On the complexity of quantified linear systems

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Abstract

In this paper, we explore the computational complexity of the conjunctive fragment of the first-order theory of linear arithmetic. Quantified propositional formulas of linear inequalities with $(k-1)$ quantifier alternations are log-space complete in Σ_k^P or Π_k^P depending on the initial quantifier. We show that when we restrict ourselves to quantified conjunctions of linear inequalities, i.e., *quantified linear systems*, the complexity classes collapse to polynomial time. In other words, the presence of universal quantifiers does not alter the complexity of the linear programming problem, which is known to be in $\mathbb{P}$. Our result reinforces the importance of sentence formats from the perspective of computational complexity.

Keywords: quantified linear systems, theory of real numbers with addition, complexity classes

1. Introduction

In the first-order theory of linear arithmetic, the sentences are arbitrary propositional combinations of linear inequalities. The theory of linear arithmetic is itself a sub-theory of the theory of real closed fields. Both these theories have been widely studied in the literature [1, 2, 3], since they find applications in a number of domains including but not limited to constraint databases [4] and model checking [5]. This paper examines the computational complexity of the conjunctive fragment of the first-order theory of linear arithmetic. Consider the sub-class of formulas in the theory of reals with addition and order starting with an existential quantifier and with $(k-1)$ quantifier alternations:

$$\exists \mathbf{x}_1 \ \forall \mathbf{x}_2 \ \ldots \ Q\mathbf{x}_k \ F(\mathbf{x}_1, \ldots, \mathbf{x}_k)$$

where Q is $\exists$ for k odd and $\forall$ for k even, and $F(\mathbf{x}_1, \ldots, \mathbf{x}_k)$ is a *propositional combination* of linear inequalities over variable sets $\mathbf{x}_1, \ldots, \mathbf{x}_k$. This sub-class has been shown to be log-space complete in $\Sigma_k^{\mathbf{P}}$ [6]. It is legitimate to ask ourselves whether this complexity result is attributable to the fact that F is a *propositional* combination or, rather, to an intrinsic complexity of *quantification* over

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[1] This research has been co-financed by the European Union (European Social Fund – ESF) and Greek national funds through the Operational Program "Education and Lifelong Learning" of the National Strategic Reference Framework (NSRF) - Research Funding Program: Thales. Investing in knowledge society through the European Social Fund.

[2] This research was supported in part by the National Science Foundation through Award CCF-0827397.

linear inequalities. We answer this very question in this paper. We show that when F is a *conjunction* of linear inequalities, i.e., a linear system, then the complexity of deciding whether the formula above holds is in $\mathbb{P}$. It is well-known that linear programming is decidable in $\mathbb{P}$ [7, 8, 9]. Our results in this paper, extend the work in [7] to the case of quantified linear systems. All the proofs provided in this paper are constructive, with algorithms based on solving a number of linear programming problems.

The rest of the paper is organized as follows. We first recall background notation in Section 1.1, and related work from the literature in Section 1.2. Section 2 provides a formal description of the problem under consideration. Section 3 discusses the principal complexity result in this paper. We conclude in Section 4, by summarizing our contribution and describing avenues for future work.

1.1. Background

We adhere to standard notation of linear algebra [10]. $\mathbb{R}$ is the set of real numbers. Small bold letters ($\mathbf{a}$, $\mathbf{b}$, …) denote column vectors, while capital bold letters ($\mathbf{A}$, $\mathbf{B}$, …) denote matrices. $\mathbf{0}$ is a column vector with all elements equal to 0. The transposed vector of $\mathbf{a}$ is denoted by $\mathbf{a}^T$. The inner product is denoted by $\mathbf{a}^T \cdot \mathbf{b}$. We write $\mathbf{A} \cdot \mathbf{x} \leq \mathbf{b}$ for a system of linear inequalities over the variables in $\mathbf{x}$, also called a *linear system*. We assume that the dimensions of vectors and matrices in inner products and linear systems are of the appropriate size. A *polyhedron* is the set of solution points of a linear system: $Sol(\mathbf{A} \cdot \mathbf{x} \leq \mathbf{b}) = \{\mathbf{x}_0 \in \mathbb{R}^{|\mathbf{x}|} \mid \mathbf{A} \cdot \mathbf{x}_0 \leq \mathbf{b}\}$. Polyhedra are convex sets. A linear system $\mathbf{A} \cdot \mathbf{x} \leq \mathbf{b}$ is *satisfiable* if its polyhedron $Sol(\mathbf{A} \cdot \mathbf{x} \leq \mathbf{b})$ is non-empty. A linear program:

$$\begin{gathered} max\, \mathbf{c}^T \cdot \mathbf{x} \\ \mathbf{A} \cdot \mathbf{x} \leq \mathbf{b} \end{gathered}$$

is the problem of finding $max\,\{\mathbf{c}^T \cdot \mathbf{x}_0 \mid \mathbf{x}_0 \in Sol(\mathbf{A} \cdot \mathbf{x} \leq \mathbf{b})\}$. If the polyhedron $Sol(\mathbf{A} \cdot \mathbf{x} \leq \mathbf{b})$ is empty, the problem is said to be *infeasible*. If the linear function $\mathbf{c}^T \cdot \mathbf{x}$ has no upper bound, the problem is said to be *unbounded*.

1.2. Related Work

Since Tarski [1] established a decision procedure for the *full elementary theory of real closed fields* with addition (+), multiplication ($\cdot$) and order ($<$, $=$), a number of quantifier elimination methods have been proposed [11, 12] together with approaches efficient-in-practice, including partial cylindrical algebraic decomposition [2] provided in the QEPCAD/QEPCAD-B systems [13], virtual substitution of test terms [14] provided in the REDLOG system [15], and the RSolver algorithm [16], whose implementation is publicly available [17]. Although it has been improved several times [3, 18], the complexity of quantifier elimination is, in the worst case, doubly exponential in the number of quantifier alternations and exponential in the number of variables [12, 19].

It is important to note that the real numbers themselves cannot be fully axiomatized by a first-order theory. This can be seen in Tarski's axiomatization of the reals, which requires a non-first-order axiom to express the Dedekind completeness of the real numbers. Briefly, Dedekind completeness is the property that all bounded subsets of real numbers must have a real least upper bound and real greatest lower bound. The axiom in question involves universal quantification over subsets of the real numbers, which cannot be done with first-order logic.

Any field which satisfies the same first-order properties as the real numbers is called a real closed field. One example of a real closed field is the set of real algebraic numbers, i.e., the set of all real numbers that are the roots of non-zero polynomials with rational coefficients. While the real algebraic numbers are a real closed field, they are not Dedekind complete, since it is possible to construct a set of rational numbers, which are all algebraic, that have a supremum of π which is not. It is also easy to see that the rationals do not form a real closed field; for instance, the solution of the algebraic equation $x^2 - 2 = 0$ is not a rational number.

The complexity of sub-classes of the full elementary theory of the reals has been considered.

The *existential theory of the reals* is obtained by restricting to existentially quantified formula $\exists \mathbf{x}\ F(\mathbf{x})$ where $F(\mathbf{x})$ is a quantifier-free formula. The computational complexity of the decision problem is singly exponential in the number of quantified variables $\mathbf{x}$ [20].

The *theory of reals with addition and order* is obtained by restricting to the function symbol $+$ and predicate symbols $<$ and $=$. In [21], a quantifier elimination procedure is provided for sentences in this theory that is singly exponential in space and doubly exponential in time. The procedure replaces unbounded quantified variables with variables that are quantified over a finite set of rational numbers. An exponential time lower bound is shown in [22], where Berman studies Pressburger arithmetic (which is the first-order theory of natural numbers with addition) and determines the time and space complexity of both theories at various levels of quantifier alternation. Consider a formula in the theory of reals with addition and order in prenex normal form with $(k-1)$ quantifier alternations

$$\exists \mathbf{x}_1\ \forall \mathbf{x}_2 \ldots Q\mathbf{x}_k\ F(\mathbf{x}_1, \ldots, \mathbf{x}_k)$$

where Q is $\exists$ for k odd and $\forall$ for k even, while $F(\mathbf{x}_1, \ldots, \mathbf{x}_k)$ is a quantifier-free formula. Sontag [6] shows that this class of formulas is log-space complete in $\Sigma_k^{\mathbf{P}}$. He does so by converting any such formula into a disjunction of conjunctions of equalities and strict inequalities, and using the fact that the extreme points of polyhedra are polynomial in the size of the input.

In this paper, we deal with a sub-theory of the reals with addition and order, where the $F(\mathbf{x}_1, \ldots, \mathbf{x}_k)$ above consists of conjunctions of linear inequalities, namely a linear system. For $k = 1$, the decision problem for such a formula boils down to to satisfiability of linear systems, which is known to be in $\mathbb{P}$.

2. Problem Statement

A *quantified linear system* (QLS) is a first-order formula of the form:

$$\exists \mathbf{x}_1\ \forall \mathbf{y}_1\ \ldots \exists \mathbf{x}_n\ \forall \mathbf{y}_n\ \ \mathbf{A} \cdot \mathbf{x} \leq \mathbf{b} \tag{1}$$

where $n \geq 0$, and $\mathbf{x}_1 \ldots \mathbf{x}_n, \mathbf{y}_1 \ldots \mathbf{y}_n$ is a partition of the vector of variables $\mathbf{x}$, and where $\mathbf{x}_1$ and/or $\mathbf{y}_n$ may be empty. A QLS *holds* if it is true as a first-order formula over the domain of the reals. The *theory of QLSes* is the set of QLSes that hold, and it is a subset of the theory of reals with addition and order. The *decision problem* for a QLS consists of checking whether it holds or not.

3. The Decision Problem for QLSes is in $\mathbb{P}$

The decision problem for QLSes with only existential quantification, namely for $\exists \mathbf{x}\ \ \mathbf{A} \cdot \mathbf{x} \leq \mathbf{b}$, boils down to the satisfiability problem for the linear system $\mathbf{A} \cdot \mathbf{x} \leq \mathbf{b}$, which is in $\mathbb{P}$ [7]. In this section, we will show that the same holds for the generic class of QLSes.

Our first result illustrates that universal quantifiers at the end of a quantifier string are superfluous, in the sense that we only need to examine cases where these quantifiers correspond to variables not occurring in the linear system; otherwise, the formula trivially does not hold.

Lemma 3.1. *Let* $\mathbf{Qx}$ *be a sequence of quantifiers over* $\mathbf{x}$. *A formula* $\mathbf{Qx}\ \forall \mathbf{y}\ \mathbf{A}\cdot\mathbf{x}+\mathbf{B}\cdot\mathbf{y} \leq \mathbf{c}$ *holds if and only if* $\mathbf{B} = \mathbf{0}$ *and* $\mathbf{Qx}\ \mathbf{A}\cdot\mathbf{x} \leq \mathbf{c}$ *holds.*

Proof. If part. Immediate, since $\mathbf{B} = \mathbf{0}$ implies that $\mathbf{Qx}\ \forall \mathbf{y}\ \mathbf{A}\cdot\mathbf{x}+\mathbf{B}\cdot\mathbf{y} \leq \mathbf{c}$ reduces to $\mathbf{Qx}\ \mathbf{A}\cdot\mathbf{x} \leq \mathbf{c}$.

Only-if part. Assume that $\mathbf{Qx}\ \forall \mathbf{y}\ \mathbf{A}\cdot\mathbf{x}+\mathbf{B}\cdot\mathbf{y} \leq \mathbf{c}$ holds. By instantiating $\mathbf{y} = \mathbf{0}$, we obtain that $\mathbf{Qx}\ \mathbf{A}\cdot\mathbf{x} \leq \mathbf{c}$ holds. We will show now that $\mathbf{B} \neq \mathbf{0}$ is impossible. Let $\mathbf{x}_0$ be any instance of $\mathbf{x}$ such that $\forall \mathbf{y}\ \mathbf{A}\cdot\mathbf{x}_0+\mathbf{B}\cdot\mathbf{y} \leq \mathbf{c}$ holds (note that such an $\mathbf{x}_0$ exists, since $\mathbf{Qx}\ \forall \mathbf{y}\ \mathbf{A}\cdot\mathbf{x}+\mathbf{B}\cdot\mathbf{y} \leq \mathbf{c}$, and thus $\exists \mathbf{x}\ \forall \mathbf{y}\ \mathbf{A}\cdot\mathbf{x}+\mathbf{B}\cdot\mathbf{y} \leq \mathbf{c}$ holds by assumption). If $\mathbf{B} \neq \mathbf{0}$, there must exist some inequality $\mathbf{a}^T\cdot\mathbf{x}_0+\mathbf{b}^T\cdot\mathbf{y} \leq c$ with $\mathbf{b}^T \neq \mathbf{0}$ such that $\forall \mathbf{y}\ \mathbf{a}^T\cdot\mathbf{x}_0+\mathbf{b}^T\cdot\mathbf{y} \leq c$ holds. Since $\mathbf{b}^T \neq \mathbf{0}$, there exists at least one element of $\mathbf{b}^T$ (say b_i, i.e., the i^{th} element of $\mathbf{b}$) for which $b_i \neq 0$. Now consider an instance $\mathbf{y}_0$ of $\mathbf{y}$ such that $\mathbf{y}_0$ contains 0s in every row except row i, which is set to $(c+1-\mathbf{a}^T\cdot\mathbf{x}_0)/b_i$. However, this means that $\mathbf{a}^T\cdot\mathbf{x}_0+\mathbf{b}^T\cdot\mathbf{y}_0 = c+1 > c$, which contradicts the fact that $\forall \mathbf{y}\ \mathbf{a}^T\cdot\mathbf{x}_0+\mathbf{b}^T\cdot\mathbf{y} \leq c$ holds. $\square$

An immediate consequence of this lemma is that universal quantifiers at the end of a quantifier string do not affect the class of complexity of QLSes. Hence, we obtain our first complexity result on QLSes.

Corollary 3.1. *Deciding whether* $\forall \mathbf{y}\ \mathbf{B}\cdot\mathbf{y} \leq \mathbf{c}$ *holds, or whether* $\exists \mathbf{x}\ \forall \mathbf{y}\ \mathbf{A}\cdot\mathbf{x}+\mathbf{B}\cdot\mathbf{y} \leq \mathbf{c}$ *holds are problems in* $\mathbb{P}$.

Proof. By Lemma 3.1, $\forall \mathbf{y}\ \mathbf{B}\cdot\mathbf{y} \leq \mathbf{c}$ holds if and only if $\mathbf{B} = \mathbf{0}$ and $\mathbf{0} \leq \mathbf{c}$. Both conditions can be readily checked in polynomial (linear) time. Analogously, $\exists \mathbf{x}\ \forall \mathbf{y}\ \mathbf{A}\cdot\mathbf{x}+\mathbf{B}\cdot\mathbf{y} \leq \mathbf{c}$ holds if and only if $\mathbf{B} = \mathbf{0}$ and $\exists \mathbf{x}\ \mathbf{A}\cdot\mathbf{x} \leq \mathbf{c}$. Since checking $\exists \mathbf{x}\ \mathbf{A}\cdot\mathbf{x} \leq \mathbf{c}$ is a problem in $\mathbb{P}$ [7], the conclusion follows. $\square$

Next, we examine the class of $\forall\exists$ quantified linear systems. First, let us recall the Fourier-Motzkin variable elimination method (e.g., see [10]). For a linear system with n variables $x_1, \ldots, x_n$, the method eliminates these variables by yielding an equivalent system, where any variable x_i (where $i = 1, \ldots, n$) is bounded by (zero or more) linear functions over variables $x_{i+1}, \ldots, x_n$. Example 3.1 illustrates the application of the Fourier-Motzkin elimination method.

Example 3.1. *Consider the linear system* $P = x+y \leq z, w \leq y, z \leq x-y$. *Using the Fourier-Motzkin method, we eliminate the variable* y *first, followed by* x, w, *and finally* z.

$$\begin{array}{rcl} & y & \leq x-z \\ & y & \leq z-x \\ w \leq & y & \\ \\ \\ \\ & \textbf{(a)} & \end{array} \qquad \begin{array}{rcl} & y & \leq x-z \\ & y & \leq z-x \\ w \leq & y & \\ z+w \leq & x & \\ & x & \leq z-w \\ \\ & \textbf{(b)} & \end{array} \qquad \begin{array}{rcl} & y & \leq x-z \\ & y & \leq z-x \\ w \leq & y & \\ z+w \leq & x & \\ & x & \leq z-w \\ & w & \leq 0 \\ & \textbf{(c)} & \end{array}$$

First, we isolate y *as shown in system* (**a**). *Then we add inequalities* $\mathbf{c}_1^T\cdot\mathbf{x} \leq \mathbf{c}_2^T\cdot\mathbf{x}$ *for every pair*

$\mathbf{c}_1^T \cdot \mathbf{x} \leq y$ and $y \leq \mathbf{c}_2^T \cdot \mathbf{x}$ of bounds on y. This leads to the equivalent system (**b**)*, where x has also been isolated. Finally, we add inequalities derived from the bounds on x, thus obtaining the final system* (**c**)*, where w has also been isolated. Notice that the single bound on w does not yield any inequality. Thus z is free to assume any value.*

The form of the final system after variable elimination is useful in establishing the correctness of a decision procedure for checking $\forall\exists$ quantified linear systems.

Theorem 3.1. *Deciding whether $\forall \mathbf{y}\ \exists \mathbf{x}\ \mathbf{A} \cdot \mathbf{x} + \mathbf{B} \cdot \mathbf{y} \leq \mathbf{c}$ holds is a problem in $\mathbb{P}$.*

Proof. Let P denote the linear system $\mathbf{A} \cdot \mathbf{x} + \mathbf{B} \cdot \mathbf{y} \leq \mathbf{c}$. If $Sol(P) = \emptyset$ (which we can check in polynomial time [7]), then it readily follows that $\forall \mathbf{y}\ \exists \mathbf{x}\ P$ does not hold. Otherwise (i.e, P is satisfiable), we distinguish two cases with respect to the size of $\mathbf{y} = (y_1, \dots, y_n)$:
($n = 0$). Then $\forall \mathbf{y}\ \exists \mathbf{x}\ P$ boils down to $\exists \mathbf{x}\ \mathbf{A} \cdot \mathbf{x} \leq \mathbf{c}$, which holds since P is satisfiable.
($n > 0$). Consider the linear programming problems:

$$M = max\,\{y_n \mid P\} \qquad m = min\,\{y_n \mid P\}. \tag{2}$$

If one or both are bounded, then for an instance $\mathbf{y}_0 = (y_1^0, \dots, y_n^0)$ of $\mathbf{y}$ such that $y_n^0 > M$ (or, respectively, $y_n^0 < m$), we obtain $Sol(\mathbf{A} \cdot \mathbf{x} + \mathbf{B} \cdot \mathbf{y}_0 \leq \mathbf{c}) = \emptyset$; hence, $\forall \mathbf{y}\ \exists \mathbf{x}\ \mathbf{A} \cdot \mathbf{x} + \mathbf{B} \cdot \mathbf{y} \leq \mathbf{c}$ does not hold. We will now assume that the two problems in (2) are unbounded. Let $\mathbf{B}' \cdot \mathbf{y}'$ be obtained from $\mathbf{B} \cdot \mathbf{y}$ by replacing y_n with 0, i.e., removing every occurrence of y_n in P. We claim that $\forall \mathbf{y}\ \exists \mathbf{x}\ \mathbf{A} \cdot \mathbf{x} + \mathbf{B} \cdot \mathbf{y} \leq \mathbf{c}$ holds if and only if $\forall \mathbf{y}'\ \exists \mathbf{x}\ \mathbf{A} \cdot \mathbf{x} + \mathbf{B}' \cdot \mathbf{y}' \leq \mathbf{c}$ holds; in such case, a decision procedure consists of a recursive call[3] on $\forall \mathbf{y}'\ \exists \mathbf{x}\ \ \mathbf{A} \cdot \mathbf{x} + \mathbf{B}' \cdot \mathbf{y}' \leq \mathbf{c}$. Next, we prove our claim.

Only-if part. We will prove this by showing the contrapositive. Hence, consider the case where $\forall \mathbf{y}'\ \exists \mathbf{x}\ \mathbf{A} \cdot \mathbf{x} + \mathbf{B}' \cdot \mathbf{y}' \leq \mathbf{c}$ does not hold and let $\mathbf{y}'_0$ be an instance of $\mathbf{y}'$ such that $Sol(\mathbf{A} \cdot \mathbf{x} + \mathbf{B}' \cdot \mathbf{y}'_0 \leq \mathbf{c}) = \emptyset$. By defining $\mathbf{y}_0 = (\mathbf{y}'_0, 0)$, it is readily checked that $\mathbf{B} \cdot \mathbf{y}_0 = \mathbf{B}' \cdot \mathbf{y}'_0$. Thus, $Sol(\mathbf{A} \cdot \mathbf{x} + \mathbf{B} \cdot \mathbf{y}_0 \leq \mathbf{c}) = \emptyset$. Therefore, $\forall \mathbf{y}\ \exists \mathbf{x}\ \mathbf{A} \cdot \mathbf{x} + \mathbf{B} \cdot \mathbf{y} \leq \mathbf{c}$ also does not hold.

If part. Consider a Fourier-Motzkin elimination procedure for P in which variables in $\mathbf{x}$ are eliminated first, followed by variables in $\mathbf{y} = (y_1, \dots, y_n)$. Let P' be the resulting system. In particular, for any $i = 1, \dots, n-1$, the variable y_i may be bounded in P' by (zero or more) inequalities of the form $f(y_{i+1}, \dots, y_n) \leq y_i$ or of the form $y_i \leq g(y_{i+1}, \dots, y_n)$ (where f, g denote linear functions). Note that y_n does not have any such bounds, since the problems in (2) are both unbounded. We prove the if-part by showing the contrapositive. Consider the case where $\forall \mathbf{y}\ \exists \mathbf{x}\ \mathbf{A} \cdot \mathbf{x} + \mathbf{B} \cdot \mathbf{y} \leq \mathbf{c}$ does not hold and let $\mathbf{y}_0 = (y_1^0, \dots, y_n^0)$ be such that $Sol(\mathbf{A} \cdot \mathbf{x} + \mathbf{B} \cdot \mathbf{y}_0 \leq \mathbf{c}) = \emptyset$. We will now show that we can set $y_n^0 = 0$ and make it easy to eliminate. Note that since P and P' are equivalent (i.e., $Sol(P) = Sol(P')$), there will exist at least one $i \in \{1, \dots, n-1\}$ such that $y_i^0 < f(y_{i+1}^0, \dots, y_n^0)$ or $y_i^0 > g(y_{i+1}^0, \dots, y_n^0)$; that is, y_i^0 does not satisfy at least one inequality in P' of the form $f(y_{i+1}, \dots, y_n) \leq y_i$ or of the form $y_i \leq g(y_{i+1}, \dots, y_n)$. Now let us consider the maximum index k for which y_k^0 is bounded by at least one inequality in P'; then, for $j > k$, there is no bound on y_j in P'. Note that $k < n$, since y_n is unbounded, hence $k \in \{1, \dots, n-1\}$ and there exists at least one unbounded y_j with $j > k$ in P'. Since for each $j > k$, y_j is free in P', we

[3]Since the problems in (2) are unbounded, there exists $a > 0$ and $b < 0$ such that $Sol((P, y_n = a)) \neq \emptyset$ and $Sol((P, y_n = b)) \neq \emptyset$. Since $Sol(P)$ is a convex set, we can conclude $Sol((P, y_n = 0)) = Sol(\mathbf{A} \cdot \mathbf{x} + \mathbf{B}' \cdot \mathbf{y}' \leq \mathbf{c}) \neq \emptyset$. Hence, the recursive call may skip the initial check on satisfiability of the linear system. This observation is used in coding the decision procedure of Algorithm 1 that will be shown later on.

can arbitrarily fix its value to 0 (i.e., $y_j = 0$ for all $j > k$). Moreover, by assigning y_k any value v such that $v < f(0, \ldots, 0)$ or $v > g(0, \ldots, 0)$, we obtain $\mathbf{y}_1 = (y_1^0, \ldots, y_{k-1}^0, v, 0, \ldots, 0)$ such that $Sol(\mathbf{A}\cdot\mathbf{x}+\mathbf{B}\cdot\mathbf{y}_1 \leq \mathbf{c}) = \emptyset$. Since $\mathbf{y}_1$ sets $y_n = 0$, we also have that $Sol(\mathbf{A}\cdot\mathbf{x}+\mathbf{B}'\cdot\mathbf{y}'_1 \leq \mathbf{c}) = \emptyset$, where $\mathbf{y}'_1$ is obtained by removing the last 0 from $\mathbf{y}_1$ (i.e., $\mathbf{y}_1 = (\mathbf{y}'_1, 0)$). Therefore, $\forall\mathbf{y}'\ \exists\mathbf{x}\ \mathbf{A}\cdot\mathbf{x}+\mathbf{B}'\cdot\mathbf{y}' \leq \mathbf{c}$ also does not hold.

Finally, note that the decision procedure described above runs in polynomial time, since it consists of solving at most $2n + 1$ linear programming problems, each one having polynomial time complexity [7]. □

It is worth noting that the Fourier-Motzkin elimination method is used within the proof to establish the correctness of the decision procedure, but it is not used by the procedure itself. This is important, since the Fourier-Motzkin elimination procedure runs in exponential time in the worst case [10]. Algorithm 1 codes in a pseudo-programming language the decision procedure *QLS_forall_exists*() devised in the proof of the theorem. Algorithm 1 takes as input a QLS of the form $\forall\mathbf{y}\ \exists\mathbf{x}\ P$, where $P = \mathbf{A}\cdot\mathbf{x} + \mathbf{B}\cdot\mathbf{y} \leq \mathbf{c}$, and returns true if and only if $\forall\mathbf{y}\ \exists\mathbf{x}\ P$ holds. First, it checks whether P is satisfiable (line 1). If this is the case, it proceeds with checking whether each variable in $\mathbf{y}$ is bounded (line 6) and, if it is not, with setting its value to 0 (line 9). In the special case that the input is $\exists\mathbf{x}\ \mathbf{A}\cdot\mathbf{x} \leq \mathbf{c}$, i.e., for an instance with no $\mathbf{y}$ variables, the procedure *QLS_forall_exists*() boils down to checking satisfiability of P. In the special case $\forall\mathbf{y}\ \mathbf{B}\cdot\mathbf{y} \leq \mathbf{c}$, i.e., for an instance with no $\mathbf{x}$ variables, the syntactic checking "$\mathbf{B} = \mathbf{0}$ and $\mathbf{0} \leq \mathbf{c}$" from the proof of Corollary 3.1 will be more efficient in an actual implementation. We have refrained from including such a sub-procedure within Algorithm 1 for simplicity of exposition.

Example 3.2. *Consider the system P of Example 3.1. Let us apply the QLS_forall_exists() procedure on $\forall x z\ \exists y w\ P$. Clearly, P is satisfiable, i.e., $Sol(P) \neq \emptyset$. We have that $max\{z \mid P\}$ is unbounded and $min\{z \mid P\}$ is unbounded. This can be readily checked from the equivalent system* **(c)** *of Example 3.1. After replacing z by 0, we obtain the system $P' = x + y \leq 0, w \leq y, 0 \leq x - y$. The variable x is unbounded in P': for every $x = x_0$, we can set $y = w = x_0$ if $x_0 \leq 0$ and $y = w = -x_0$ if $x_0 > 0$ to obtain a solution of P'. The QLS_forall_exists() procedure then returns that $\forall z x\ \exists y w\ P$ holds. Notice that after replacing x by 0 in P', we obtain $P'' = y \leq 0, w \leq y$, where both y and w are clearly bounded. Hence, if either y or w were universally quantified over, the corresponding QLS would not hold; e.g., $\forall z x y\ \exists w\ P$ does not hold.*

Example 3.3. *In a bi-level linear programming problem [23], a planner at level one of a hierarchy has its objective function* and *decision space determined by a planner at level two of the hierarchy:*

$$P_1 \left\{ \begin{array}{l} max\ \mathbf{c}_{1,1}^T \cdot \mathbf{x}_1 + \mathbf{c}_{1,2}^T \cdot \mathbf{x}_2 \\ \qquad \text{where } \mathbf{x}_2 \text{ solves} \\ \qquad P_2 \left\{ \begin{array}{l} max\ \mathbf{c}_{2,1}^T \cdot \mathbf{x}_1 + \mathbf{c}_{2,2}^T \cdot \mathbf{x}_2 \\ \mathbf{A}\cdot\mathbf{x}_1 + \mathbf{B}\cdot\mathbf{x}_2 \leq \mathbf{c} \end{array} \right. \end{array} \right.$$

The planner at level one maximizes its objective function by controlling only vector $\mathbf{x}_1$. For any fixed choice of $\mathbf{x}_1$, the planner at level two chooses a value of $\mathbf{x}_2$ to maximize its objective function. In this context, the QLS formula $\forall\mathbf{x}_1\ \exists\mathbf{x}_2\ \mathbf{A}\cdot\mathbf{x}_1 + \mathbf{B}\cdot\mathbf{x}_2 \leq \mathbf{c}$ holds if and only if any choice $\mathbf{x}_1$ at level one admits a possible choice $\mathbf{x}_2$ at level two. Checking whether this condition holds is in $\mathbb{P}$ by Theorem 3.1. In general, bi-level linear programming is $\mathbb{NP}$-hard [24, 25, 26].

Algorithm 1 Decision procedure *QLS_forall_exists*($\forall\mathbf{y}\ \exists\mathbf{x}\ P$).

if $Sol(P) = \emptyset$ **then**
  return false;
end if
let $(y_1, \ldots, y_n) = \mathbf{y}$;
while $n > 0$ **do**
  if $max\{y_n \mid P\}$ is bounded or $min\{y_n \mid P\}$ is bounded **then**
    return false;
  end if
  replace y_n with 0 in P;
  $n \leftarrow n - 1$;
end while
return true;

By the decomposition theorem of polyhedra, a non-empty polyhedron $Sol(\mathbf{A} \cdot \mathbf{x} + \mathbf{B} \cdot \mathbf{y} \leq \mathbf{c})$ can be expressed as the Minkowski sum of its characteristic cone $Sol(\mathbf{A} \cdot \mathbf{x} + \mathbf{B} \cdot \mathbf{y} \leq \mathbf{0})$ and a polytope (a bounded polyhedron) [10]. Hence, the characteristic cone represents the "infinite" part of the polyhedron. As a consequence, a linear function is bounded over a polyhedron if and only if it is bounded over its characteristic cone. By looking through the decision procedure *QLS_forall_exists*(), we notice that it relies on checking upper and lower bounds over $Sol(\mathbf{A} \cdot \mathbf{x} + \mathbf{B} \cdot \mathbf{y} \leq \mathbf{c}) \neq \emptyset$, which, as observed, is equivalent to checking upper and lower bounds over $Sol(\mathbf{A} \cdot \mathbf{x} + \mathbf{B} \cdot \mathbf{y} \leq \mathbf{0})$. Hence, the constant terms $\mathbf{c}$ play no role, as we now formally state.

Corollary 3.2. *Let* $Sol(\mathbf{A} \cdot \mathbf{x} + \mathbf{B} \cdot \mathbf{y} \leq \mathbf{c})$ *be non-empty. Then* $\forall\mathbf{y}\ \exists\mathbf{x}\ \mathbf{A} \cdot \mathbf{x} + \mathbf{B} \cdot \mathbf{y} \leq \mathbf{c}$ *holds if and only if* $\forall\mathbf{y}\ \exists\mathbf{x}\ \mathbf{A} \cdot \mathbf{x} + \mathbf{B} \cdot \mathbf{y} \leq \mathbf{0}$ *holds.*

Proof. Let P be $\mathbf{A} \cdot \mathbf{x} + \mathbf{B} \cdot \mathbf{y} \leq \mathbf{c}$, and P' be its homogeneous version $\mathbf{A} \cdot \mathbf{x} + \mathbf{B} \cdot \mathbf{y} \leq \mathbf{0}$. Since $Sol(P) \neq \emptyset$, we have that $max\{y_n \mid P\}$ is upper bounded if and only if $max\{y_n \mid P'\}$ is upper bounded – see e.g., [27, Corollary 3.1] for a proof. Similarly, $min\{y_n \mid P\}$ is lower bounded if and only if $min\{y_n \mid P'\}$ is lower bounded. Therefore, the decision procedure *QLS_forall_exists*() for $\forall\mathbf{y}\ \exists\mathbf{x}\ P$ is equivalent to the decision procedure for $\forall\mathbf{y}\ \exists\mathbf{x}\ P'$. The result follows. □

This result is useful in proving the following crucial decomposition property of a general QLS problem into a $\forall\exists$ problem and a QLS problem with fewer quantifier alternations.

Lemma 3.2. *Let* $\mathbf{Qz}$ *be a sequence of quantifiers over* $\mathbf{z}$*. Then* $\mathbf{Qz}\ \forall\mathbf{y}\ \exists\mathbf{x}\ \mathbf{A} \cdot \mathbf{x} + \mathbf{B} \cdot \mathbf{y} + \mathbf{C} \cdot \mathbf{z} \leq \mathbf{d}$ *holds if and only if both* $\forall\mathbf{y}\ \exists\mathbf{x}\ \mathbf{A} \cdot \mathbf{x} + \mathbf{B} \cdot \mathbf{y} \leq \mathbf{0}$ *and* $\mathbf{Qz}\ \exists\mathbf{y}\ \exists\mathbf{x}\ \mathbf{A} \cdot \mathbf{x} + \mathbf{B} \cdot \mathbf{y} + \mathbf{C} \cdot \mathbf{z} \leq \mathbf{d}$ *hold.*

Proof. If part. Let $\mathbf{z}_0$ be such that $\exists\mathbf{y}\ \exists\mathbf{x}\ \mathbf{A} \cdot \mathbf{x} + \mathbf{B} \cdot \mathbf{y} + \mathbf{C} \cdot \mathbf{z}_0 \leq \mathbf{d}$ holds. Notice that this implies $Sol(\mathbf{A}\cdot\mathbf{x}+\mathbf{B}\cdot\mathbf{y} \leq \mathbf{d}-\mathbf{C}\cdot\mathbf{z}_0) \neq \emptyset$. By Corollary 3.2 (if part) and the hypothesis $\forall\mathbf{y}\ \exists\mathbf{x}\ \mathbf{A}\cdot\mathbf{x}+\mathbf{B}\cdot\mathbf{y} \leq \mathbf{0}$, we then have that $\forall\mathbf{y}\ \exists\mathbf{x}\ \mathbf{A} \cdot \mathbf{x} + \mathbf{B} \cdot \mathbf{y} \leq \mathbf{d} - \mathbf{C} \cdot \mathbf{z}_0$ holds. Since $\mathbf{Qz}\ \exists\mathbf{y}\ \exists\mathbf{x}\ \mathbf{A} \cdot \mathbf{x} + \mathbf{B} \cdot \mathbf{y} + \mathbf{C} \cdot \mathbf{z} \leq \mathbf{d}$ holds, the instances $\mathbf{z}_0$ of $\mathbf{z}$ for which the previous reasoning applies satisfy the $\mathbf{Qz}$ quantification. We can then reintroduce quantifications over $\mathbf{z}$ to conclude that $\mathbf{Qz}\ \forall\mathbf{y}\ \exists\mathbf{x}\ \mathbf{A} \cdot \mathbf{x} + \mathbf{B} \cdot \mathbf{y} + \mathbf{C} \cdot \mathbf{z} \leq \mathbf{d}$ holds.

Only-if part. $\mathbf{Qz}\ \forall\mathbf{y}\ \exists\mathbf{x}\ \mathbf{A}\cdot\mathbf{x}+\mathbf{B}\cdot\mathbf{y}+\mathbf{C}\cdot\mathbf{z} \leq \mathbf{d}$ clearly implies $\mathbf{Qz}\ \exists\mathbf{y}\ \exists\mathbf{x}\ \mathbf{A}\cdot\mathbf{x}+\mathbf{B}\cdot\mathbf{y}+\mathbf{C}\cdot\mathbf{z} \leq \mathbf{d}$. Let $\mathbf{z}_0$ be such that $\forall\mathbf{y}\ \exists\mathbf{x}\ \mathbf{A} \cdot \mathbf{x} + \mathbf{B} \cdot \mathbf{y} + \mathbf{C} \cdot \mathbf{z}_0 \leq \mathbf{d}$ holds (at least one such $\mathbf{z}_0$ exists since $\mathbf{Qz}\ \forall\mathbf{y}\ \exists\mathbf{x}\ \mathbf{A} \cdot \mathbf{x} + \mathbf{B} \cdot \mathbf{y} + \mathbf{C} \cdot \mathbf{z} \leq \mathbf{d}$ holds). We can rewrite such a formula as $\forall\mathbf{y}\ \exists\mathbf{x}\ \mathbf{A} \cdot \mathbf{x} + \mathbf{B} \cdot \mathbf{y} \leq$

Algorithm 2 Decision procedure *QLS_decision*(Φ).

1: **if** $\Phi = \mathbf{Q}\mathbf{x}\ \forall\mathbf{y}\ \mathbf{A}\cdot\mathbf{x}+\mathbf{B}\cdot\mathbf{y}\leq\mathbf{c}$ **then**
2:   **if** $\mathbf{B}\neq\mathbf{0}$ **then**
3:     **return** false;
4:   **end if**
5:   $\Phi\leftarrow\mathbf{Q}\mathbf{x}\ \mathbf{A}\cdot\mathbf{x}\leq\mathbf{c}$;
6: **end if**
7: **while** Φ contains $\forall$ quantifiers **do**
8:   **let** $\Phi = \mathbf{Q}\mathbf{z}\ \forall\mathbf{y}\ \exists\mathbf{x}\ \mathbf{A}\cdot\mathbf{x}+\mathbf{B}\cdot\mathbf{y}+\mathbf{C}\cdot\mathbf{z}\leq\mathbf{d}$;
9:   **if** not *QLS_forall_exists*($\forall\mathbf{y}\ \exists\mathbf{x}\ \mathbf{A}\cdot\mathbf{x}+\mathbf{B}\cdot\mathbf{y}\leq\mathbf{0}$) **then**
10:     **return** false;
11:   **end if**
12:   $\Phi\leftarrow\mathbf{Q}\mathbf{z}\ \exists\mathbf{y}\ \exists\mathbf{x}\ \mathbf{A}\cdot\mathbf{x}+\mathbf{B}\cdot\mathbf{y}+\mathbf{C}\cdot\mathbf{z}\leq\mathbf{d}$;
13: **end while**
14: **let** $\Phi = \exists\mathbf{x}\ \mathbf{A}\cdot\mathbf{x}\leq\mathbf{b}$;
15: **return** $Sol(\mathbf{A}\cdot\mathbf{x}\leq\mathbf{b})\neq\emptyset$;

$\mathbf{d}-\mathbf{C}\cdot\mathbf{z}_0$. Notice that this implies $Sol(\mathbf{A}\cdot\mathbf{x}+\mathbf{B}\cdot\mathbf{y}\leq\mathbf{d}-\mathbf{C}\cdot\mathbf{z}_0)\neq\emptyset$. Hence, by Corollary 3.2 (only-if part), $\forall\mathbf{y}\ \exists\mathbf{x}\ \mathbf{A}\cdot\mathbf{x}+\mathbf{B}\cdot\mathbf{y}\leq\mathbf{0}$ also holds. □

We are now in the position to state our main result.

Theorem 3.2. *The decision problem for the class of QLSes is in* $\mathbb{P}$.

Proof. Consider a QLS formula and let $\mathbf{Q}$ be the sequence of quantifiers in it. If $\mathbf{Q}$ ends with $\forall$, we can apply Lemma 3.1. This would either lead to the conclusion that the formula does not hold (if any variable in the last universal quantifier actually appears in the linear system) or that the universal quantifier can be removed. In the latter case, we proceed as when $\mathbf{Q}$ ends with $\exists$.

Assume now that $\mathbf{Q}$ ends with $\exists$. Then, the QLS is of the form

$$\exists\mathbf{x}_0\ \forall\mathbf{y}_1\ \exists\mathbf{x}_1\ldots\forall\mathbf{y}_{n-1}\ \exists\mathbf{x}_{n-1}\ \forall\mathbf{y}_n\ \exists\mathbf{x}_n\ \mathbf{A}_0\cdot\mathbf{x}_0+\mathbf{B}_1\cdot\mathbf{y}_1+\mathbf{A}_1\cdot\mathbf{x}_1+\ldots+\mathbf{B}_n\cdot\mathbf{y}_n+\mathbf{A}_n\cdot\mathbf{x}_n\leq\mathbf{c} \quad (3)$$

where $n\geq 0$ and $\mathbf{x}_0$ may be empty. We will prove our claim by induction on the number n of universal quantifiers in $\mathbf{Q}$. If $n=0$ (i.e., if $\mathbf{Q}$ contains no universal quantifiers), then formula (3) reduces to $\exists\mathbf{x}_0\ \mathbf{A}_0\cdot\mathbf{x}_0\leq\mathbf{c}$, hence solvable in polynomial time [7]. Otherwise, if $n>0$, $\mathbf{Q}$ is of the form $\mathbf{Q}'\forall\exists$ and the corresponding QLS is of the form (3). By Lemma 3.2, such a decision problem can be reduced to a decision problem for a $\forall\exists$ formula:

$$\forall\mathbf{y}_n\ \exists\mathbf{x}_n\ \mathbf{A}_n\cdot\mathbf{x}_n+\mathbf{B}_n\cdot\mathbf{y}_n\leq\mathbf{0}$$

which is in $\mathbb{P}$ by Theorem 3.1, and to a decision problem for a formula whose prefix $\mathbf{Q}'\exists$ has $(n-1)$ universal quantifiers:

$$\exists\mathbf{x}_0\ \forall\mathbf{y}_1\ \exists\mathbf{x}_1\ldots\forall\mathbf{y}_{n-1}\ \exists\mathbf{x}_{n-1}\ \exists\mathbf{y}_n\ \exists\mathbf{x}_n\ \mathbf{A}_0\cdot\mathbf{x}_0+\mathbf{B}_1\cdot\mathbf{y}_1+\mathbf{A}_1\cdot\mathbf{x}_1+\ldots+\mathbf{B}_n\cdot\mathbf{y}_n+\mathbf{A}_n\cdot\mathbf{x}_n\leq\mathbf{c}$$

which is in $\mathbb{P}$ by the inductive hypothesis. □

Algorithm 2 illustrates the pseudo-code for the decision procedure *QLS_decision*() devised in the proof of Theorem 3.2. Lines 1-6 implement the base case (Lemma 3.1). Lines 7-13 implement the recursive elimination of all universal quantifiers (Theorem 3.2). Finally, lines 14-15 check whether the resulting existentially quantified formula holds by checking satisfiability of the corresponding linear system.

4. Conclusion

We have investigated the computational complexity of the conjunctive fragment of the theory of linear arithmetic, called quantified linear systems. This class extends the satisfiability problem of linear systems, which is known to be in $\mathbb{P}$, and is included in the theory of real numbers with addition and order, which is $\mathbb{PSPACE}$-hard. We have shown that the decision problem for QLSes is in $\mathbb{P}$. Our proofs are constructive, in that we Turing-reduce the QLS decision problem to linear programming. This result demonstrates that the complexity of the theory of real numbers with addition and order is attributable to the presence of propositional connectives *and* of quantifiers, rather than *only* to quantifiers. Hence, it reinforces the importance of sentence formats from the perspective of computational complexity.

In quantified linear systems, variables are unbounded on both sides, i.e., there is no upper or lower bound imposed on their values. A question that arises naturally is whether the presented complexity results change, when bounds are placed on the variables. The case where each universally quantified variable has an upper and a lower bound has been studied in [28]. The class of $\forall\exists$ formulas for this problem is co$\mathbb{NP}$-complete and the problem itself is in $\mathbb{PSPACE}$; establishing its hardness however remains open. We are particularly interested in the computational complexity of Quantified Linear Systems, in which each universally quantified variable has an upper bound or a lower bound (but not both).

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