Models of Computation

Written Exam on June 8, 2011

(First part: Exercises 1 and 2, 90 minutes Second part: Exercises 3 and 4, 90 minutes)
(Previous TSD students: Exercises 1, 2 and 3, 135 minutes)

Exercise 1 (8)

Extend IMP with a restricted version of pointers as follows:
(i) $\sigma : \Sigma = Loc \to (N + Loc \times Loc)$, namely a location can contain either an integer or a pair of (pointers to) locations;
(ii) $a ::= \ (x, y) \ \ | \ a ::= \ fst(a) \ \ | \ \ a ::= \ snd(a)$, namely an arithmetic expression can be a pair of locations or can return *the content* of the first (second) location of a pair;
(iii) the denotational semantics of arithmetic expressions is modified as follows:
$\mathcal{A} : Aexp \to \Sigma \to (N + Loc \times Loc)$;
$\mathcal{A}[\![n]\!]\sigma = n$, $\mathcal{A}[\![(x, y)]\!]\sigma = (x, y)$, $\mathcal{A}[\![x]\!]\sigma = \sigma(x)$,
$\mathcal{A}[\![a_0 + a_1]\!]\sigma = \textbf{case } (\mathcal{A}[\![a_0]\!]\sigma, \mathcal{A}[\![a_1]\!]\sigma) : (n_0, n_1) \to n_0 + n_1, otherwise \to 0$ and similarly for $\times$, etc.,
$\mathcal{A}[\![fst(a)]\!]\sigma = \textbf{case } \mathcal{A}[\![a]\!]\sigma : n \to 0, (x, y) \to \sigma(x)$,
$\mathcal{A}[\![snd(a)]\!]\sigma = \textbf{case } \mathcal{A}[\![a]\!]\sigma : n \to 0, (x, y) \to \sigma(y)$,
$\mathcal{B}[\![a_0 \neq a_1]\!]\sigma = \textbf{case } (\mathcal{A}[\![a_0]\!]\sigma, \mathcal{A}[\![a_1]\!]\sigma) : (n_0, n_1) \to n_0 \neq n_1, otherwise \to true$ and similarly for $\leq$, etc.

Define the operational semantics of the new/modified constructs and evaluate the command:
$x := (y, x);\ y := (x, x);\ \textbf{while } x \neq 0 \textbf{ do } x := fst(x)$ according to operational semantics. Finally, prove the equivalence between operational and denotational semantics.

Exercise 2 (7)

A *preorder* is a pair $(\mathcal{S}, \sqsubseteq)$, where $\sqsubseteq$ is a relation on set $\mathcal{S}$ with the properties of reflexivity and transitivity (not necessarily antisymmetry). Let $S_1 \equiv S_2$ iff $S_1 \sqsubseteq S_2$ e $S_2 \sqsubseteq S_1$ and prove that $\equiv$ is an equivalence relation.

Preorders on the same $\mathcal{S}$ are themselves ordered by inclusion, namely: $(\mathcal{S}, \sqsubseteq) \subseteq (\mathcal{S}, \sqsubseteq')$ if $\sqsubseteq \subseteq \sqsubseteq'$. Prove that preorders ordered by inclusion form a complete partial ordering with bottom. Given two preorders, their lub (least upper bound) and their glb (greatest lower bound) are always defined? How?

Exercise 3 (7)

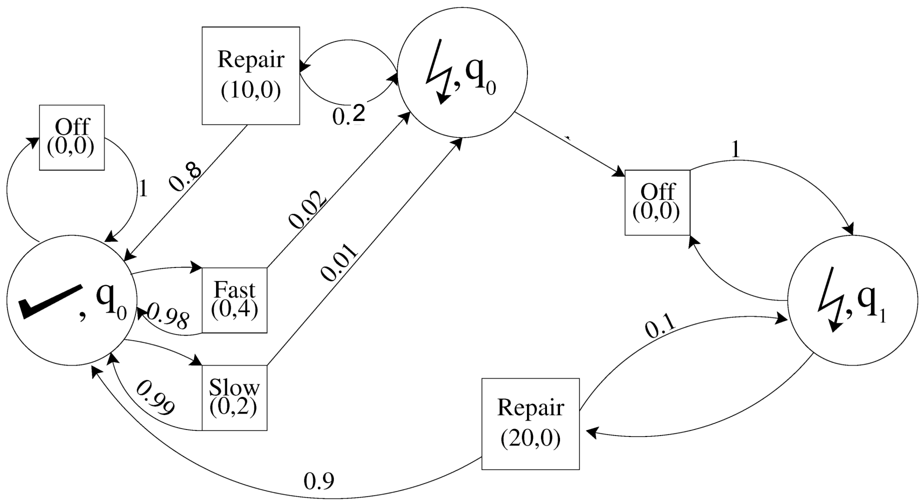
Let us consider CCS with prefix, sum, nil, parallel composition and the additional operations $\rfloor$ and $\|$. The axiomatization of this calculus consists of the following axioms:

$E_1|E_2 = E_1\rfloor E_2 + E_2\rfloor E_1 + E_1\|E_2 \qquad E_1|E_2 = E_2|E_1 \qquad (E_1|E_2)|E_3 = E_1|(E_2|E_3)$
$\mu.E_1\rfloor E_2 = \mu.(E_1|E_2) \qquad \lambda.E_1\|\overline{\lambda}.E_2 = \tau(E_1|E_2) \qquad \mu_1.E_1\|\mu_2.E_2 = nil \text{ if } \mu_1 \neq \overline{\mu_2}$
$(E_1 + E_2)\rfloor E = E_1\rfloor E + E_2\rfloor E \qquad (E_1 + E_1')\|(E_2 + E_2') = E_1\|E_2 + E_1\|E_2' + E_1'\|E_2 + E_1'\|E_2'$
$nil\rfloor E = nil\|E = E\|nil = nil$
$E + nil = E \qquad E_1 + E_2 = E_2 + E_1 \qquad (E_1 + E_2) + E_3 = E_1 + (E_2 + E_3) \qquad E + E = E.$

(i) Prove using the axioms that $a.b.nil|c.nil + c.a.b.nil = a.b.nil|c.nil$, where a, b and c are all different;
(ii) define inference rules also for the new operators $\rfloor$ and $\|$;
(iii) prove using the rules just defined that the axioms $E_1|E_2 = E_1\rfloor E_2 + E_2\rfloor E_1 + E_1\|E_2$ and $(E_1 + E_2)\rfloor E = E_1\rfloor E + E_2\rfloor E$ do *bisimulate*. Namely, given generic transitions of E_1, E_2 and E (in the second case), every transition of the left hand side of the axiom can be matched by a transition of the right hand side (and viceversa) with the same action and with continuations which can be proved equivalent using (all) the axioms.

Exercise 4(8)

A *water pump* can be described by the following control model.



Round boxes describe the states of the pump (working or not working) and of the control (q_0 or q_1). Arcs out of the round boxes represent the possible control actions. While unlabeled, they can be assumed as labelled by the action in the square box they end up in. For instance, the pump can be made working *Fast*, or *Slow*, or it can be turned *Off*. Out of the square boxes there are probability distributions: e.g. if the pump works fast it is more likely to break down. Number pairs in the square boxes represent (repair) cost and gain (amount of water pumped) respectively, in each condition. Notice that, in the presence of a failure, there are two possibilities: either try to repair the pump while keeping it operative, or turn it off. In the former case, the cost of the repair is smaller (10 instead of 20), but so it is the likelihood that the repair is effective (80% instead of 90%).

(i) Define a Markov Decision Process, MDP (also *reactive probabilistic labeled transition system* in the terminology used in the course) corresponding to the given control model.

A *strategy* is an assignment of a unique control action to every round box. Given any strategy, the MDP is straightforwardly transformed in a DTMC with costs and gains associated to every state. For instance, if a strategy assigns the action *Slow* to the state *(working, q_0)*, the corresponding state of the DTMC has a transition with probability 1% to the state *(not working, q_0)* and a probability of 99% of remaining in the same state.

(ii) Define the DTMC corresponding to the strategy which assigns *Slow* to the state *(working, q_0)*, *Off* to *(not working, q_0)* and *Repair* to the remaining state. Compute the stationary probabilities (ergodic?) and the average global cost and gain.

(iii) Do the same assigning the action *Fast* to the state *(working, q_0)* and *Repair* to the state *(not working, q_0)*.

(iv) Discuss which is the best strategy.