Models of Computation

Written Exam on February 1, 2012

(MOD students: Exercises 1-5, 180 minutes   Previous TSD students: Exercises 1-3, 120 minutes)

Exercise 1 (7)

Extend IMP with the construct **either** c_0 **or** c_1 having the following operational semantics:

$$\frac{< c_0, \sigma > \rightarrow \sigma'}{< \textbf{either } c_0 \textbf{ or } c_1, \sigma > \rightarrow \sigma'} \qquad \frac{< c_1, \sigma > \rightarrow \sigma'}{< \textbf{either } c_0 \textbf{ or } c_1, \sigma > \rightarrow \sigma'}.$$

Consider then the command:

$$w_n \;=\; \textbf{while } x \neq n \textbf{ do either } x := 2 \times x;\; y := y \times y \textbf{ or } x := 2 \times x + 1;\; y := k \times y \times y.$$

Proof that for every $\sigma, \sigma' \in \Sigma$ with $\langle w_n, \sigma \rangle \rightarrow \sigma'$

$$\text{if } \sigma(y) = k^{\sigma(x)} \qquad \text{then } \sigma' = \sigma[n/x, k^n/y].$$

Exercise 2 (6)

Prove that the relation $R_S = (\mathcal{P}(\omega) \backslash S, \subseteq)$ is a partial ordering for every class S of sets of natural numbers. Then exhibit a class $\overline{S}$ such that $R_{\overline{S}}$ is not complete and has no minimum. Finally, prove that R_S is complete if S contains only finite sets.

Exercise 3 (7)

Modify HOFL, assigning the following semantics to the conditional construct:

$$\frac{t_0 \rightarrow 0 \;\; t_1 \rightarrow c_1 \;\; t_2 \rightarrow c_2}{\textbf{if } t_0 \textbf{ then } t_1 \textbf{ else } t_2 \rightarrow \;\; c_1} \qquad \frac{t_0 \rightarrow n \;\; n \neq 0 \;\; t_1 \rightarrow c_1 \;\; t_2 \rightarrow c_2}{\textbf{if } t_0 \textbf{ then } t_1 \textbf{ else } t_2 \rightarrow \;\; c_2}.$$

Define the corresponding denotational semantics and proof that also for the modified semantics $t \rightarrow c$ implies $[\![t]\!] = [\![c]\!]$. Finally, derive both the operational and the denotational semantics of $(fact\ 0)$, with $fact = rec\ f.\lambda x.$ **if** x **then** 1 **else** $x \times (f\ x - 1)$, checking if they coincide or not.

Exercise 4 (5)

Define by structural induction the free names $fn(p)$ of a CCS agent p. Prove by rule induction on the operational semantics that $p \xrightarrow{\mu} q$ implies $fn(q) \subseteq fn(p)$. Show an example where $fn(q) \subset fn(p)$. Now consider the π-calculus, and show an example of $p \xrightarrow{\alpha} q$ where $fn(p) \subset fn(q)$.

Exercise 5 (5)

A reactive, probabilistic labelled transition system representing the options of a taxi driver has states s_1 (parking), s_2 (moving slowly looking for costumers) and s_3 (busy with clients); actions p (stay in the parking) and m (move); and transition function α with $\alpha s_1 p s_3 = 0.2$, $\alpha s_1 p s_1 = 0.8$, $\alpha s_1 m s_2 = 1$, $\alpha s_2 x s_2 = 0.5$, $\alpha s_2 x s_3 = 0.5$, $\alpha s_3 x s_1 = 0.6$, and $\alpha s_3 x s_3 = 0.4$, where x is any action p or m. Consider the two Markov chains obtained by letting the taxi driver choose always p or always m (in the former case disregard state s_2). Are they ergodic? In both cases compute the stationary probability distributions and the average costs and gains, knowing that the costs of states s_1, s_2 and s_3 are 0, 2 and 5, while the gains are 0, 0 and 10 respectively. What is the most convenient choice?

(1)

Correzione Prova Scritta del 01/02/2012

Esercizio 1

Dimostrazione per induzione sulle prove

$$P(\langle w_n, \sigma\rangle \to \sigma') \overset{def}{=} \sigma(y) = K^{\sigma(x)} \Rightarrow \sigma' = \sigma[^n/_x, ^{K^n}/_y]$$

$$\frac{\sigma(x) = n}{\langle w_n \sigma\rangle \to \sigma} \qquad P(\langle w_n, \sigma\rangle \to \sigma) \overset{def}{=} \sigma(y) = K^{\sigma(x)}$$

$$\Rightarrow \sigma = \sigma[^n/_x, ^{K^n}/_y] \quad \text{ovvio da } \sigma(x) = n$$

CVD.

- Prima regola di either_or_

$$\langle \text{either}(x := 2\times x; y := y\times y) \text{ or } (x := 2x+1; y := K\times y \times y, \sigma\rangle \to \sigma'' \leftarrow$$

$$\langle x := 2\times x; y := y\times y, \sigma\rangle \to \sigma'' \qquad \sigma'' = \sigma[^{2\sigma(x)}/_x, ^{\sigma(y)^2}/_y]$$

$$\frac{\sigma(x) \neq n \quad \sigma'' = \sigma[^{2\sigma(x)}/_x, ^{\sigma(y)^2}/_y] \quad \langle w_n, \sigma''\rangle \to \sigma'}{\langle w_n, \sigma\rangle \to \sigma'}$$

Assumiamo: $\sigma(x) \neq n$ (che però non servirà)

$\sigma''(y) = K^{\sigma''(x)} \Rightarrow \sigma' = \sigma''[^n/_x, ^{K^n}/_y]$ ipotesi induttiva

Dobbiamo dimostrare

$\sigma(y) = K^{\sigma(x)} \overset{?}{\Rightarrow} \sigma' = \sigma[^n/_x, ^{K^n}/_y]$

Assumiamo la premessa: $\sigma(y) = K^{\sigma(x)}$

Resta da dimostrare $\sigma' \overset{?}{=} \sigma[^n/_x, ^{K^n}/_y]$

$\sigma''(y) = \sigma(y)^2 \qquad \sigma''(x) = 2\sigma(x)$

$\sigma(y)^2 = \left(K^{\sigma(x)}\right)^2$ ancora la premessa

$\sigma''(y) = K^{2\sigma(x)} = \sigma''(x)$ quindi $\sigma' = \sigma''[^n/_x, ^{K^n}/_y]$ dalla ipotesi indut

CVD.

2

- Seconda regola di either-or-

$$\sigma'' = \sigma\left[{}^{2\sigma(x)+1}/_{x},\ {}^{K\,\sigma(y)^2}/_{y}\right]$$

$$\frac{\sigma(x) \neq n \quad \sigma'' = \sigma\left[{}^{2\sigma(x)+1}/_{x},\ {}^{K\cdot\sigma(y)^2}/_{y}\right] \quad \langle w_n, \sigma'' \rangle \to \sigma'}{\langle w_n, \sigma \rangle \to \sigma'}$$

Cambia solo:

$$\sigma''(y) = K\,\sigma(y)^2 \qquad \sigma''(x) = 2\sigma(x)+1$$

$$\sigma(y)^2 = \left(K^{\sigma(x)}\right)^2 \qquad K\,\sigma(y)^2 = K^{2\sigma(x)+1}$$

$$\sigma''(y) = K^{\sigma''(x)}$$

quindi $\sigma' = \sigma''\left[{}^{n}/_{x},\ {}^{K^n}/_{y}\right]$ dall'ipotesi induttiva CVD

Esercizio 2

Le proprietà:

Riflessiva $\forall I.\ I \subseteq I$

Antisimmetrica $\forall I, J.\ I \subseteq J,\ J \subseteq I \Rightarrow I = J$

Transitiva $\forall I, J, K.\ I \subseteq J,\ J \subseteq K \Rightarrow I \subseteq K$

che valgono per $I, J, K \in P(\omega)$, valgono anche per $I, J, K \in P(\omega) \setminus S$, essendo I, J e K quantificati universalmente.

Poniamo $\bar{S} = \{\perp, \omega\}$. Allora $P(\omega) \setminus S$ non ha minimo, essendo tutti i singleton $\{0\}, \{1\}, \{2\}, \dots$ minimali ma non minimi.

Inoltre la catena

$\{0\} \subseteq \{0,1\} \subseteq \{0,1,2\} \subseteq \dots$

non ha nessun maggiorante e quindi non ha limite.

Dimostriamo che se $\bar{S}$ contiene solo insiemi finiti allora $(P(\omega)/S, \subseteq)$ è completo.

Sia $I_0 \subseteq I_1 \subseteq I_2 \subseteq \dots$ una catena. Se $\cup I_i$ in $P(\omega)$ è un insieme finito, allora $\{I_i\}$ è una catena finita, e quindi ha limite anche in $P(\omega)/S$. Invece se $\{I_i\}$ è una catena infinita, allora $\cup I_i$ è un insieme infinito, e quindi esiste in $P(\omega)/S$ per costruzione.

Esercizio 3

Semantica denotazionale modificata

$$[\![\text{if } t_0 \text{ then } t_1 \text{ else } t_2]\!]\rho = Condd([\![t_0]\!]\rho, [\![t_1]\!]\rho, [\![t_2]\!]\rho)$$

$$Condd : N_\perp \times D_\perp \times D_\perp \to D_\perp$$

$$Condd(z_0, z_1, z_2) = \begin{cases} \perp_{D_\perp} & \text{se } z_0 = \perp_{N_\perp} \text{ o } z_1 = \perp_{D_\perp} \text{ o } z_2 = \perp_{D_\perp} \\ z_1 & \text{se } z_0 = \lfloor 0 \rfloor \text{ e } z_2 \neq \perp_{D_\perp} \\ z_2 & \text{se } z_0 = \lfloor n \rfloor,\ n \neq 0, \text{ e } z_1 \neq \perp_{D_\perp} \end{cases}$$

basta che un elemento sia $\perp$, e vale $\perp$

Condd è continua essendo continua su ogni argomento:

$Cond(\perp_{N_\perp}, z_1, z_2) = \perp_{D_\perp}$

$Cond(\lfloor 0 \rfloor, z_1, \perp_{D_\perp}) = \perp_{D_\perp}$

$Cond(\lfloor 0 \rfloor, z_1, \lfloor y_2 \rfloor) = z_1$

$Cond(\lfloor n \rfloor, \perp_{D_\perp}, z_2) = \perp_{D_\perp}$

$Cond(\lfloor n \rfloor, \lfloor y_1 \rfloor, z_1) = z_2$

ogni funzione è una costante oppure l'identità

$z_1 \backslash z_2$	$\perp_{D_\perp}$	$\lfloor y_2 \rfloor$
$\perp_{D_\perp}$	$\perp_{D_\perp}$	$\perp_{D_\perp}$
$\lfloor y_1 \rfloor$	$\perp_{D_\perp}$	$\perp_{D_\perp}$

$z_0 = \perp_{N_\perp}$

$z_1 \backslash z_2$	$\perp_{D_\perp}$	$\lfloor y_2 \rfloor$
$\perp_{D_\perp}$	$\perp_{D_\perp}$	$\perp_{D_\perp}$
$\lfloor y_1 \rfloor$	$\perp_{D_\perp}$	$\lfloor y_1 \rfloor$

$z_1 = \lfloor 0 \rfloor$

$z_1 \backslash z_2$	$\perp_{D_\perp}$	$\lfloor y_2 \rfloor$
$\perp_{D_\perp}$	$\perp_{D_\perp}$	$\perp_{D_\perp}$
$\lfloor y_1 \rfloor$	$\perp_{D_\perp}$	$\lfloor y_2 \rfloor$

$z_1 = \lfloor 1 \rfloor$

$P(t \to c) \overset{def}{=} [\![t]\!]\rho = [\![c]\!]\rho \qquad t : \tau$

$$\frac{t_0 \to 0 \quad t_1 \to c_1 \quad t_2 \to c_2}{\text{if } t_0 \text{ then } t_1 \text{ else } t_2 \to c_1}$$

$P(t_0 \to 0) \overset{def}{=} [\![t_0]\!]\rho = \lfloor 0 \rfloor \qquad P(t_2 \to c_2) \overset{def}{=} [\![t_2]\!]\rho = [\![c_2]\!]\rho$

$P(t_1 \to c_1) \overset{def}{=} [\![t_1]\!]\rho = [\![c_1]\!]\rho$

$P(\text{if } t_0 \text{ then } t_1 \text{ else } t_2 \to c_1) = [\![\text{if } t_0 \text{ then } t_1 \text{ else } t_2]\!]\rho \overset{?}{=} [\![c_1]\!]\rho$

$[\![\text{if } t_0 \text{ then } t_1 \text{ else } t_2]\!]\rho = Condd([\![t_0]\!]\rho, [\![t_1]\!]\rho, [\![t_2]\!]\rho)$

Essendo $[\![t_0]\!]\rho = \lfloor 0 \rfloor$ e $[\![c_2]\!] \neq \perp_{(V_\tau)_\perp}$ abbiamo

$Condd(\lfloor 0 \rfloor, [\![t_1]\!]\rho, \lfloor v_2 \rfloor) = [\![t_1]\!]\rho = [\![c_1]\!]\rho$

Stessa prova per la 2^a regola CVD.

$(fact\ 0) \to c \leftarrow fact \to \lambda x.t \quad t[0/x] \to c \leftarrow$

$\text{if } x \text{ then } 1 \text{ else } x \times (fact\ x-1))\ [0/x] \to c \leftarrow$

$0 \to 0 \quad 1 \to c \quad 0 \times (fact\ -1) \to c' \xleftarrow{c=1} 0 \times (fat\ -1) \to c'$

$\xleftarrow{c = c_1 \times c_2} 0 \to c_1 \ (fact\ -1) \to c_2 \xleftarrow{c_1 = 0} (f\ -1) \to c_2 \leftarrow \ldots$

non c'è prova: $(fact\ 0)$ non ha forma canonica

$$[\![fact\ 0]\!]\rho = \text{let } \varphi \Leftarrow [\![fact]\!]\rho \,.\, \varphi\lfloor 0 \rfloor$$

$$[\![fact]\!] = fix\ \lambda d.\ \lfloor \lambda d'.\ Condd\left(d', \lfloor 1 \rfloor, d' \times_{\perp} \text{let } \varphi \Leftarrow d.\ \varphi\left(d' -_{\perp} \lfloor 1 \rfloor\right)\right) \rfloor$$

$$d_0 = \perp_{[N_\perp \to N_\perp]_\perp}$$

$$d_1 = \lfloor \lambda d'.\ Condd\left(d', \lfloor 1 \rfloor, \perp_{N_\perp}\right) \rfloor = \lfloor \perp_{[N_\perp \to N_\perp]} \rfloor$$

basta che un argomento sia $\perp$

$$d_2 = \lfloor \lambda d'.\ Condd\left(d', \lfloor 1 \rfloor, d' \times_{\perp} \text{let } \varphi \Leftarrow \lfloor \perp \rfloor . \varphi(d' -_{\perp} \lfloor 1 \rfloor)\right) \rfloor$$
$$= \lfloor \lambda d'.\ Condd(d', \lfloor 1 \rfloor, \perp_{N_\perp}) \rfloor = \lfloor \perp_{[N_\perp \to N_\perp]} \rfloor$$

$$[\![fact\ 0]\!]\rho = \left(\lambda d'.\ \perp_{[N_\perp \to N_\perp]}\right) 0 = \perp_{N_\perp} \qquad \text{CVD.}$$

Exercise 4

$fn(x) = fn(nil) = \phi$

$fn(rec\ x.p) = fn(\tau.p) = fn(p)$ $\qquad fn(p+q) = fn(p|q) = fn(p) \cup fn(q)$

$fn(\alpha.p) = fn(\bar{\alpha}.p) = \{\alpha\} \cup fn(p)$

$fn(p\backslash\alpha) = fn(p) - \{\alpha\}$

$fn(p[\phi]) = \phi(fn(p)) = \{\phi(\alpha) \mid \alpha \in fn(p)\}$

$P(p \xrightarrow{\mu} q) \overset{def}{=} fn(p) \supseteq fn(q)$

- $\mu.p \xrightarrow{\mu} p$ obvious, $fn(\mu.p) = fn(p) \cup \ldots$

- $\dfrac{p \xrightarrow{\mu} q}{p\backslash\alpha \xrightarrow{\mu} q\backslash\alpha}$ obvious, $fn(p) \supseteq fn(q)$ implies $fn(p) - \alpha \supseteq fn(q) - \alpha$

- $\dfrac{p \xrightarrow{\mu} q}{p[\phi] \xrightarrow{\phi(\mu)} q[\phi]}$ obvious $fn(p) \supseteq fn(q)$ implies $\phi(fn(p)) \supseteq \phi(fn(q))$, ϕ injective

- $\dfrac{p \xrightarrow{\mu} q}{p+r \xrightarrow{\mu} q}$ obvious $fn(p) \supseteq fn(q)$ implies $fn(p) \cup \ldots \supseteq fn(q)$ and symmetric

- $\dfrac{p \xrightarrow{\mu} q}{p|r \xrightarrow{\mu} q|r}$ obvious $fn(p) \supseteq fn(q)$ implies $fn(p) \cup fn(r) \supseteq fn(q) \cup fn(r)$ and symmetric

- $\dfrac{p_1 \xrightarrow{\alpha} q_1 \quad p_2 \xrightarrow{\bar{\alpha}} q_2}{p_1|p_2 \xrightarrow{\tau} q_1|q_2}$ $\begin{matrix} fn(p_1) \supseteq fn(q_1) \\ fn(p_2) \supseteq fn(q_2) \end{matrix}$ implies $fn(p_1) \cup fn(p_2) \supseteq fn(q_1) \cup fn(q_2)$

- $\dfrac{p[rec\ x.p/x] \xrightarrow{\mu} q}{rec\ x.p \xrightarrow{\mu} q}$ notice that $fn(p[rec\ x.p/x]) = fn(p) \cup fn(rec\ x.p) = fn(p)$

thus $fn(p[rec\ x.p/x]) \supseteq fn(q)$ implies $fn(rec\ x.p) = fn(p) \supseteq fn(q)$

An example: $\alpha . nil \xrightarrow{\alpha} nil$

$fn(\alpha . nil) = \{\alpha\} \supset fn(nil) = \phi$

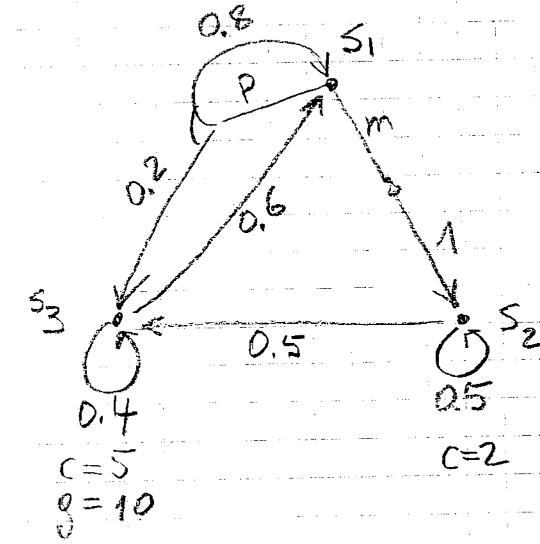
The π-calculus:

$(x)\bar{y}x . \bar{x}y . nil \xrightarrow{\bar{y}(z)} \bar{z}y . nil$

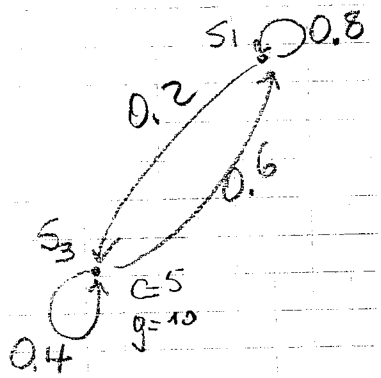
$fn((x)\bar{y}x . \bar{x}y . nil) = \{y\}$

$fn(\bar{z}y . nil) = \{y, z\}$

Exercise 5



Case p is chosen:



$x_1 = 0.8x_1 + 0.6x_3 \qquad x_1 + x_3 = 1$

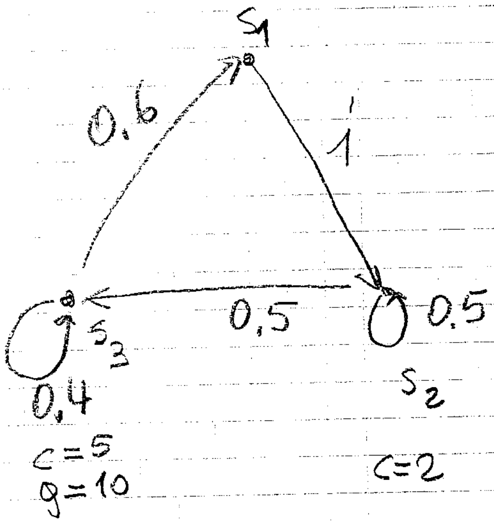
$x_3 = 0.2x_1 + 0.4x_3$

$0.2x_1 = 0.6x_3 \qquad x_1 = 3x_3$

$x_1 = 0.75 \qquad x_3 = 0.25$

$c_3 = 1.25 \qquad g_3 = 2.5 \qquad \boxed{\text{diff} = 1.25}$

Case m is chosen



$x_1 = 0.6x_3 \qquad x_1 + x_2 + x_3 = 1$

$x_2 = x_1 + 0.5x_2$

$x_3 = 0.5x_2 + 0.4x_3$

$0.6x_3 = 0.5x_2 \qquad x_2 = 1.2x_3$

$0.6x_3 + 1.2x_3 + x_3 = 1 \qquad x_3 = 10/28$

$x_2 = \frac{12}{28} = \frac{3}{7} \qquad x_1 = \frac{0.6}{2.8} = \frac{6}{28}$

$c_3 = \frac{50}{28} \qquad g_3 = \frac{100}{28} \qquad c_2 = \frac{24}{28} \qquad \text{diff} = \frac{26}{28} = \boxed{0.93}$

$\boxed{\text{P is better}}$