Models of Computation

Midterm Exam on April 9, 2010

Exercise 1 (12)

Extend IMP with: (i) the new syntactic category S (with variables s) of *expressions with side effects*, shortly *elat*, having $\langle s, \sigma \rangle \rightarrow (\sigma', n)$ as well formed formulas for the operational semantics and semantic domain $\Sigma \rightarrow (\Sigma \times N)_\perp$; and (ii) the new productions

$$s ::= \; c\,;a \qquad\qquad c ::= \; x := s$$

with the following informal semantics: for the first production, the meaning is to return the memory produced by the command c as *modified memory* by elat s; and the evaluation of the arithmetic expression a in such a memory as *result* of s. For the second production, the meaning is first to evaluate s and then to execute the assignment, employing the memory modified by s as memory, and the result of s as value to assign to x.

The denotational semantics is as follows:

$$\begin{aligned}
\mathcal{S}[\![c;a]\!]\sigma = \;& (\lambda\sigma'.(\sigma', \mathcal{A}[\![a]\!]\sigma'))^*(\mathcal{C}[\![c]\!]\sigma)\\
\mathcal{C}[\![x := s]\!]\sigma = \;& \textbf{case } \mathcal{S}[\![s]\!]\sigma \textbf{ of} :\\
& \perp_{(\Sigma\times N)_\perp} : \perp_{\Sigma_\perp}\\
& (\sigma', n) \quad : \sigma'[n/x].
\end{aligned}$$

Define the operational semantics of the two new constructs and evaluate the command $x := (x := 3; 5)$ according to both the operational and the denotational semantics. Prove the operational - denotational equivalence and show (employing either the operational or the denotational semantics) that $x := (c\,;a)$ is equivalent to $c\,;x := a$.

Exercise 2 (8)

Consider the context free (more precisely, right linear) grammar $S ::= \;\alpha S \mid \beta$, where the terminal alphabet is $V = \{a, b\}$, with $\alpha, \beta \in V^*$. Derive the inference rules R corresponding to such a grammar and prove (in both ways) that the set of theorems is $I_R = \{\alpha^n\beta \in S \mid n \in \omega\}$.

Then consider the immediate consequences operator $\hat{R}$ and check that I_R is its minimal fixpoint. Finally, discuss if other fixpoints exist for particular values of α and β.

Exercise 3 (10)

The *binomial coefficients* $\binom{n}{k}$, with $n, k \in \omega$ e $0 \leq k \leq n$, asre defined by:

$$\binom{n}{0} = \binom{n}{n} = 1 \qquad\qquad \binom{n+1}{k+1} = \binom{n}{k} + \binom{n}{k+1}.$$

Prove that the above definition is given by well-founded recursion, specifying the well-founded relation and the corresponding function $F(b, h)$.

Then consider the HOFL program:

$$t = rec\; f.\lambda n.\lambda k.\textbf{if } k \textbf{ then } 1 \textbf{ else if } n - k \textbf{ then } 1 \textbf{ else } ((f\; n-1)\; k) + ((f\; n-1)\; k-1),$$

compute its type and evaluate the canonical form of the term $((t\; 2)\; 1)$.