is* **coNP-complete**.

The relation between QLPs and QLIs is discussed in the following section, with respect to the 2-person game semantics that are presented for QLIs.

## 7 Semantics

In this section, we interpret a quantified linear implication problem as a 2-person game. Such a game includes an existential player $\mathbf{X}$, who chooses values for the existentially quantified variables, and a universal player $\mathbf{Y}$, who chooses values for the universally quantified variables. Consider the generic form of QLI (System (2)), and assume, without loss of generality, that $\mathbf{x}_1$ and $\mathbf{y}_n$ are not empty (dummy variables can be added, if necessary). Let the *initial board configuration* of the game be:

$$[\mathbf{A} \cdot \mathbf{x} + \mathbf{N} \cdot \mathbf{y} \leq \mathbf{b} \rightarrow \mathbf{C} \cdot \mathbf{x} + \mathbf{M} \cdot \mathbf{y} \leq \mathbf{d}] \tag{3}$$

The game is played in a sequence of $2 \cdot n$ rounds. Let $i = 1, \ldots, n$. In round $(2 \cdot i - 1)$, $\mathbf{X}$ makes his $i$th move (by choosing values for the variables in the set $\mathbf{x}_i$). Then, $\mathbf{Y}$ makes his $i$th move (by choosing values for the variables in the set $\mathbf{y}_i$) in round $2 \cdot i$. Hence, $\mathbf{X}$ and $\mathbf{Y}$ make their moves by selecting values for their respective variable sets. The moves are strictly alternating: $\mathbf{X}$ will make his $i$th move, which will be followed by $\mathbf{Y}$'s $i$th move, after which $\mathbf{X}$ will make his $(i+1)$th move and so on.

Every move that $\mathbf{X}$ or $\mathbf{Y}$ make changes the board configuration of the game by replacing variables (existentially quantified or universally quantified, depending on the round of the game) with their given values. For instance, assume that any $\mathbf{x_i}$ consists of a single variable $x_i$. In the first round, say that the existential player $\mathbf{X}$ chooses to give to $x_1$ the value $x_1' \in \Re$, i.e., $\mathbf{X}$ sets $x_1 = x_1'$. Then, the board configuration will change from its initial state (see (3)) to the following:

$$[\mathbf{A}' \cdot \mathbf{x}' + \mathbf{N} \cdot \mathbf{y} \leq \mathbf{b} - x_1'\mathbf{a}_1 \rightarrow \mathbf{C}' \cdot \mathbf{x}' + \mathbf{M} \cdot \mathbf{y} \leq \mathbf{d} - x_1'\mathbf{c}_1] \tag{4}$$

After $\mathbf{X}$'s first move, (4) is the *current board configuration* of the game. In this configuration, $\mathbf{A}'$ and $\mathbf{C}'$ are derived from $\mathbf{A}$ and $\mathbf{C}$ respectively by removing the first column (corresponding to $x_1$). Vector $\mathbf{x}'$ is derived from $\mathbf{x}$ by removing variable $x_1$, that is, $\mathbf{x}' = [x_2, \ldots, x_n]^T$. Note also that $x_1'\mathbf{a}_1$ and $x_1'\mathbf{c}_1$ are subtracted from $\mathbf{b}$ and $\mathbf{d}$ from the Left-Hand Side (LHS) and the Right-Hand Side (RHS) of the implication respectively. Vectors $\mathbf{a}_1$ and $\mathbf{c}_1$ denote the first column of $\mathbf{A}$ and $\mathbf{C}$ respectively, while recall that $x_1'$ is a constant (since it represents $\mathbf{X}$'s choice for his first move).

Each move made by $\mathbf{X}$ or $\mathbf{Y}$ depends on the current board configuration and on the previous moves made by the opponent. Hence, the $i$th move made by $\mathbf{X}$, namely $\mathbf{x}_i$, may depend on the first $(i-1)$ moves made by $\mathbf{Y}$ and the board configuration after round $2 \cdot i - 2$. Similarly, $\mathbf{y}_i$ may depend on the first $i$ moves made by $\mathbf{X}$ and the board configuration after round $2 \cdot i - 1$.

In any game of this form, the goals of the players are the following: $\mathbf{X}$ selects the values of the existentially quantified variables so as to violate the constraints in the LHS or to satisfy the constraints in the RHS of the implication. On the other hand, $\mathbf{Y}$ selects the values of the universally quantified variables so as to satisfy the constraints of the LHS and on the same time to violate the constraints of the RHS of the implication. We say that $\mathbf{X}$ *wins the game* if at the end of the game (i.e., after the $2 \cdot n$ rounds) the board configuration is such that its LHS is false (as a conjunction of inequalities) or its RHS is true. Otherwise, we say that $\mathbf{Y}$ wins the game (i.e., if the LHS is satisfied and the RHS is falsified).

It is important to note that the game as described above is non-deterministic in nature, in that we have not specified *how* $\mathbf{X}$ and $\mathbf{Y}$ make their moves. We say that $\mathbf{X}$ *has a winning strategy* if it is possible for $\mathbf{X}$ to win the game, i.e., if there is a sequence of moves such that $\mathbf{X}$ wins the game. Otherwise, we say that $\mathbf{Y}$ has a winning strategy. The QLI holds precisely when player $\mathbf{X}$ has a winning strategy.

*Remark 1* A QLI holds if and only if the existential player has a winning strategy.

Let us now show that the proposed game is a conservative extension of the game semantics of QLP problems (Subramani, 2007). There, an existential player $\mathbf{X}$ and a universal player $\mathbf{Y}$ also choose their moves according to the order of the variables in the corresponding quantifier string. If, at the end, the instantiated linear system in the QLP is true, then $\mathbf{X}$ wins the game (and we say that $\mathbf{X}$ has a winning strategy). Otherwise, $\mathbf{Y}$ wins the game. Based on these semantics, we explore the relation between QLPs and QLIs. Consider a generic QLP as described by System (1). Now consider the following QLI:

$$\begin{array}{l} \exists \mathbf{x}_1\ \forall \mathbf{y}_1 \ldots \exists \mathbf{x}_n\ \forall \mathbf{y}_n \\ \qquad \left.\begin{array}{c} \mathbf{l}_1 \le \mathbf{y}_1 \le \mathbf{u}_1 \\ \ldots \\ \mathbf{l}_n \le \mathbf{y}_n \le \mathbf{u}_n \end{array}\right\} \rightarrow \mathbf{A} \cdot \mathbf{x} + \mathbf{N} \cdot \mathbf{y} \le \mathbf{b} \end{array} \tag{5}$$

where $\mathbf{l}_1, \ldots, \mathbf{l}_n$ and $\mathbf{u}_1, \ldots, \mathbf{u}_n$ are partitions of $\mathbf{l}$ and $\mathbf{u}$ and correspond to the lower and upper bounds respectively on the variables in $\mathbf{y}_1, \ldots, \mathbf{y}_n$ of $\mathbf{y}$ that appear in the quantifier string of System (1).

**Theorem 4** *The existential player has a winning strategy in System (5) if and only if the existential player has a winning strategy in System (1).*

*Proof* Note that the interval constraints on the universal variables that are in the quantifier string of the QLP (see System (1)) have been placed within the

LHS of (5). These bounds are restrictive for the universal player in System (5) as well, although they are not within the quantifier string. Recall that the universal player $\mathbf{Y}$ wants the implication not to hold in order to win the game. Therefore, $\mathbf{Y}$ must choose values for the universally quantified variables so that the LHS is satisfied (otherwise, the existential player $\mathbf{X}$ trivially wins the game). Hence, we can safely assume that $\mathbf{Y}$ will satisfy the interval constraints in the LHS (while also trying to falsify $\mathbf{A} \cdot \mathbf{x} + \mathbf{N} \cdot \mathbf{y} \leq \mathbf{b}$).

*If part.* Assume that $\mathbf{X}$ has a winning strategy for System (1). Let $\mathbf{x}_0$ be the vector of Skolem functions that represent this strategy. Thus for any $\mathbf{y}_0 \in [\mathbf{l}, \mathbf{u}]$ of $\mathbf{y}$, $\mathbf{A} \cdot \mathbf{x}_0 + \mathbf{N} \cdot \mathbf{y}_0 \leq \mathbf{b}$ is true. But then, since the universally quantified variables are restricted by $\mathbf{y} \in [\mathbf{l}, \mathbf{u}]$ due to the interval constraints in the LHS of (5), the existential player can use the same functions from $\mathbf{x}_0$ and satisfy the RHS of the implication, i.e., making System (5) hold and hence winning the game.

*Only-if part.* Assume that $\mathbf{X}$ has a winning strategy for System (5) with $\mathbf{x}_0$ the vector of Skolem functions representing that strategy. Since the universal player wants to satisfy the LHS in order to win, this means that $\mathbf{y}$ is instantiated to $\mathbf{y}_0$ such that the LHS necessarily holds (i.e., $\mathbf{y}_0 \in [\mathbf{l}, \mathbf{u}]$). Hence, the same vectors $\mathbf{x}_0$, $\mathbf{y}_0$ can be then used to make $\mathbf{X}$ win System (1). $\square$

Note that the quantifier string of QLPs restricts the possible moves of the universal player through lower and upper bounds. The absence of such bounds in the quantifier string of QLIs follows from the fact that the universal player wants to satisfy the LHS of the implication. Hence, it is the satisfaction of the LHS that restricts the moves of the universal player in QLIs. If explicit interval constraints exist for the universal variables, these can also be placed within the LHS of the implication.

## 8 Arbitrary quantifier alternations

We examine first the computational complexity of the generic class of QLIs with an arbitrary number of quantifier alternations. These problems are described by System (2).

**Theorem 5** *Problem (2) is* **PSPACE-hard**.

*Proof* We will reduce the Q3SAT problem, which is **PSPACE-complete**, to an instance described by (2). Consider a Q3SAT instance $Q(\mathbf{x}, \mathbf{y})\ \phi(\mathbf{x}, \mathbf{y})$, where $Q(\mathbf{x}, \mathbf{y})$ represents the quantifier string, $\mathbf{x}$ is the set of existential variables, $\mathbf{y}$ is the set of universal variables, and $\phi$ is a conjunction of 3-literals clauses. We will construct a QLI which holds if and only if $Q(\mathbf{x}, \mathbf{y})\ \phi(\mathbf{x}, \mathbf{y})$ is satisfiable.

For each existential variable $x_i$ in the instance of Q3SAT, we introduce an existential variable $x_i$ and a universal variable $y_{x_i}$. Also, we add constraints $y_{x_i} \leq x_i$ and $y_{x_i} \leq 1 - x_i$ to the LHS and constraints $y_{x_i} \leq 0$ and $0 \leq x_i \leq 1$ to the RHS of the constructed QLI.

For each universal variable $y_i$ in the instance of Q3SAT, we introduce a universal variable $y_i$ and an existential variable $x_{y_i}$. We also add constraint $0 \leq y_i \leq 1$ to the LHS and constraints $0 \leq x_{y_i} \leq 1$, $2y_i - 1 \leq x_{y_i}$, and $x_{y_i} \leq 2y_i$ to the RHS of the constructed QLI.

Moreover, for each clause $\phi_j$ in the instance of Q3SAT, we add a corresponding constraint to the RHS of the constructed QLI. This constraint asks for the sum of the existential variables corresponding to the literals of $\phi_j$ to be greater than or equal to 1. Note that this constraint contains only existential variables. Even in the case of a universal variable $y_k$ in $\phi_j$, the constraint contains the existential variable $x_{y_k}$ of the QLI. Some indicative cases for the constraint added for a clause $\phi_j$ are presented below:

1. If $\phi_j = (x_i, y_k, x_l)$, we add the constraint $x_i + x_{y_k} + x_l \geq 1$.
2. If $\phi_j = (x_i, y_k, \bar{x_l})$, we add the constraint $x_i + x_{y_k} + (1 - x_l) \geq 1$.
3. If $\phi_j = (x_i, \bar{y_k}, \bar{x_l})$, we add the constraint $x_i + (1 - x_{y_k}) + (1 - x_l) \geq 1$.
4. If $\phi_j = (\bar{x_i}, \bar{y_k}, \bar{x_l})$, we add the constraint $(1 - x_i) + (1 - x_{y_k}) + (1 - x_l) \geq 1$.

The quantifier string of the QLI is constructed according to $Q(\mathbf{x}, \mathbf{y})$ of the corresponding Q3SAT instance. For $\exists \mathbf{x}_i$ in $Q(\mathbf{x}, \mathbf{y})$, we introduce $\exists \mathbf{x}_i \forall \mathbf{y}_{\mathbf{x}_i}$. For $\forall \mathbf{y}_i$ in $Q(\mathbf{x}, \mathbf{y})$, we introduce $\forall \mathbf{y}_i \exists \mathbf{x}_{\mathbf{y}_i}$. Let us call $\hat{Q}$ the alternation of quantifiers in $Q(\mathbf{x}, \mathbf{y})$. Then, we have that the alternation of quantifiers in the constructed QLI is $\hat{Q}\,\forall$ if $\hat{Q}$ ends with an existential quantifier, and $\hat{Q}\,\exists$ if $\hat{Q}$ ends with a universal quantifier.

Note that the constructed QLI is of the form (2). To complete the proof, we will show that this QLI holds if and only if $Q(\mathbf{x}, \mathbf{y})\ \phi(\mathbf{x}, \mathbf{y})$ is satisfiable. We start by establishing that all variables that participate in the constraints corresponding to the clauses of the Q3SAT instance are effectively restricted to values 0 or 1. To do so, we utilize the semantics introduced in Section 7.

Consider the constraints added to the QLI for each existential variable $x_i$ in the instance of Q3SAT. These constraints imply formula

$$\forall y_{x_i}\ [y_{x_i} \leq \min(x_i, 1 - x_i) \rightarrow (y_{x_i} \leq 0 \land 0 \leq x_i \leq 1)] \tag{6}$$

which leads to restricting $x_i \in \{0, 1\}$. To better see this, note that if $x_i \notin [0, 1]$, then constraint $0 \leq x_i \leq 1$ in the RHS would be violated. But then formula (6) would not hold (hence the existential player $\mathbf{X}$ would lose the game), since $\mathbf{X}$ cannot cause any constraint in the LHS to be violated as well. On the other hand, if $x_i \in (0, 1)$, then $\mathbf{Y}$ could choose to set $y_{x_i} = \min(x_i, 1 - x_i) > 0$, which would cause constraint $y_{x_i} \leq 0$ to be violated and hence formula (6) not to hold (causing $\mathbf{X}$ to lose the game). To sum up, any choice of $x_i \notin \{0, 1\}$ would cause the existential player to lose the game.

Now consider variables $y_i$ and $x_{y_i}$ added to the QLI for each universal variable $y_i$ in the instance of Q3SAT. We want to show that $x_{y_i}$ is restricted to values $\{0, 1\}$. Recall that $x_{y_i}$ appears after $y_i$ in the quantifier string of the constructed QLI. Hence, the value of $x_{y_i}$ is constrained by the value of $y_i$. Therefore, we will examine $x_{y_i}$ with respect to $y_i$. Firstly, note that $y_i \in [0, 1]$, since otherwise constraint $0 \leq y_i \leq 1$ (which is in the LHS) would be violated and the QLI would trivially hold (i.e., the universal player $\mathbf{Y}$ would lose the game). Secondly, note that if $y_i \in \{0,1\}$, the existential player $\mathbf{X}$ is forced to set $x_{y_i} = y_i$ (and thus $x_{y_i} \in \{0,1\}$). Any other choice of $x_{y_i}$ would violate at least one constraint of the RHS, causing $\mathbf{X}$ to lose the game.

Hence, we only need to show that the universal player $\mathbf{Y}$ will not choose $y_i \in (0,1)$. Suppose that $\mathbf{Y}$ can win by choosing $y_i = 0$ (hence restricting $x_{y_i} = 0$). If, instead, $y_i \in (0, \frac{1}{2}]$, then we have that $x_{y_i} \in [0, 2y_i]$, which means that the set of possible responses of the existential player $\mathbf{X}$ expands (and hence the result of the game may change as well). Therefore, $\mathbf{Y}$ has no incentive to choose $y_i \in (0, \frac{1}{2}]$ and not $y_i = 0$. Similarly, $\mathbf{Y}$ has no incentive to choose $y_i \in [\frac{1}{2}, 1)$ instead of $y_i = 1$. To sum up, since $\mathbf{Y}$ selects the values of his variables so as to win the game (by the discussion in Section 7), $\mathbf{Y}$ will choose $y_i \in \{0,1\}$ and hence $x_{y_i} \in \{0,1\}$ with $x_{y_i} = y_i$.

Let us now show that for the QLI obtained from a Q3SAT instance of the form $Q(\mathbf{x},\mathbf{y})\ \phi(\mathbf{x},\mathbf{y})$, the existential player $\mathbf{X}$ has a winning strategy for the QLI if and only if $Q(\mathbf{x},\mathbf{y})\ \phi(\mathbf{x},\mathbf{y})$ is satisfiable. Let values 0 and 1 in the QLI correspond to **false** and **true** in the Q3SAT respectively.

*If part.* Assume that $Q(\mathbf{x},\mathbf{y})\ \phi(\mathbf{x},\mathbf{y})$ is satisfiable. Then there exists a vector of Skolem functions

$$\mathbf{x}' = [c_1, f_1(y_1), f_2(y_1,y_2), \ldots, f_{n-1}(y_1,y_2,\ldots,y_{n-1})]^T$$

such that for any values $\mathbf{y}'$ given to $\mathbf{y}$, the Q3SAT expression is satisfied. Note that $f_i()$ are Skolem functions and are used to represent that the values of the elements of $\mathbf{x}'$ depend on the values of the corresponding elements of $\mathbf{y}'$. By restricting the $x_i$ variables to the set $\{0,1\}$ (as discussed above), constraints $y_{x_i} \le x_i$ and $y_{x_i} \le 1-x_i$ imply that $y_{x_i} \le 0$ (thus satisfying the corresponding constraint of the RHS of the QLI) and $0 \le x_i \le 1$ is also satisfied. On the other hand, by restricting $y_i$ variables to the set $\{0,1\}$ with $x_{y_i} = y_i$, constraints $0 \le y_i \le 1$, $0 \le x_{y_i} \le 1$, $2y_i - 1 \le x_{y_i}$, and $x_{y_i} \le 2y_i$ are all satisfied. Now consider the constraints constructed from $\phi_j$, and assume without loss of generality that $\phi_j$ is of the form $(x_i, y_k, x_l)$. Since the Q3SAT expression is satisfied (for $\mathbf{x} = \mathbf{x}'$), we must have that at least one of $x_i, y_k, x_l$ is **true**. This means that at least one of $x_i, y_k, x_l$ is equal to 1, and since $x_{y_k} = y_k$ if $y_k = 1$, we have that $x_i + x_{y_k} + x_l \ge 1$ holds. The same can be safely argued for all constraints corresponding to clauses, so the existential player has a winning strategy for the corresponding QLI.

*Only-if part.* Assume $\mathbf{X}$ has a winning strategy $U$ for the constructed QLI (representable by a vector of Skolem functions). This means that for every sequence of moves made by the universal player, the QLI holds. Recall that variables $x_i$ and $y_i$ are restricted to $\{0,1\}$. Consider the constraint constructed from $\phi_j$, and without loss of generality let that constraint be $x_i + x_{y_k} + x_l \ge 1$. Since the QLI holds (for strategy $U$), at least one of $x_i, x_{y_k}, x_l$ will be equal to 1, while also $x_{y_k} = y_k$. Hence, at least one of the literals in the original clause is **true**, causing $\phi_j$ to be **true** as well. The same can be argued for all clauses of $\phi(\mathbf{x},\mathbf{y})$. Hence, $Q(\mathbf{x},\mathbf{y})\ \phi(\mathbf{x},\mathbf{y})$ is satisfiable. □

## 9 Classes with no quantifier alternations

We commence our analysis of the complexities of various classes of QLIs by considering formulas with no quantifier alternations. Problem $\langle 0, \forall, \mathbf{B}\rangle$ is known to be in **P** (see Theorem 1). Here, we consider problem $\langle 0, \exists, \mathbf{B}\rangle$ described by:

$$\exists \mathbf{x} \ \ [\mathbf{A} \cdot \mathbf{x} \leq \mathbf{b} \rightarrow \mathbf{C} \cdot \mathbf{x} \leq \mathbf{d}] \tag{7}$$

and show that it is trivially in **P**.

**Theorem 6** *Problem* $\langle 0, \exists, \mathbf{B}\rangle$ *is in* **P**.

*Proof* By Theorem 1, we can decide in polynomial time whether $\forall \mathbf{x}\ [0 = 0 \rightarrow \mathbf{A} \cdot \mathbf{x} \leq \mathbf{b}]$ holds, i.e., whether $\forall \mathbf{x}\ \ \mathbf{A} \cdot \mathbf{x} \leq \mathbf{b}$ holds. If this is the case, System (7) reduces to showing that $\exists \mathbf{x}\ \ \mathbf{C} \cdot \mathbf{x} \leq \mathbf{d}$ holds, which is a problem in **P** (Khachiyan, 1979). Otherwise, System (7) is trivially true. □

## 10 Classes with 1-quantifier alternation

### 10.1 Problems starting with existential quantifier

Let us turn now our attention to 1-quantifier alternation classes. We start by considering problems $\langle 1, \exists, \mathbf{BB}\rangle$, described by:

$$\exists \mathbf{r} \ \ \forall \mathbf{x} \ \ [\mathbf{A} \cdot \mathbf{x} + \mathbf{N} \cdot \mathbf{r} \leq \mathbf{b} \rightarrow \mathbf{C} \cdot \mathbf{x} + \mathbf{M} \cdot \mathbf{r} \leq \mathbf{d}] \tag{8}$$

and problems $\langle 1, \exists, \mathbf{LB}\rangle$ described by:

$$\exists \mathbf{r} \ \ \forall \mathbf{x} \ \ [\mathbf{A} \cdot \mathbf{x} + \mathbf{N} \cdot \mathbf{r} \leq \mathbf{b} \rightarrow \mathbf{C} \cdot \mathbf{x} \leq \mathbf{d}] \tag{9}$$

Throughout this section, we will provide intuitions about the classes of formulas by resorting to parameterized linear systems, namely linear systems such as $\mathbf{A} \cdot \mathbf{x} + \mathbf{N} \cdot \mathbf{r} \leq \mathbf{b}$ where variables in $\mathbf{r}$ assume the role of parameters. The set of solutions of a parameterized linear system is a *parameterized polyhedron* (Loechner and Wilde, 1997), defined as the collection of polyhedra obtained by fixing the values for parameters. With this intuition, formula (9) can be interpreted as checking whether there exists a parameter instance $\mathbf{r}_0$ such that the polyhedron of $\mathbf{A} \cdot \mathbf{x} + \mathbf{N} \cdot \mathbf{r}_0 \leq \mathbf{d}$ is included in the polyhedron of $\mathbf{C} \cdot \mathbf{x} \leq \mathbf{d}$.

Let us start showing that the two classes above are reducible to each other.

**Theorem 7** *Problems* $\langle 1, \exists, \mathbf{BB}\rangle$ *and* $\langle 1, \exists, \mathbf{LB}\rangle$ *are polynomial-time reducible to each other.*

*Proof* Every instance of the class $\langle 1, \exists, \mathbf{LB}\rangle$ is clearly of the form (8), without the need of any transformation. Conversely, a formula (8) can be rewritten in linear time in the following equivalent form, which is in $\langle 1, \exists, \mathbf{LB}\rangle$:

$$\exists \mathbf{r}\ \forall \mathbf{x}, \mathbf{x}' \ \ [\mathbf{x}' = \mathbf{r} \wedge \mathbf{A} \cdot \mathbf{x} + \mathbf{N} \cdot \mathbf{r} \leq \mathbf{b} \rightarrow \mathbf{C} \cdot \mathbf{x} + \mathbf{M} \cdot \mathbf{x}' \leq \mathbf{d}] \tag{10}$$

Notice that the newly introduced variables $\mathbf{x}'$ are constrained to assume the values of the variables in $\mathbf{r}$; otherwise, the LHS is falsified and System (10) is trivially true. □

Let us now prove that the decision problem for the two classes is **NP-complete**. First, we show hardness.

**Lemma 1** *Problem* $\langle 1, \exists, \mathbf{BB} \rangle$ *is* **NP-hard**.

*Proof* The proof of Theorem 5 for Q3SAT with only existential quantifiers boils down to showing that 3SAT reduces in polynomial time to formula (8). Since 3SAT is **NP-complete**, we have that $\langle 1, \exists, \mathbf{BB} \rangle$ is **NP-hard**. □

*Example 2* Consider the following 3SAT instance.

$$\exists x_1\ \exists x_2\ \exists x_3\ \exists x_4\ \ (x_1, x_2, \bar{x}_3), (\bar{x}_2, x_3, x_4)$$

We use the reduction introduced in the proof of Theorem 5 to reduce this 3SAT instance to the following $\langle 1, \exists, \mathbf{BB} \rangle$ instance:

$$\exists x_1\ \exists x_2\ \exists x_3\ \exists x_4\ \forall y_{x_1}\ \forall y_{x_2}\ \forall y_{x_3}\ \forall y_{x_4}$$

$$\left.\begin{array}{r} y_{x_1} \leq x_1,\ y_{x_1} \leq 1 - x_1 \\ y_{x_2} \leq x_2,\ y_{x_2} \leq 1 - x_2 \\ y_{x_3} \leq x_3,\ y_{x_3} \leq 1 - x_3 \\ y_{x_4} \leq x_4,\ y_{x_4} \leq 1 - x_4 \end{array}\right\} \rightarrow \left\{\begin{array}{l} y_{x_1} \leq 0,\ 0 \leq x_1 \leq 1 \\ y_{x_2} \leq 0,\ 0 \leq x_2 \leq 1 \\ y_{x_3} \leq 0,\ 0 \leq x_3 \leq 1 \\ y_{x_4} \leq 0,\ 0 \leq x_4 \leq 1 \\ x_1 + x_2 + (1 - x_3) \geq 1 \\ (1 - x_2) + x_3 + x_4 \geq 1 \end{array}\right.$$

This $\langle 1, \exists, \mathbf{BB} \rangle$ instance holds if and only if the above 3SAT instance is satisfiable.

The following theorem completes our claim.

**Theorem 8** *Problems* $\langle 1, \exists, \mathbf{BB} \rangle$ *and* $\langle 1, \exists, \mathbf{LB} \rangle$ *are* **NP-complete**.

*Proof* By Theorem 7, we can restrict ourselves to considering only the $\langle 1, \exists, \mathbf{LB} \rangle$ class.

Let $P_1$ be $\mathbf{A} \cdot \mathbf{x} + \mathbf{N} \cdot \mathbf{r} \leq \mathbf{b}$, and $P_2$ be $\mathbf{C} \cdot \mathbf{x} \leq \mathbf{d}$ in (9). Since deciding the problem is **NP-hard** by Lemma 1, we only need to show that System (9) is in **NP**. We will do that by proving that if there exists an $\mathbf{r}_0$ such that

$$\forall \mathbf{x}\ \ [\mathbf{A} \cdot \mathbf{x} + \mathbf{N} \cdot \mathbf{r}_0 \leq \mathbf{b} \rightarrow \mathbf{C} \cdot \mathbf{x} \leq \mathbf{d}] \tag{11}$$

holds, then the representation of $\mathbf{r}_0$ is polynomial in the size of $\mathbf{A}, \mathbf{b}, \mathbf{C}, \mathbf{d}$, and $\mathbf{N}$. Since System (11) belongs to the class $\langle 0, \forall, \mathbf{B} \rangle$, which can be solved in polynomial time (Theorem 1), if such a polynomial sized $\mathbf{r}_0$ exists, then it is a valid certificate for the problem (9).

Assume that there exists an $\mathbf{r}_0$ such that (11) holds. Consider the problem

$$\forall \mathbf{r}\ \ \forall \mathbf{x}\ \ [\mathbf{A} \cdot \mathbf{x} + \mathbf{N} \cdot \mathbf{r} \leq \mathbf{b} \rightarrow \mathbf{C} \cdot \mathbf{x} \leq \mathbf{d}] \tag{12}$$

This is an instance of $\langle 0, \forall, \mathbf{B}\rangle$ which is in $\mathbf{P}$. If this implication is true, then (9) is true as well, and we are done. Assume it is false. Then there exists an $\mathbf{r}_2$ such that:

$$\exists \mathbf{x}\ \ [\mathbf{A}\cdot\mathbf{x}+\mathbf{N}\cdot\mathbf{r}_2 \leq \mathbf{b}, \mathbf{C}\cdot\mathbf{x} \not\leq \mathbf{d}]$$

Let $S_\mathbf{r}$ be the set of extreme points of $\mathbf{A}\cdot\mathbf{x}+\mathbf{N}\cdot\mathbf{r}\leq\mathbf{b}$. Thus we have that $S_{\mathbf{r}_0} \subseteq Sol(\mathbf{C}\cdot\mathbf{x}\leq\mathbf{d})$ and $S_{\mathbf{r}_2} \not\subseteq Sol(\mathbf{C}\cdot\mathbf{x}\leq\mathbf{d})$. For a constraint $\mathbf{c}_i^T\cdot\mathbf{x}\leq d_i$ in $\mathbf{C}\cdot\mathbf{x}\leq\mathbf{d}$, let $d_i'(\mathbf{r})$ be the solution of the linear program:

$$\begin{gathered}\max\ \mathbf{c}_i^T\cdot\mathbf{x}-d_i\\ \mathbf{A}\cdot\mathbf{x}+\mathbf{N}\cdot\mathbf{r}\leq\mathbf{b}\end{gathered}$$

and let $d'(\mathbf{r}) = \max_i(d_i'(\mathbf{r}))$. By (11), we have $d'(\mathbf{r}_0)\leq 0$. By (12), we have $d'(\mathbf{r}_2) > 0$. Since $\mathbf{A}\cdot\mathbf{x}+\mathbf{N}\cdot\mathbf{r}\leq\mathbf{b}$ is a convex set, there exists $0\leq\alpha\leq 1$ such that $d'(\mathbf{r}')=0$ when $\mathbf{r}'=\alpha\cdot\mathbf{r}_0+(1-\alpha)\cdot\mathbf{r}_2$. Thus, we have that $\forall\mathbf{x}\ \ [\mathbf{A}\cdot\mathbf{x}+\mathbf{N}\cdot\mathbf{r}'\leq\mathbf{b}\rightarrow\mathbf{C}\cdot\mathbf{x}\leq\mathbf{d}]$ and that for some $i$, $d_i'(\mathbf{r}')=0$. Therefore, for some $\mathbf{x}'\in S_{\mathbf{r}'}$, we have that $\mathbf{C}\cdot\mathbf{x}'\leq\mathbf{d}$ and $\mathbf{c}_i^T\cdot\mathbf{x}'=d_i$. We can now create a linear system from the constraints in $\mathbf{A}\cdot\mathbf{x}+\mathbf{N}\cdot\mathbf{r}\leq\mathbf{b}$ and $\mathbf{C}\cdot\mathbf{x}\leq\mathbf{d}$ for which $(\mathbf{x}',\mathbf{r}')$ is an extreme point as follows:

1. Add the constraint $\mathbf{a}^T\cdot\mathbf{x}+\mathbf{n}^T\cdot\mathbf{r}=b$ for each constraint $\mathbf{a}^T\cdot\mathbf{x}+\mathbf{n}^T\cdot\mathbf{r}\leq b$ of $\mathbf{A}\cdot\mathbf{x}+\mathbf{N}\cdot\mathbf{r}\leq\mathbf{b}$ such that $\mathbf{a}^T\cdot\mathbf{x}'+\mathbf{n}^T\cdot\mathbf{r}'=b$.
2. Add the constraint $\mathbf{c}_i^T\cdot\mathbf{x}=d_i$.
3. Add the remaining constraints from $\mathbf{A}\cdot\mathbf{x}+\mathbf{N}\cdot\mathbf{r}\leq\mathbf{b}$ and $\mathbf{C}\cdot\mathbf{x}\leq\mathbf{d}$.

Since $\mathbf{r}'$ is part of an extreme point solution to this system, its representation is polynomial in the size of $\mathbf{A},\mathbf{b},\mathbf{C},\mathbf{d}$, and $\mathbf{N}$. Therefore, we have our desired polynomial sized $\mathbf{r}_0=\mathbf{r}'$ satisfying (11). □

Let us turn now our attention to problems $\langle 1,\exists,\mathbf{RB}\rangle$, described by:

$$\exists\mathbf{r}\ \forall\mathbf{x}\ \ [\mathbf{A}\cdot\mathbf{x}\leq\mathbf{b}\rightarrow\mathbf{C}\cdot\mathbf{x}+\mathbf{M}\cdot\mathbf{r}\leq\mathbf{d}] \tag{13}$$

This implication can be read as whether there exist values $\mathbf{r}_0$ such that the set of solutions of $\mathbf{C}\cdot\mathbf{x}+\mathbf{M}\cdot\mathbf{r}_0\leq\mathbf{d}$ includes the set of solutions to $\mathbf{A}\cdot\mathbf{x}\leq\mathbf{b}$. (Eirinakis et al, 2012) firstly showed that problem (13) is in $\mathbf{P}$. This result is surprising in view of Theorem 8. By moving the set of existentially quantified variables from the RHS to the LHS of the implication, the corresponding decision problem from $\mathbf{P}$ becomes **NP-complete**. For completeness of exposition, we report here a (slightly revised) proof that will be useful later on for reasoning on sub-classes of $\langle 1,\exists,\mathbf{RB}\rangle$.

**Theorem 9** *Problem* $\langle 1,\exists,\mathbf{RB}\rangle$ *is in* $\mathbf{P}$.

*Proof* Consider a quantified linear implication described by System (13). Let $P_1$ denote the LHS and $P_2$ the RHS of the implication. The satisfiability of $P_1$ can be checked in polynomial time (Khachiyan, 1979). If $P_1$ is unsatisfiable, (13) is always true. Assume now it is satisfiable. We build a linear program on

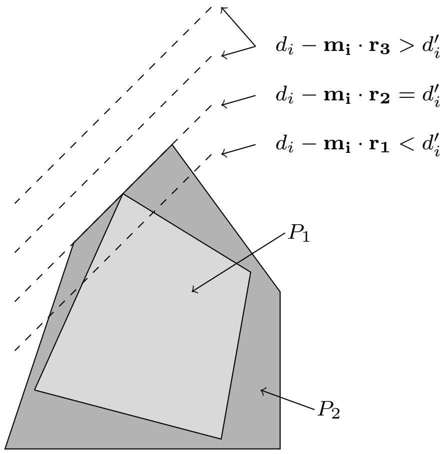


**Fig. 2** Let $P_1 = \mathbf{A} \cdot \mathbf{x} \leq \mathbf{b}$ and $P_2 = \mathbf{C} \cdot \mathbf{x} + \mathbf{M} \cdot \mathbf{r} \leq \mathbf{d}$. Then, $\exists \mathbf{r}\ \forall \mathbf{x}\ P_1 \to P_2$ if and only if $\mathbf{d} - \mathbf{M} \cdot \mathbf{r} \geq \mathbf{d'}$, with $d'_i = \max\{\mathbf{c}_i^T \cdot \mathbf{x}_0 | \mathbf{x}_0 \in Sol(\mathbf{A} \cdot \mathbf{x} \leq \mathbf{b})\}$.

$\mathbf{r}$ as follows. For every row $\mathbf{c}_i^T \cdot \mathbf{x} + \mathbf{m}_i^T \cdot \mathbf{r} \leq d_i$ in $P_2$, let $d'_i$ be the solution of the linear program:

$$\max\ \mathbf{c}_i^T \cdot \mathbf{x}$$
$$\mathbf{A} \cdot \mathbf{x} \leq \mathbf{b}$$

If the linear program is unbounded, there cannot be any variable instance $\mathbf{r}_0$ of $\mathbf{r}$ such that $\mathbf{c}_i^T \cdot \mathbf{x} + \mathbf{m}_i^T \cdot \mathbf{r}_0 \leq d_i$ for every $\mathbf{x}$. Thus, (13) does not hold. Assume now that all $d'_i$s are finite and let $\mathbf{d'}$ be the vector of all $d'_i$s. We claim that (13) holds if and only if the following linear system on $\mathbf{r}$ is feasible:

$$\mathbf{d} - \mathbf{M} \cdot \mathbf{r} \geq \mathbf{d'} \tag{14}$$

Intuitively, this means that there exists some value of $\mathbf{r}$, say $\mathbf{r}_0$, for which each hyperplane of $\mathbf{C} \cdot \mathbf{x} \leq \mathbf{d} - \mathbf{M} \cdot \mathbf{r}_0$ is either incident to some extreme point of $\mathbf{A} \cdot \mathbf{x} \leq \mathbf{b}$ or not in $Sol(\mathbf{A} \cdot \mathbf{x} \leq \mathbf{b})$ (e.g., see Figure 2, where $\mathbf{r_1}, \mathbf{r_2}$, and $\mathbf{r_3}$ are different values of $\mathbf{r}$).

*If part.* Let $\mathbf{r}_0$ be such that $\mathbf{d} - \mathbf{M} \cdot \mathbf{r}_0 \geq \mathbf{d'}$. Since by construction of $\mathbf{d'}$, we have that $\forall \mathbf{x}\ [\mathbf{A} \cdot \mathbf{x} \leq \mathbf{b} \to \mathbf{C} \cdot \mathbf{x} \leq \mathbf{d'}]$ holds, by transitivity the following holds:

$$\forall \mathbf{x}\ \ [\mathbf{A} \cdot \mathbf{x} \leq \mathbf{b} \to \mathbf{C} \cdot \mathbf{x} + \mathbf{M} \cdot \mathbf{r}_0 \leq \mathbf{d}]$$

By re-introducing existential quantifiers on $\mathbf{r}$, we have (13).

*Only-if part.* By hypothesis, there exists $\mathbf{r}_0$ such that

$$\forall \mathbf{x}\ \ [\mathbf{A} \cdot \mathbf{x} \leq \mathbf{b} \to \mathbf{C} \cdot \mathbf{x} + \mathbf{M} \cdot \mathbf{r}_0 \leq \mathbf{d}]$$

Let us show that $\mathbf{r}_0$ is a solution of (14). Consider a row $\mathbf{c}_i^T \cdot \mathbf{x} + \mathbf{m}_i^T \cdot \mathbf{r} \leq d_i$ of $P_2$. By definition of $\mathbf{r}_0$, the following holds: $\forall \mathbf{x}\ \ [\mathbf{A} \cdot \mathbf{x} \leq \mathbf{b} \to \mathbf{c}_i^T \cdot \mathbf{x} \leq d_i - \mathbf{m}_i^T \cdot \mathbf{r}_0]$, which implies:

$$d_i - \mathbf{m}_i^T \cdot \mathbf{r}_0 \geq \max\{\mathbf{c}_i^T \cdot \mathbf{x} \mid \mathbf{A} \cdot \mathbf{x} \leq \mathbf{b}\}$$

By definition of $\mathbf{d}'$, we conclude $\mathbf{d} - \mathbf{M} \cdot \mathbf{r}_0 \geq \mathbf{d}'$, hence (14) is satisfiable. $\square$

As an immediate consequence, the sub-classes $\langle 1, \exists, \mathbf{RR}\rangle$ and $\langle 1, \exists, \mathbf{RL}\rangle$ are also in $\mathbf{P}$. In particular, problems from $\langle 1, \exists, \mathbf{RR}\rangle$ allow for extending Theorem 2 to the case of unbounded universal quantification.

**Corollary 1** *The decision problem for* $\exists \mathbf{y}\ \forall \mathbf{x}\ \mathbf{A}\mathbf{x} + \mathbf{N} \cdot \mathbf{y} \leq \mathbf{b}$ *is in* $\mathbf{P}$.

*Proof* The formula can be rewritten into the equivalent form $\exists \mathbf{y}\ \forall \mathbf{x}\ [0 = 0 \rightarrow \mathbf{A}\mathbf{x} + \mathbf{N} \cdot \mathbf{y} \leq \mathbf{b}]$ which belongs to $\langle 1, \exists, \mathbf{RR}\rangle$. $\square$

It is interesting to apply the procedure from the proof of Theorem 9 to the formula $\exists \mathbf{y}\ \forall \mathbf{x}\ [0 = 0 \rightarrow \mathbf{A}\mathbf{x} + \mathbf{N} \cdot \mathbf{y} \leq \mathbf{b}]$. First, we check whether $\mathbf{A} = \mathbf{0}$. If this is the case, the problem boils down to $\exists \mathbf{y}\ \mathbf{N} \cdot \mathbf{y} \leq \mathbf{b}$, which is in $\mathbf{P}$. Otherwise, the implication is false, since for some $\mathbf{c}^T \cdot \mathbf{x} + \mathbf{m}^T \cdot \mathbf{y} \leq b$ in the RHS of the implication, we have that $max\{\mathbf{c}^T \cdot \mathbf{x} \mid 0 = 0\}$ is unbounded.

An instance of $\langle 1, \exists, \mathbf{BR}\rangle$ is described by:

$$\exists \mathbf{r}\ \ \forall \mathbf{x}\ \ [\mathbf{N} \cdot \mathbf{r} \leq \mathbf{b} \rightarrow \mathbf{C} \cdot \mathbf{x} + \mathbf{M} \cdot \mathbf{r} \leq \mathbf{d}] \tag{15}$$

Such a class and, *a fortiori*, its sub-class $\langle 1, \exists, \mathbf{LR}\rangle$ are decidable in polynomial time.

**Theorem 10** *Problem* $\langle 1, \exists, \mathbf{BR}\rangle$ *is in* $\mathbf{P}$.

*Proof* By Theorem 1, we can check in polynomial time whether $\forall \mathbf{r}\ [0 = 0 \rightarrow \mathbf{N} \cdot \mathbf{r} \leq \mathbf{b}]$, i.e., whether $\forall \mathbf{r}\ \mathbf{N} \cdot \mathbf{r} \leq \mathbf{b}$ holds. If it does not hold, then System (15) trivially holds. If it holds, then System (15) reduces to the equivalent form $\exists \mathbf{r}\ \forall \mathbf{x}\ \mathbf{C} \cdot \mathbf{x} + \mathbf{M} \cdot \mathbf{r} \leq \mathbf{d}$, which can be checked in polynomial time by Corollary 1. $\square$

Finally, an instance of $\langle 1, \exists, \mathbf{BL}\rangle$ is described by:

$$\exists \mathbf{r}\ \ \forall \mathbf{x}\ \ [\mathbf{A} \cdot \mathbf{x} + \mathbf{N} \cdot \mathbf{r} \leq \mathbf{b} \rightarrow \mathbf{M} \cdot \mathbf{r} \leq \mathbf{d}] \tag{16}$$

If $\exists \mathbf{r}\ \mathbf{M} \cdot \mathbf{r} \leq \mathbf{d}$ holds, formula (16) is trivially true. Otherwise, the formula boils down to $\exists \mathbf{y}\ \forall \mathbf{x}\ [\mathbf{A} \cdot \mathbf{x} + \mathbf{N} \cdot \mathbf{y} \leq \mathbf{b} \rightarrow 0 = 1]$, which is a formula in $\langle 1, \exists, \mathbf{LL}\rangle$. This formula can be rewritten as the negation of a QLP with unbounded universal quantification, i.e., $\neg \forall \mathbf{r}\ \exists \mathbf{x}\ \mathbf{A} \cdot \mathbf{x} + \mathbf{N} \cdot \mathbf{r} \leq \mathbf{b}$. A partial extension of Theorem 3 to QLP with unbounded universal quantification is reported next.

**Corollary 2** *The decision problem for a formula* $\forall \mathbf{y}\ \ \exists \mathbf{x}\ \ \mathbf{A} \cdot \mathbf{x} + \mathbf{N} \cdot \mathbf{y} \leq \mathbf{b}$ *is in* $\mathbf{coNP}$.

*Proof* The formula can be rewritten into the equivalent form $\neg \exists \mathbf{y}\ \forall \mathbf{x}\ [\mathbf{A} \cdot \mathbf{x} + \mathbf{N} \cdot \mathbf{y} \leq \mathbf{b} \rightarrow 0 = 1]$. Its negation belongs to $\langle 1, \exists, \mathbf{LL}\rangle$, which is in $\mathbf{NP}$ since it is a subclass of the **NP-complete** class $\langle 1, \exists, \mathbf{LB}\rangle$ (Theorem 8). Hence, the decision problem for our formula is in $\mathbf{coNP}$. $\square$

From the above reasoning, it follows that the classes $\langle 1, \exists, \mathbf{BL}\rangle$ and $\langle 1, \exists, \mathbf{LL}\rangle$ are reducible to each other in polynomial time. Both are in $\mathbf{NP}$, since they are sub-classes of the **NP-complete** class $\langle 1, \exists, \mathbf{BB}\rangle$ (Theorem 8). Unfortunately, their hardness is unknown.

Figure 3 summarizes our results for $\exists\ \forall$ classes of QLIs.

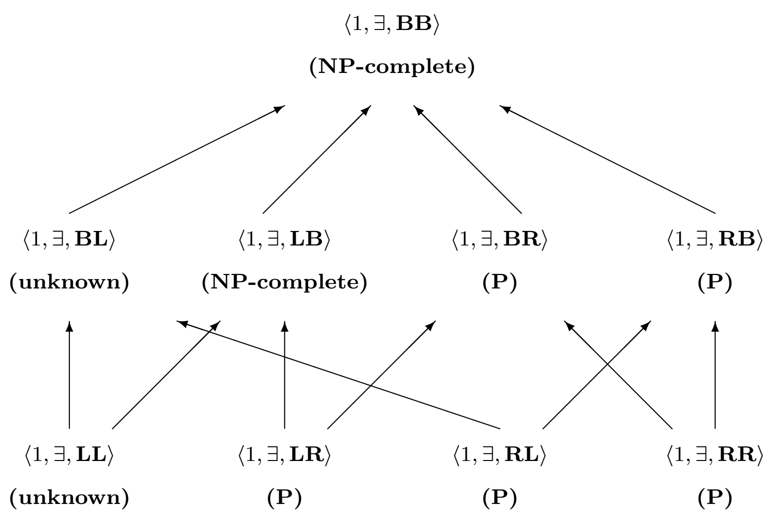


**Fig. 3** Complexities of $\exists\ \forall$ classes of QLIs. Arrows denote reducibilities. Classes labelled **(unknown)** are in **NP** but their hardness is unknown.

## 10.2 Problems starting with universal quantifier

Let us turn our attention to 1-quantifier alternations that start with universal quantification. Problems in $\langle 1, \forall, \mathbf{BR} \rangle$ are described by:

$$\forall \mathbf{r}\ \ \exists \mathbf{x}\ \ [\mathbf{N} \cdot \mathbf{r} \leq \mathbf{b} \rightarrow \mathbf{C} \cdot \mathbf{x} + \mathbf{M} \cdot \mathbf{r} \leq \mathbf{d}] \tag{17}$$

Such formulas can be intuitively interpreted as checking whether for every parameter $\mathbf{r}_0$ in the space of solutions of $\mathbf{N} \cdot \mathbf{r} \leq \mathbf{b}$ the polyhedron of $\mathbf{C} \cdot \mathbf{x} + \mathbf{M} \cdot \mathbf{r}_0 \leq \mathbf{d}$ is non-empty. The class $\langle 1, \forall, \mathbf{BR} \rangle$ and its super-class $\langle 1, \forall, \mathbf{BB} \rangle$ are **coNP-hard**.

**Theorem 11** *Problems* $\langle 1, \forall, \mathbf{BB} \rangle$ *and* $\langle 1, \forall, \mathbf{BR} \rangle$ *are* **coNP-hard**.

*Proof* Consider an instance of the **F**-QLP problem: $\forall \mathbf{r} \in [\mathbf{l}, \mathbf{u}]\ \exists \mathbf{x}\ \mathbf{C} \cdot \mathbf{x} + \mathbf{M} \cdot \mathbf{r} \leq \mathbf{d}$. It can be rewritten into an equivalent formula (17) by elementary logic properties:

$$\forall \mathbf{r}\ \exists \mathbf{x}\ [\mathbf{l} \leq \mathbf{r} \leq \mathbf{u} \rightarrow \mathbf{C} \cdot \mathbf{x} + \mathbf{M} \cdot \mathbf{r} \leq \mathbf{b}]$$

Since **F**-QLP is **coNP-complete** (Theorem 3), the class of problems $\langle 1, \forall, \mathbf{BR} \rangle$ is **coNP-hard**; a fortiori, its super-class $\langle 1, \forall, \mathbf{BB} \rangle$ is **coNP-hard** as well. □

The exact complexities of the above problems are open. Next, we explore some $\forall\exists$-QLI classes that are in **P**. Problems in $\langle 1, \forall, \mathbf{LB} \rangle$ are described by:

$$\forall \mathbf{r}\ \exists \mathbf{x}\ [\mathbf{A} \cdot \mathbf{x} + \mathbf{N} \cdot \mathbf{r} \leq \mathbf{b} \rightarrow \mathbf{C} \cdot \mathbf{x} \leq \mathbf{d}] \tag{18}$$

**Theorem 12** *Problem* $\langle 1, \forall, \mathbf{LB} \rangle$ *is in* **P**.

*Proof* Consider any instance of System (18). In order to decide such a problem we can first check in polynomial time (Khachiyan, 1979) whether $\exists \mathbf{x}\ \mathbf{C} \cdot \mathbf{x} \leq \mathbf{d}$ holds. If it holds, then System (18) also holds. Otherwise, the problem reduces to deciding whether $\exists \mathbf{r}\ \forall \mathbf{x}\ \mathbf{A} \cdot \mathbf{x} + \mathbf{N} \cdot \mathbf{r} \leq \mathbf{b}$ holds, which is also in $\mathbf{P}$ by Corollary 1. If it holds, System (18) does not hold; otherwise, System (18) holds. Summarizing, problem $\langle 1, \forall, \mathbf{LB} \rangle$ is in $\mathbf{P}$. $\square$

As an immediate consequence, the sub-classes $\langle 1, \forall, \mathbf{LL} \rangle$ and $\langle 1, \forall, \mathbf{LR} \rangle$ are in $\mathbf{P}$ as well. The same result can be obtained for $\langle 1, \forall, \mathbf{BL} \rangle$:

$$\forall \mathbf{r}\ \exists \mathbf{x}\ [\mathbf{A} \cdot \mathbf{x} + \mathbf{N} \cdot \mathbf{r} \leq \mathbf{b} \rightarrow \mathbf{M} \cdot \mathbf{r} \leq \mathbf{d}] \tag{19}$$

and, a fortiori, for its sub-class $\langle 1, \forall, \mathbf{RL} \rangle$.

**Theorem 13** *Problem* $\langle 1, \forall, \mathbf{BL} \rangle$ *is in* $\mathbf{P}$.

*Proof* We distinguish two cases. If $\mathbf{A} = \mathbf{0}$, formula (19) boils down to a $\langle 0, \forall, \mathbf{B} \rangle$ formula, whose decision problem is in $\mathbf{P}$ by Theorem 1. Assume now that there is a row $\mathbf{a}^T \cdot \mathbf{x} + \mathbf{n}^T \cdot \mathbf{r} \leq b$ in the LHS of (19) such that $\mathbf{a} \neq \mathbf{0}$. We claim that (19) is always true. Let $\mathbf{r}_0$ be a fixed value for $\mathbf{r}$. Since $\mathbf{a} \neq \mathbf{0}$, there exists some $\mathbf{x}_0$ such that $\mathbf{a}^T \cdot \mathbf{x}_0 + \mathbf{n}^T \cdot \mathbf{r}_0 > b$. Thus, the implication $\mathbf{A} \cdot \mathbf{x}_0 + \mathbf{N} \cdot \mathbf{r}_0 \leq \mathbf{b} \rightarrow \mathbf{M} \cdot \mathbf{r}_0 \leq \mathbf{d}$ is true, since its LHS is false. $\square$

Problems in $\langle 1, \forall, \mathbf{RB} \rangle$ are described by:

$$\forall \mathbf{r}\ \exists \mathbf{x}\ [\mathbf{A} \cdot \mathbf{x} \leq \mathbf{b} \rightarrow \mathbf{C} \cdot \mathbf{x} + \mathbf{M} \cdot \mathbf{r} \leq \mathbf{d}]$$

This formula can be reduced in polynomial time to a $\langle 1, \forall, \mathbf{RR} \rangle$ formula as follows. First, we check whether $\forall \mathbf{x}\ \mathbf{A} \cdot \mathbf{x} \leq \mathbf{b}$ holds, which can be done in polynomial time by Theorem 1. If it does not hold, then the formula above is always true. If it holds, then the formula above reduces to the $\langle 1, \forall, \mathbf{RR} \rangle$ formula:

$$\forall \mathbf{r}\ \exists \mathbf{x}\ [0 = 0 \rightarrow \mathbf{C} \cdot \mathbf{x} + \mathbf{M} \cdot \mathbf{r} \leq \mathbf{d}]$$

or, equivalently, to the unbounded QLP formula $\forall \mathbf{r}\ \exists \mathbf{x}\ \mathbf{C} \cdot \mathbf{x} + \mathbf{M} \cdot \mathbf{r} \leq \mathbf{d}$. By Corollary 2, we know that deciding whether it holds is in $\mathbf{coNP}$. However, its hardness is unknown. Therefore, the hardness of problems $\langle 1, \forall, \mathbf{RB} \rangle$ and $\langle 1, \forall, \mathbf{RR} \rangle$ is unknown as well. Figure 4 summarizes our results for $\forall\,\exists$ classes of QLIs. The symmetry with $\exists\,\forall$ classes (see Figure 3) is apparent.

## 11 Classes with 2-quantifier alternations and beyond

### 11.1 Problems $\langle k, \exists, \mathbf{B}^{k+1} \rangle$ and $\langle k, \forall, \mathbf{B}^{k+1} \rangle$

We generalize Lemma 1 to the case of $k$ alternations of quantifiers, with $k$ odd. In the following, we write $\mathbf{B}^{k+1}$ to denote the string $\underbrace{\mathbf{B} \ldots \mathbf{B}}_{k+1}$.

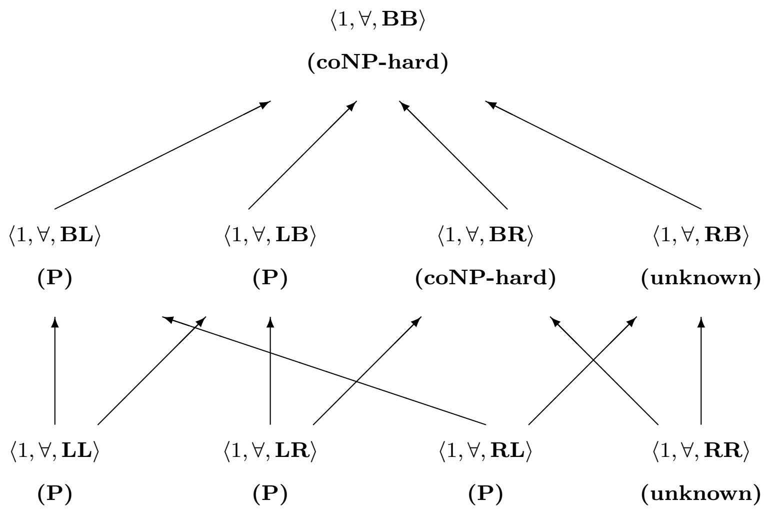


**Fig. 4** Complexities of ∀ ∃ classes of QLIs. Arrows denote reducibilities. Classes labelled **(unknown)** are in **coNP** but their hardness is unknown.

**Theorem 14** *Problem* $\langle k, \exists, \mathbf{B}^{k+1}\rangle$ *with* $k$ *odd is* $\mathbf{\Sigma_k^P}$**-hard**.

*Proof* Consider the class of Q3SAT formulas with $k$ quantifiers starting with an existential one, i.e., with the quantifier string of the form $\exists\forall\ldots\exists$. Such a class is $\mathbf{\Sigma_k^P}$**-complete** (the assumption that $k$ is odd is essential) (Stockmeyer, 1977, Theorem 4.1). The proof of Theorem 5 reduces such a class to a QLI with a quantifier string obtained by adding a universal quantifier at the end, namely to a $\langle k, \exists, \mathbf{B}^{k+1}\rangle$ formula. Hence, the result. □

Similarly, we generalize Theorem 11.

**Theorem 15** *Problem* $\langle k, \forall, \mathbf{B}^{k+1}\rangle$ *with* $k$ *even is* $\mathbf{\Pi_k^P}$**-hard**.

*Proof* Consider the class of Q3SAT formulas with $k$ quantifiers starting with a universal one, i.e., with the quantifier string of the form $\forall\exists\ldots\exists$. Such a class is $\mathbf{\Pi_k^P}$**-complete** (the assumption that $k$ is even is essential) (Stockmeyer, 1977, Theorem 4.1). The proof of Theorem 5 reduces such a class to a QLI with a quantifier string obtained by adding a universal quantifier at the end, namely to a $\langle k, \forall, \mathbf{B}^{k+1}\rangle$ formula. Hence, the result. □

## 11.2 Problems $\langle 2, \forall, \mathbf{BBB}\rangle$ and $\langle 2, \forall, \mathbf{LLB}\rangle$

Let us focus now on some classes with 2-quantifier alternations with intuitive interpretations in terms of parameterized systems. Problem $\langle 2, \forall, \mathbf{LLB}\rangle$ is described by:

$$\forall \mathbf{s}\ \exists \mathbf{r}\ \forall \mathbf{x}\ \ [\mathbf{A}\cdot\mathbf{x} + \mathbf{P}\cdot\mathbf{s} + \mathbf{N}\cdot\mathbf{r} \le \mathbf{b} \rightarrow \mathbf{C}\cdot\mathbf{x} \le \mathbf{d}] \tag{20}$$

This formula can be read as whether for every parameter values $\mathbf{s}$ (e.g., sensors values in some reactive system) there exists at least a value of the remaining parameters $\mathbf{r}$ (control factors) such that the set of solutions of the LHS system (future states of the reactive system) is included in the set of solutions of the non-parameterized RHS system (safe states of the reactive system). First, let us show that the problem is equivalent to the general class $\langle 2, \forall, \mathbf{BBB} \rangle$.

**Theorem 16** *Problems* $\langle 2, \forall, \mathbf{BBB} \rangle$ *and* $\langle 2, \forall, \mathbf{LLB} \rangle$ *are polynomial-time reducible to each other.*

*Proof* Every formula in $\langle 2, \forall, \mathbf{LLB} \rangle$ is also a formula in $\langle 2, \forall, \mathbf{BBB} \rangle$. For the opposite, consider a formula of the form $\langle 2, \forall, \mathbf{BBB} \rangle$:

$$\forall \mathbf{s}\ \exists \mathbf{r}\ \forall \mathbf{x}\ \ [\mathbf{A} \cdot \mathbf{x} + \mathbf{P} \cdot \mathbf{s} + \mathbf{N} \cdot \mathbf{r} \leq \mathbf{b} \rightarrow \mathbf{C} \cdot \mathbf{x} + \mathbf{T} \cdot \mathbf{s} + \mathbf{M} \cdot \mathbf{r} \leq \mathbf{d}]$$

It is readily checked that it holds if and only if the following formula holds:

$$\begin{array}{l} \forall \mathbf{s}\ \exists \mathbf{r}\ \forall \mathbf{x}, \mathbf{x}', \mathbf{x}'' \\ \quad [\mathbf{x}' = \mathbf{s} \wedge \mathbf{x}'' = \mathbf{r} \wedge \mathbf{A} \cdot \mathbf{x} + \mathbf{P} \cdot \mathbf{s} + \mathbf{N} \cdot \mathbf{r} \leq \mathbf{b} \rightarrow \mathbf{C} \cdot \mathbf{x} + \mathbf{T} \cdot \mathbf{x}' + \mathbf{M} \cdot \mathbf{x}'' \leq \mathbf{d}] \end{array}$$

where the newly introduced variables $\mathbf{x}'$ (resp., $\mathbf{x}''$) are constrained to assume the values of $\mathbf{s}$ (resp., $\mathbf{r}$). Since the formula above is of the form (20), and it is obtained in linear time, $\langle 2, \forall, \mathbf{BBB} \rangle$ reduces in polynomial time to $\langle 2, \forall, \mathbf{LLB} \rangle$. □

The following is immediately derived by combining Theorem 15 for $k = 2$ and Theorem 16.

**Corollary 3** *Problems* $\langle 2, \forall, \mathbf{BBB} \rangle$ *and* $\langle 2, \forall, \mathbf{LLB} \rangle$ *are* $\mathbf{\Pi_2^P}$**-hard**.

*Example 3* Consider the following Q3SAT instance.

$$\forall y_1\ \forall y_2\ \exists x_1\ \exists x_2\ \ (y_1, y_2, x_2), (\bar{y}_2, x_1, \bar{x}_2)$$

We use the approach from the proof of Theorem 5 to reduce this Q3SAT instance to the following $\langle 2, \forall, \mathbf{BBB} \rangle$ instance:

$$\forall y_1\ \forall y_2\ \exists x_{y_1}\ \exists x_{y_2}\ \exists x_1\ \exists x_2\ \forall y_{x_1}\ \forall y_{x_2}$$

$$\left.\begin{array}{r} y_{x_1} \leq x_1,\ y_{x_1} \leq 1 - x_1 \\ y_{x_2} \leq x_2,\ y_{x_2} \leq 1 - x_2 \\ 0 \leq y_1 \leq 1 \\ 0 \leq y_2 \leq 1 \end{array}\right\} \rightarrow \left\{\begin{array}{l} y_{x_1} \leq 0,\ 0 \leq x_1 \leq 1 \\ y_{x_2} \leq 0,\ 0 \leq x_2 \leq 1 \\ 0 \leq x_{y_1} \leq 1 \\ 0 \leq x_{y_2} \leq 1 \\ x_{y_1} \geq 2y_1 - 1, x_{y_1} \leq 2y_1 \\ x_{y_2} \geq 2y_2 - 1, x_{y_2} \leq sy_2 \\ x_{y_1} + x_{y_2} + x_2 \geq 1 \\ (1 - x_{y_2}) + x_1 + (1 - x_2) \geq 1 \end{array}\right.$$

This $\langle 2, \forall, \mathbf{BBB} \rangle$ instance holds if and only if the above Q3SAT instance is satisfiable.

### 11.3 Problem $\langle 2, \forall, \mathbf{LRB} \rangle$

Problem $\langle 2, \forall, \mathbf{LRB} \rangle$ is described by:

$$\forall \mathbf{s} \, \exists \mathbf{r} \, \forall \mathbf{x} \, [\mathbf{A} \cdot \mathbf{x} \le \mathbf{b} + \mathbf{N} \cdot \mathbf{s} \rightarrow \mathbf{C} \cdot \mathbf{x} \le \mathbf{d} + \mathbf{M} \cdot \mathbf{r}] \tag{21}$$

This implication can be read as whether for all values $\mathbf{s}_0$ for parameters $\mathbf{s}$ there exists values $\mathbf{r}_0$ for parameters $\mathbf{r}$ such that $\mathbf{C} \cdot \mathbf{x} \le \mathbf{d} + \mathbf{M} \cdot \mathbf{r}_0$ includes $\mathbf{A} \cdot \mathbf{x} \le \mathbf{b} + \mathbf{N} \cdot \mathbf{s}_0$ – a form of inclusion of parameterized polyhedra. To stress such an intuition, in this subsection we write $\mathbf{s}$ and $\mathbf{r}$ in the RHS of inequalities of linear systems.

**Theorem 17** *Problem* $\langle 2, \forall, \mathbf{LRB} \rangle$ *is* **coNP-hard***.*

*Proof* Consider an instance of the $\mathbf{F}$-QLP problem:

$$\forall \mathbf{s} \in [\mathbf{l}, \mathbf{u}] \, \exists \mathbf{r} \; \mathbf{C} \cdot \mathbf{s} \le \mathbf{d} + \mathbf{M} \cdot \mathbf{r}$$

This problem (which is **coNP-complete** by Theorem 3) is reduced to a problem described by System (21) as follows. For each $s_i$ of $\mathbf{s}$, add $x_i$ such that $\mathbf{s} = \mathbf{x}$. We create the following equivalent implication using elementary logical properties:

$$\forall \mathbf{s} \, \exists \mathbf{r} \, \forall \mathbf{x} \;\; [(\mathbf{x} = \mathbf{s} \wedge \mathbf{l} \le \mathbf{s} \le \mathbf{u}) \rightarrow \mathbf{C} \cdot \mathbf{x} \le \mathbf{d} + \mathbf{M} \cdot \mathbf{r}]$$

Note that the interval constraints of the universally quantified variables that appear in the quantifier string of the $\mathbf{F}$-QLP problem are placed in the LHS of the implication. It is easy to see that the initial $\mathbf{F}$-QLP problem is satisfied if and only if the resultant $\langle 2, \forall, \mathbf{LRB} \rangle$ problem is satisfied: If there exists $\mathbf{s}$, with $\mathbf{s} \in [\mathbf{l}, \mathbf{u}]$, such that $\mathbf{x} = \mathbf{s}$ is included in $\mathbf{C} \cdot \mathbf{x} \le \mathbf{d} + \mathbf{M} \cdot \mathbf{r}$ (i.e., $\mathbf{C} \cdot \mathbf{s} \le \mathbf{d} + \mathbf{M} \cdot \mathbf{r}$ is satisfied for $\mathbf{s} \in [\mathbf{l}, \mathbf{u}]$), then the corresponding $\mathbf{F}$-QLP will be satisfied as well. On the other hand, if the $\mathbf{F}$-QLP is satisfied, then since $\mathbf{x} = \mathbf{s}$, the corresponding implication will also be satisfied. □

Next, we present a special case of problem $\langle 2, \forall, \mathbf{LRB} \rangle$ that is still **coNP-hard**, although the LHS of the implication consists only of difference constraints. Recall that a difference constraint is a constraint of the form $x_i - x_j \le b_{ij}$ (Section 2).

**Lemma 2** *Problem (21) is* **coNP-hard** *even if* $\mathbf{A} \cdot \mathbf{x} \le \mathbf{b} + \mathbf{N} \cdot \mathbf{s}$ *is restricted to difference constraints with respect to variables in* $\mathbf{x}$ *and* $\mathbf{s}$*.*

*Proof* Consider an instance of the $\mathbf{F}$-QLP problem:

$$\forall \mathbf{s} \in [\mathbf{l}, \mathbf{u}] \, \exists \mathbf{r} \; \mathbf{C} \cdot \mathbf{s} \le \mathbf{d} + \mathbf{M} \cdot \mathbf{r} \tag{22}$$

Recall that this problem is **coNP-complete** (Theorem 3). We will reduce this problem to an instance of (21) with its LHS being restricted to difference constraints. The QLI of the form (21) is constructed as follows:

1. We introduce variables $x_0$ and $s_0$ and add constraints $x_0 - s_0 \leq 0$ and $s_0 - x_0 \leq 0$ to the LHS of the QLI (these constraints imply $x_0 = s_0$).
2. For each element $s_i$ of $\mathbf{s}$ in (22), we introduce variables $x_i$ and $s_i$ and add constraints $x_i - s_i \leq 0$ and $s_i - x_i \leq 0$ (these constraints imply $x_i = s_i$) as well as $s_i - s_0 \leq u_i$ and $s_i - s_0 \geq l_i$ to the LHS of the QLI .
3. In the constructed QLI, let vector $\mathbf{x}$ be the vector containing all $x_i$ variables of the QLI (including $x_0$) and $\mathbf{x}'$ be such that $x'_i = x_i - x_0$.
4. We introduce the constraints of (22) to the QLI by adding constraints $\mathbf{C} \cdot \mathbf{x}' \leq \mathbf{d} + \mathbf{M} \cdot \mathbf{r}$ to the RHS of the QLI.
5. In the constructed QLI, let vector $\mathbf{s}$ be the vector containing all $s_i$ variables of the QLI (including $s_0$) and $\mathbf{s}'$ be such that $s'_i = s_i - s_0$.
6. We add $\forall \mathbf{s}\ \exists \mathbf{r}\ \forall \mathbf{x}, \mathbf{x}'$ as the quantifier string of the constructed QLI.

Hence, the constructed QLI instance has the form:

$$\forall \mathbf{s}\ \exists \mathbf{r}\ \forall \mathbf{x}, \mathbf{x}'\ [\mathbf{A} \cdot \mathbf{x} \leq \mathbf{b} + \mathbf{N} \cdot \mathbf{s} \to \mathbf{C} \cdot \mathbf{x}' \leq \mathbf{d} + \mathbf{M} \cdot \mathbf{r}] \tag{23}$$

Note that this is a QLI of the form (21) with its LHS consisting only of difference constraints with respect to variables in $\mathbf{x}$ and $\mathbf{s}$.

Let us show that (23) holds iff (22) holds.

Assume that (23) holds. Then, for $\mathbf{A} \cdot \mathbf{x} \leq \mathbf{b} + \mathbf{N} \cdot \mathbf{s}$ to be satisfied, constraints $\mathbf{x} = \mathbf{s}$ and $l_i \leq s_i - s_0 = s'_i \leq u_i$ must all be satisfied (otherwise, the existential player trivially wins the game). Thus, for the constructed instance to hold, $\mathbf{C} \cdot \mathbf{x}' \leq \mathbf{d} + \mathbf{M} \cdot \mathbf{r}$ must also hold when $\mathbf{x}' = \mathbf{s}'$ and $l_i \leq s'_i \leq u_i$. By replacing $\mathbf{x}'$ with $\mathbf{s}'$ in this system of constraints, we get that $\forall \mathbf{s}' \in [\mathbf{l}, \mathbf{u}]\ \exists \mathbf{r}\ \ \mathbf{C} \cdot \mathbf{s}' \leq \mathbf{d} + \mathbf{M} \cdot \mathbf{r}$ (i.e., (22)) must holds as well.

Assume now that (23) does not hold. Then there must exist a vector of values $\bar{\mathbf{s}}$ (and correspondingly a vector of values $\bar{\mathbf{s}}'$ such that $\bar{s}'_i = \bar{s}_i - \bar{s}_0$) for which the following formula holds:

$$\begin{aligned} \forall \mathbf{r}\ \exists \mathbf{x} \quad & \mathbf{A} \cdot \mathbf{x} \leq \mathbf{b} + \mathbf{N} \cdot \bar{\mathbf{s}} \\ & \mathbf{C} \cdot \mathbf{x}' \not\leq \mathbf{d} + \mathbf{M} \cdot \mathbf{r} \end{aligned}$$

Recall that $l_i \leq \bar{s}'_i \leq u_i$ and that the only value of $\mathbf{x}$ which can satisfy $\mathbf{A} \cdot \mathbf{x} \leq \mathbf{b} +\ \mathbf{N} \cdot \bar{\mathbf{s}}$ is $\mathbf{x} = \bar{\mathbf{s}}$ (which also implies $\mathbf{x}' = \bar{\mathbf{s}}'$). Hence, we must have that $\forall \mathbf{r}\ \mathbf{C} \cdot \bar{\mathbf{s}}' \not\leq \mathbf{d} + \mathbf{M} \cdot \mathbf{r}$, which means that $\forall \mathbf{s}' \in [\mathbf{l}, \mathbf{u}]\ \exists \mathbf{r}\ \mathbf{C} \cdot \mathbf{s}' \leq \mathbf{d} + \mathbf{M} \cdot \mathbf{r}$ (i.e., System (22)) does not hold.

Summarizing, we have reduced in polynomial time the **coNP-complete** problem (22) to an instance of (21) with its LHS being restricted to difference constraints. This implies our conclusion. □

QLIs of the form (21) become tractable in the presence of interval constraints in the LHS of the implication. Recall that an interval constraint on a variable $x$ is of the form $l \leq x \leq u$ (Section 2).

**Lemma 3** *Problem (21) is in* $\mathbf{P}$ *if each constraint in* $\mathbf{A} \cdot \mathbf{x} \leq \mathbf{b} + \mathbf{N} \cdot \mathbf{s}$ *is an interval constraint.*

*Proof* Note that if constraints in $\mathbf{A} \cdot \mathbf{x} \leq \mathbf{b} + \mathbf{N} \cdot \mathbf{s}$ contain only one variable of $\mathbf{x}$ or $\mathbf{s}$, then constraints involving only variables of $\mathbf{s}$ can be ignored. This is because variables of $\mathbf{s}$ exist only in the LHS of the implication. Hence, their corresponding interval constraints are trivially satisfied (recall that the universal player has to satisfy the LHS of the implication).

Thus, we have that each element $x_i$ of $\mathbf{x}$ is simply bound by an interval $l_i \leq x_i \leq u_i$ which means that $\mathbf{A} \cdot \mathbf{x} \leq \mathbf{b} + \mathbf{N} \cdot \mathbf{s}$ is equivalent to $\mathbf{l} \leq \mathbf{x} \leq \mathbf{u}$. So $\forall \mathbf{s}\ \exists \mathbf{r}\ \forall \mathbf{x}\ [\mathbf{A} \cdot \mathbf{x} \leq \mathbf{b} + \mathbf{N} \cdot \mathbf{s} \rightarrow \mathbf{C} \cdot \mathbf{x} \leq \mathbf{d} + \mathbf{M} \cdot \mathbf{r}]$ is equivalent to

$$\exists \mathbf{r}\ \forall \mathbf{x}\ [\mathbf{l} \leq \mathbf{x} \leq \mathbf{u} \rightarrow \mathbf{C} \cdot \mathbf{x} \leq \mathbf{d} + \mathbf{M} \cdot \mathbf{r}]$$

which corresponds to the **E**-QLP problem $\exists \mathbf{r}\ \forall \mathbf{x} \in [\mathbf{l}, \mathbf{u}]\ \mathbf{C} \cdot \mathbf{x} \leq \mathbf{d} + \mathbf{M} \cdot \mathbf{r}$. The latter is in **P** by Theorem 2. □

### 11.4 Problems $\langle 2, \forall, \mathbf{LRL} \rangle$ and $\langle 2, \forall, \mathbf{LRR} \rangle$

Finally, we explore the trivial sub-classes $\langle 2, \forall, \mathbf{LRL} \rangle$ and $\langle 2, \forall, \mathbf{LRR} \rangle$, showing that both of them are in **P**. Problems in $\langle 2, \forall, \mathbf{LRL} \rangle$ are described by:

$$\forall \mathbf{s}\ \exists \mathbf{r}\ \forall \mathbf{x}\ [\mathbf{A} \cdot \mathbf{x} + \mathbf{P} \cdot \mathbf{s} \leq \mathbf{b} \rightarrow \mathbf{M} \cdot \mathbf{r} \leq \mathbf{d}] \tag{24}$$

**Theorem 18** *Problem* $\langle 2, \forall, \mathbf{LRL} \rangle$ *is in* **P**.

*Proof* Consider any instance of System (24). In order to decide such a problem, we can first check whether $\exists \mathbf{r}\ \mathbf{M} \cdot \mathbf{r} \leq \mathbf{d}$ holds. If it holds, then System (24) also holds. Otherwise, the problem reduces to deciding whether $\exists \mathbf{s}\ \exists \mathbf{x}\ \mathbf{A} \cdot \mathbf{x} + \mathbf{P} \cdot \mathbf{s} \leq \mathbf{b}$ holds. If it holds, System (24) does not hold; otherwise, System (24) holds. The result follows from the fact that both such checks can be done in polynomial time (Khachiyan, 1979). □

The same result can be obtained for $\langle 2, \forall, \mathbf{LRR} \rangle$:

$$\forall \mathbf{s}\ \exists \mathbf{r}\ \forall \mathbf{x}\ [\mathbf{P} \cdot \mathbf{s} \leq \mathbf{b} \rightarrow \mathbf{C} \cdot \mathbf{x} + \mathbf{M} \cdot \mathbf{r} \leq \mathbf{d}] \tag{25}$$

**Theorem 19** *Problem* $\langle 2, \forall, \mathbf{LRR} \rangle$ *is in* **P**.

*Proof* To decide an instance of the form (25), we need to first decide problem $\exists \mathbf{r}\ \forall \mathbf{x}\ \mathbf{C} \cdot \mathbf{x} + \mathbf{M} \cdot \mathbf{r} \leq \mathbf{d}$, which is in **P** by Corollary 1. If it holds, then System (25) holds. If not, then we need to decide problem $\exists \mathbf{s}\ \mathbf{P} \cdot \mathbf{s} \leq \mathbf{b}$, which is also in **P** (Khachiyan, 1979). If it holds, then System (25) does not hold; otherwise, System (25) holds. □

## 12 Conclusions

In this paper, we analyzed quantified linear implications from the perspective of computational complexity . First, we provided a 2-person game semantics for QLIs, which clarified how existentially and universally quantified variables obtain their values. Using these 2-person game semantics, we showed that the decision problem for QLIs with arbitrary number of quantifier alternations is **PSPACE-hard**. This result is particularly interesting when contrasted with the long-standing open problem of whether the class of QLPs is **PSPACE-hard**. Moreover, we established the computational complexities of several QLI classes. While 0-quantifier alternation QLIs are all in **P**, for 1-quantifier alternation, some classes were shown to be **NP-complete**, while some others were shown to be **coNP-hard**. For 2-quantifier alternations, the classes we explored range from **P** to $\mathbf{\Pi_2^P}$**-hard**. We also extended our results to QLIs with 2 or more quantifier alternations, where we established the hardness of QLI classes with respect to the first quantifier in the quantifier string and the number of quantifier alternations. Finally, we investigated the effect of the structure of the LHS on the computational complexity of the corresponding decision problem, for the QLI class $\langle 2, \forall, \mathbf{LRB}\rangle$. While in the case of difference constraints the decision problem remains **coNP-hard**, in the case of interval constraints the decision problem is in **P**.

Future work includes establishing the exact computational complexities of classes for which partial results were obtained. An important open issue is establishing the exact computational complexity of deciding the feasibility of a QLI with an arbitrary number of quantifier alternations. As discussed above, the problem is **PSPACE-hard**, but not known to be in **PSPACE**. Other open issues include showing completeness for the 1-quantifier alternation classes that are currently open. For instance it is not known whether the class $\langle 1, \exists, \mathbf{BL}\rangle$ is **NP-complete**, and whether the class $\langle 1, \forall, \mathbf{BB}\rangle$ is **coNP-complete**. It would also be desirable to characterize the exact complexity hierarchy of classes with more than 2-quantifier alternations (since only hardness results were established in this paper).

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