Models of Computation

Written Exam on July 12, 2011

(First part: Exercises 1 and 2, 90 minutes   Second part: Exercises 3, 4 and 5, 90 minutes)
(Previous TSD students: Exercises 1, 2 and 3, 120 minutes)

Exercise 1 (7)

Extend IMP with rational numbers and consider the command:

$$w \;=\; \textbf{while}\; y < n \;\textbf{do}\,(x := x/2;\; z := z + x;\; y := y + 1) \quad n \geq 0.$$

Prove by rule induction that whenever $\langle w, \sigma\rangle \to \sigma'$, if $0 \leq \sigma(y) \leq n$, $\sigma(x) = 2^{-\sigma(y)}$, $\sigma(z) = 1 - 2^{-\sigma(y)}$, then $\sigma'(y) = n$, $\sigma'(x) = 2^{-n}$, $\sigma'(z) = 1 - 2^{-n}$.

Observe that, as a consequence, if $\sigma(y) = 0$, $\sigma(x) = 1$, $\sigma(z) = 0$, then $\sigma'(y) = n$, $\sigma'(x) = 2^{-n}$, $\sigma'(z) = 1 - 2^{-n}$.

Exercise 2 (8)

Consider the monotone functions $f : \omega \to \omega$, where ω is ordered with $\leq$. Two such functions are related as $f \sqsubseteq g$ iff $\forall n. f(n) \leq g(n)$. Is $\sqsubseteq$ a partial ordering? Complete? With bottom? Is function $F = \lambda f.\lambda n.f(f(n))$ monotone? Continuous? If not, give conterexamples.

Exercise 3 (5)

Extend CCS with the operation of *sequential composition* ; defined by the following inference rules:

$$\frac{q \xrightarrow{\mu} q'}{nil; q \xrightarrow{\mu} q'} \qquad \frac{p \xrightarrow{\mu} p'}{p; q \xrightarrow{\mu} p'; q}.$$

Show that $q_1 \simeq q_2 \Rightarrow p; q_1 \simeq p; q_2$, while bisimilarity is not a congruence with respect to the second argument: $p_1 \simeq p_2 \not\Rightarrow p_1; q \simeq p_2; q$ (Hint: consider agents *nil* and $(\alpha.nil)\backslash\alpha$.). Consider now the alternative definition:

$$\frac{p \stackrel{\mu'}{\not\to} \quad q \xrightarrow{\mu} q'}{p; q \xrightarrow{\mu} q'} \qquad \frac{p \xrightarrow{\mu} p'}{p; q \xrightarrow{\mu} p'; q}.$$

According to the new definition, is bisimilarity a congruence with respect to both arguments of ;? Notice that property $p \stackrel{\mu'}{\not\to}$ means that there is no proof that $p \xrightarrow{\mu'} p'$. Can such a property be finitely axiomatizable? Remember that CCS is Turing-equivalent.

Exercise 4 (5)

In the π-calculus, infinite branching is a serious drawback for finite verification. Show that agents $x(y).\overline{y}y.nil$ and $(y)\overline{x}y.\overline{y}y.nil$ are infinite branching. Modify the input axiom, the open rule, and possibly the parallel composition rule by limiting to one the number of different fresh names which can be assigned to the new name. Modify also the input clause for the early bisimulation by limiting the set of possible continuations by substituting all the free names and only one fresh name. Discuss the possible criteria for choosing the fresh name, e.g. the first, in some ordering, name which is not free in the agent. Check if your criteria make agents $x(y).\overline{y}y.nil$ and $x(y).(\overline{y}y.nil|(z)z\overline{w}.nil)$ bisimilar or not.

Exercise 5(5)

Consider the following PEPA program P with infinite states $\{A_\alpha, B_\beta\}$, with $\alpha, \beta \in \{0, 1\}^*$:

$$A_\epsilon \;\text{ where }\; A_\alpha = (a, \lambda)B_{\alpha 0} + (a, \lambda)B_{\alpha 1} \;\; B_\alpha = (b, \lambda)A_{\alpha 0} + (b, \lambda)A_{\alpha 1}.$$

Draw (!) the transition system of P, find the reachable states fromA_ϵ and determine the bisimilar states. Finally, find the smallest PEPA program bisimilar to P.

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Exercise 1

$$P(\langle w,\sigma\rangle \to \sigma') \overset{def}{\iff} 0 \le \sigma(y) \le n, \sigma(x) = 2^{-\sigma(y)}, \sigma(z) = 1-2^{-\sigma(y)}$$
$$\implies \sigma'(y) = n, \sigma'(x) = 2^{-n}, \sigma'(z) = 1-2^{-n}$$

Proof by rule induction

$$\frac{\langle y<n,\sigma\rangle \to false}{\langle w,\sigma\rangle \to \sigma}$$

We assume $\sigma(y) \geq n$

We assume also:

$0 \le \sigma(y) \le n$ thus $\sigma(y) = n$

$\sigma(x) = 2^{-n}$ $\sigma(z) = 1-2^{-n}$

we have to prove $\sigma(y) = n$ $\sigma(x) = 2^{-n}$ $\sigma(z) = 1-2^{-n}$

QED.

$$\frac{\langle y<n,\sigma\rangle \to true \quad \langle x:=x/2\ z:=z+x; y:=y+1,\sigma\rangle \to \sigma'' \quad \langle w,\sigma''\rangle \to \sigma'}{\langle w,\sigma\rangle \to \sigma'}$$

$$\sigma'' = \sigma\left[\sigma(x)/2/x, \sigma(z)+\sigma(x)/2/z, \sigma(y)+1/y\right]$$

We assume $\sigma(y) < n$ and also

$0 \le \sigma(y)$ $\sigma(x) = 2^{-\sigma(y)}$ $\sigma(z) = 1-2^{-\sigma(y)}$

(2)

Thus we have:

$$\sigma''(y) = \sigma(y)+1 \quad \text{thus} \quad 0 < \sigma''(y) \leq n$$
$$\sigma''(x) = \sigma(x)/2 = 2^{-\sigma(y)}/2 = 2^{-(\sigma(y)+1)} = 2^{-\sigma''(y)}$$
$$\sigma''(z) = \sigma(z) + \sigma(x)/2 = 1-2^{-\sigma(y)} + 2^{-\sigma(y)}/2$$
$$= 1-2^{-(\sigma(y)+1)} = 1-2^{-\sigma''(y)}$$

We thus have all the premises of $P(\langle w, \sigma''\rangle \to \sigma')$

Thus we conclude:
$\sigma'(y) = n$, $\sigma'(x) = 2^{-n}$, $\sigma'(z) = 1-2^{-n}$ Q.E.D.

Finally, the values $\sigma(y)=0$, $\sigma(x)=1$, $\sigma(z)=0$ satisfy the premises of $P(\langle w, \sigma\rangle \to \sigma')$:

$0 \leq 0 \leq n$, $1 = 2^{-0}$, $0 = 1-2^{-0}$

Q.E.D.

Exercise 2

$\sqsubseteq$ Partial ordering

- $f \overset{?}{\sqsubseteq} f \qquad \forall x.\ f(x) \leq f(x)$
- $f \sqsubseteq g \quad g \sqsubseteq f \overset{?}{\Rightarrow} f = g \qquad \forall x.\ f(x) \leq g(x),\ g(x) \leq f(x)$

  $f(x) = g(x)$, namely $f = g$ QED
- $f \sqsubseteq g \quad g \sqsubseteq h \overset{?}{\Rightarrow} f \sqsubseteq h \qquad \forall x.\ f(x) \leq g(x) \leq h(x)$ QED

Not Complete

Counterexample: $f_i(x) = i$ constant

$f_0 \sqsubseteq f_1 \sqsubseteq f_2 \sqsubseteq \dots$ There is no lub.

Infact $\left(\bigsqcup f_i\right)(x)$ should be larger than any number.

With bottom $\quad f(x) = 0$ constant

F monotone:

$\forall x.\ f(x) \leq g(x) \overset{?}{\Rightarrow} \forall x.\ f(f(x)) \leq g(g(x))$

In fact $f(x) \leq g(x) \Rightarrow f(f(x)) \leq f(g(x)) \leq g(g(x))$ QED

F continuous: meaningless since the PO is not complete

Exercise 3

$q_1 \sim q_2$ implies R bisimulation $q_1 R q_2$

Let $R' = R \cup \{(p;q_1, p;q_2) \mid \forall p\}$

We must prove $p;q_1 \, R' \, p;q_2$ (obvious), R' bisim.

Infact, if $q_1 R' q_2$ and $q_1 R q_2$ obvious

If $p;q_1 \, R' \, p;q_2$ $\quad p \neq nil$ obvious

If $nil;q_1 \, R' \, nil;q_2$;

$nil;q_1 \xrightarrow{\mu} q_1' \Rightarrow q_1 \xrightarrow{\mu} q_1' \Rightarrow q_2 \xrightarrow{\mu} q_2' \quad q_1' R q_2'$

$\Rightarrow p;q_2 \xrightarrow{\mu} p;q_2' \quad p;q_1' \, R' \, p;q_2'$ QED.

$nil;\beta nil \xrightarrow{\beta}$ $\qquad nil \simeq (\alpha nil)\backslash\alpha$

$(\alpha nil)\backslash\alpha \, ; \beta nil \not\xrightarrow{\beta}$ $\qquad \downarrow \quad \not\downarrow$

With the new definition we have

$P_1 \simeq P_2 \Rightarrow P_1;q \simeq P_2;q$

The same as before $P_1 R P_2$ $\quad R' = R \cup \{(P_1;q, P_2;q) \mid \forall q\}$

Infact if $P_1 R' P_2$ and $P_1 R P_2$ obvious

If $P_1;q \, R \, P_2;q$ and $P_1;q \xrightarrow{\mu} P_1';q$ obvious

If $P_1;q \, R \, P_2;q$ and $P_1 \not\rightarrow$ then also $P_2 \not\rightarrow$

then $q \xrightarrow{\mu} q'$ for both agents $\qquad$ QED.

Nontermination is not semidecidable for T-equivalent formalism,

Theorems of inference systems are semidecidable.

Exercise 4

$$x(y).\overline{y}y.nil \xrightarrow{x(w)} \overline{w}w.nil \quad \forall w \neq x$$

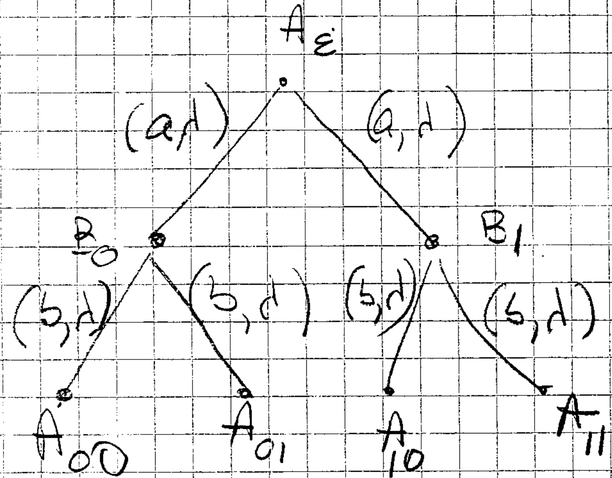
$$(y)\overline{x}y.\overline{y}y.nil \xrightarrow{\overline{x}(w)} \overline{w}w.nil \quad \forall w \neq x$$

- Input axiom $\quad x(y)P \xrightarrow{x(w)} P\{w/y\} \quad w = choose(x(y).P)$
- Open rule $\quad \dfrac{P \xrightarrow{\overline{x}y} P'}{(y)P \xrightarrow{\overline{x}(w)} P'\{w/y\}} \quad w = choose(x(y).P)$
- parellel rule $\quad \dfrac{P \xrightarrow{\alpha} P'}{P|Q \xrightarrow{\sigma(\alpha)} \sigma(P')|Q}$ where σ renames $choose(P)$ into $choose(P|Q)$
- if $P \xrightarrow{x(y)} P'$ with $y \notin fn(P,Q)$ then for all $w \in fn(P) \cup fn(Q) \cup choose(P,Q)$ there exist Q' such that $Q \xrightarrow{x(z)} Q'$ and $P'\{w/y\} S Q'\{w/z\}$

Functions $choose(P)$ and $choose(P,Q)$ are arbitrary, but they must choose a name which is not free in P or P,Q

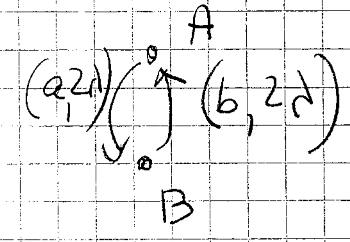
Also the case $P \xrightarrow{\overline{x}(y)} P'$ must be modified.

Exercise 5



The reachable states from A_ε are A_α with $|\alpha|$ even and B_α with $|\alpha|$ odd.
All A_α for all α and B_α for all α are bisimilar

The smallest PePa program is



A

$A \stackrel{def}{=} (a, 2\lambda)B$

$B \stackrel{def}{=} (b, 2\lambda)A$