Models of Computation

Written Exam on September 4, 2012

(MOD students: Exercises 1-4, 180 minutes   Previous TSD students: Exercises 1-3, 140 minutes)

Exercise 1 (8)

1. Prove by structural induction on arithmetic expressions a the property:

$$x \notin VAR(a) \quad \mathcal{A}[\![a]\!]\sigma = n \quad \mathcal{A}[\![a]\!]\sigma[k/x] = m \quad \Rightarrow \quad n = m.$$

2. For the commands $c_1 = x_1 := a_1$ e $c_2 = x_2 := a_2$, give sufficient conditions for $c_1; c_2$ and $c_2; c_1$ to be denotationally equivalent and prove it.

Exercise 2 (8)

Given a set A, consider the partial order relations $\sqsubseteq$ on A. Since relations are sets, they are ordered by inclusion $\subseteq$ and thus they form themselves a partial ordering $(PO, \subseteq)$. Prove that $(PO, \subseteq)$ is complete with bottom. Finally, prove that $(CPO, \subseteq)$, which includes only the partial orderings which are complete, has bottom, but, if A is infinite, it is not complete.

Exercise 3 (7)

Consider a closed HOFL term t and prove, employing both the denotational and the lazy operational semantics (but not their equivalence!), that

$$rec\ x.t \equiv t \qquad ((\lambda x.t)\ rec\ x.x) \equiv t$$

while the second property does not hold for the eager operational semantics.

Exercise 4 (7)

Determine the CCS processes reachable from $p = rec\, x.\alpha.(x|x)$ considering the parallel composition operator NOT associative/commutative, compute the bisimilarity relation on them and find a minimal process q bisimilar to p. Then define bisimilarity for *labeled* CTMC. Finally, in full analogy with the CCS case, determine the PEPA processes reachable from A, with $A \stackrel{def}{=} (\alpha, \lambda).(A|_\phi A)$ (again considering the parallel composition operator not associative/commutative), compute the bisimilarity relation on them and find a minimal process B bisimilar to A.

Correzione Prova Scritta del 040912

Esercizio 1

$a ::= x \mid a \; op \; a$

$VAR(x) = \{x\}$

$VAR(a_1 \; op \; a_2) = Var(a_1) \cup VAR(a_2)$

$$P(a) \stackrel{def}{=} x \notin VAR(a) \quad \mathcal{A}[\![a]\!]\sigma = n \quad \mathcal{A}[\![a]\!]\sigma[k/x] = m \implies n = m$$

(1a) $a = y \qquad VAR(a) = y$

$x \neq y \quad \mathcal{A}[\![y]\!]\sigma = n \quad \mathcal{A}[\![y]\!]\sigma[k/x] = m \stackrel{?}{\implies} n = m$

$x \neq y \quad \sigma(y) = n \quad \sigma[k/x](y) = m \stackrel{?}{\implies} n = m$

Ovvio, essendo $\sigma[k/x](y) = \sigma(y)$ se $x \neq y$. CVD.

(1b) Assumiamo le premesse induttive:

$x \notin VAR(a_1) \quad \mathcal{A}[\![a_1]\!]\sigma = n_1 \quad \mathcal{A}[\![a_1]\!]\sigma[k/x] = m_1 \implies n_1 = m_1$

$x \notin VAR(a_2) \quad \mathcal{A}[\![a_2]\!]\sigma = n_2 \quad \mathcal{A}[\![a_2]\!]\sigma[k/x] = m_2 \implies n_2 = m_2$

Dobbiamo dimostrare:

$x \notin VAR(a_1 \; op \; a_2) \quad \mathcal{A}[\![a_1 \; op \; a_2]\!]\sigma = n \quad \mathcal{A}[\![a_1 \; op \; a_2]\!]\sigma[k/x] = m \stackrel{?}{\implies} n = m$

Assumiamo le premesse della tesi:

$x \notin VAR(a_1 \; op \; a_2)$ e quindi $x \notin VAR(a_1) \quad x \notin VAR(a_2)$

$\underbrace{\mathcal{A}[\![a_1]\!]\sigma}_{n_1} \; op \; \underbrace{\mathcal{A}[\![a_2]\!]\sigma}_{n_2} = n \quad \underbrace{\mathcal{A}[\![a_1]\!]\sigma[k/x]}_{m_1} \; op \; \underbrace{\mathcal{A}[\![a_2]\!]\sigma[k/x]}_{m_2} = m$

Dalle premesse si ricava $n_1 = m_1$ e $n_2 = m_2$, quindi $n = m$

CVD.

(2) $c_1 \stackrel{def}{=} x_1 := a_1 \qquad c_2 \stackrel{def}{=} x_2 := a_2$

Le condizioni di non-interferenza sono:

$x_1 \neq x_2 \quad x_1 \notin VAR(a_2) \quad x_2 \notin VAR(a_1)$

Dobbiamo dimostrare:

$c_1 ; c_2 \stackrel{?}{\equiv}_{den} c_2 ; c_1$

cioè

$$\mathcal{C}[\![x_2 := a_2]\!]\sigma[\mathcal{A}[\![a_1]\!]\sigma/x_1] \stackrel{?}{=} \mathcal{C}[\![x_1 := a_1]\!]\sigma[\mathcal{A}[\![a_2]\!]\sigma/x_2]$$

$$\sigma[\mathcal{A}[\![a_1]\!]\sigma/x_1][\mathcal{A}[\![a_2]\!]\sigma[\mathcal{A}[\![a_1]\!]\sigma/x_1]/x_2]$$

$$\stackrel{?}{=} \sigma[\mathcal{A}[\![a_2]\!]\sigma/x_2][\mathcal{A}[\![a_1]\!]\sigma[\mathcal{A}[\![a_2]\!]\sigma/x_2]/x_1]$$

Dato che $x_1 \notin VAR(a_2)$ abbiamo $\mathcal{A}[\![a_2]\!]\sigma[\cdots/x_1] = \mathcal{A}[\![a_2]\!]\sigma$
e similmente $\mathcal{A}[\![a_1]\!]\sigma[\cdots/x_2] = \mathcal{A}[\![a_1]\!]\sigma$ quindi

$$\sigma[\mathcal{A}[\![a_1]\!]\sigma/x_1][\mathcal{A}[\![a_2]\!]\sigma/x_2] \stackrel{?}{=} \sigma[\mathcal{A}[\![a_2]\!]\sigma/x_2][\mathcal{A}[\![a_1]\!]\sigma/x_1]$$

Essendo però $x_1 \neq x_2$ entrambi i membri valgono σ':

$$\sigma'(x) = \begin{cases} \mathcal{A}[\![a_1]\!]\sigma & \text{se } x = x_1 \\ \mathcal{A}[\![a_2]\!]\sigma & \text{se } x = x_2 \\ \sigma(x) & \text{se } x \neq x_1, x_2 \end{cases}$$

CVD

Una condizione leggermente più generale è ammettere $x_1 = x_2$ sotto la condizione $\mathcal{A}[\![a_1]\!]\sigma = \mathcal{A}[\![a_2]\!]\sigma$ (ovvio).

Esercizio 2

L'insieme $2^{A\times A}$ ordinato per inclusione (di coppie) $\sqsubseteq$ è ovviamente un ordinamento parziale.
Essendo la def di ordinamento parziale data con proprietà definite universalmente, ogni sottoinsieme di $2^{A\times A}$, con lo stesso ordinamento è ancora un ordinamento parziale. Questo vale per OP (le relazioni che sono $OP_\perp$) e CPO (solo $CPO_\perp$).

La relazione $\{(a,a) \mid a \in A\}$ è il bottom sia di PO sia di CPO.

Sia $\sqsubseteq_0 \subseteq \sqsubseteq_1 \subseteq \dots$ una catena in PO
Sia $\bigcup_{i\in\omega} \sqsubseteq_i$ il lub in $2^{A\times A}$. È anche in PO:

infatti (i) $(a,a) \in \sqsubseteq_i \ \forall i$, quindi anche a $\bigcup_{i\in\omega} \sqsubseteq_i$

(ii) se $(x,y), (y,x) \in \bigcup_{i\in\omega} \sqsubseteq_i$ allora $\exists_k (x,y),(y,x) \in \sqsubseteq_k$, assurdo.

(iii) se $(x,y), (y,z) \in \bigcup_{i\in\omega} \sqsubseteq_i$ allora $\exists_k (x,y),(y,z) \in \sqsubseteq_k$
quindi $(x,z) \in \sqsubseteq_k$ quindi $(x,z) \in \bigcup_{i\in\omega} \sqsubseteq_i$ CVD

Sia $\sqsubseteq_0 \subseteq \sqsubseteq_1 \subseteq \dots$ una catena in CPO se A è finito anche $2^{A\times A}$ è finito, quindi la catena è finita, quindi il suo lub è il suo massimo elemento.
Altrimenti siano $a_0, a_1, \dots$ infiniti elementi di A
Sia $a_i \sqsubseteq_k a_j$ $0 \le i \le j \le k$ con $\sqsubseteq_k$ in CPO perché finita
Ma $\bigcup_{k\in\omega} \sqsubseteq_k$ è $a_i \sqsubseteq_k a_j$ $0 \le i \le j$ non è in CPO
dato che $a_0 \sqsubseteq_{\cup\sqsubseteq_k} a_1 \sqsubseteq_{\cup\sqsubseteq_k} \dots$ non ha lub.
L'insieme $\{\sqsubseteq \mid \sqsubseteq \in CPO, \bigcup_{k\in\omega} \sqsubseteq_k \subseteq \sqsubseteq\}$ non ha minimo.

Esercizio 3

Se t è chiuso, $x \notin FV(t)$. Quindi $t[t'/x] = t$.

Operazionale lazy

$$\mathrm{rec}\ x.t \to c \Leftarrow t[\mathrm{rec}\ x.t/x] \to c \Leftarrow t \to c \qquad \text{CVD.}$$

$$((\lambda x.t)\ \mathrm{rec}\ x.x) \to c \Leftarrow t[\mathrm{rec}\ x.x/x] \to c \Leftarrow t \to c \qquad \text{CVD.}$$

Operazionale eager

$$((\lambda x.t)\ \mathrm{rec}\ x.x) \to c \Leftarrow \underbrace{\mathrm{rec}\ x.x \to c'}_{\text{non risolubile}} \quad t[c'/x] \to c \qquad \text{CVD.}$$

Denotazionale

$$[\![\mathrm{rec}\ x.t]\!]\rho = fix\ \lambda d.[\![t]\!]\rho[d/x] = fix\ \lambda d.[\![t]\!]\rho$$

$$d_0 = \bot \quad d_1 = (\lambda d.[\![t]\!]\rho)\bot = [\![t]\!]\rho \quad d_2 = (\lambda d.[\![t]\!]\rho)[\![t]\!]\rho =$$
$$[\![t]\!]\rho = d_1 = fix \quad \text{CVD}$$

$$[\![(\lambda x.t)\ \mathrm{rec}\ x.x]\!]\rho = \mathrm{let}\ \varphi \Leftarrow \lfloor \lambda d.[\![t]\!]\rho[d/x] \rfloor .\ \varphi\ \bot$$
$$= (\lambda d.[\![t]\!]\rho)\bot = [\![t]\!]\rho \qquad \text{CVD.}$$

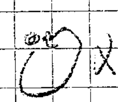
Exercise 4

$$\text{rec } x.\ \alpha.\ (x|x) \xrightarrow{\alpha} (\text{rec } x.\ \alpha.\ (x|x)) \mid \text{rec } x.\ \alpha.\ (x|x)$$

$$p \xrightarrow{\alpha} p|p \begin{array}{l} \xrightarrow{\alpha} (p|p)|p \\ \xrightarrow{\alpha} p|(p|p) \end{array}$$ and so on

In general $p \xrightarrow{\alpha^n} q$
where q is any term obtained composing n processes p with operation $|$ non associative.
Also if q contains n processes p, then there are exactly n q' with $q \xrightarrow{\alpha} q'$, since any of the p's can progress into $p|p$ and the resulting processes q' are all different.

Let us call $\mathbb{P}$ the set of all reachable processes; then $R_0 = \mathbb{P} \times \mathbb{P}$ and also $R_1 = \mathbb{P} \times \mathbb{P} = fix$, since all process can only execute α.

Thus the minimal LTS is 
which corresponds
to $\text{rec } x.\ \alpha.\ x$.

Also PEPA process A, with $A = (\alpha,\lambda).(A|_\phi A)$

can progress similarly:

$$A \xrightarrow{(\alpha,\lambda)} A|_\phi A \quad \begin{array}{l} \nearrow (A|_\phi A)|_\phi A \\ \searrow A|(A|_\phi A) \end{array} \quad \text{and so on}$$

Again $A \xrightarrow{(\alpha,\lambda)^n} q$ with q any $|$-term with n A's.

Let us call $\mathcal{Q}_n$ the set of processes with n A's. If $q \in \mathcal{Q}_n$ then there are n q' in $\mathcal{Q}_{n+1}$ such that $q \xrightarrow{\alpha} q'$.

Given a labeled CTMC $\alpha: S \times L \times S \to \mathbb{R}$ an equivalence relation/partition R is a bisimulation if

$$p R q \Rightarrow \forall l \in L.\ \forall I \in R.\ \sum_r \alpha(p,l,r) = \sum_r \alpha(q,l,r)$$

and bisimilarity is the maximal bisimulation

In our case $R_0 = \underline{P} \times \underline{P}$ but $R_1 = \{\mathcal{Q}_i\}_{i \in \omega}$ since $P_n \in \mathcal{Q}_n$ implies $\alpha(P_n, \alpha, \underline{P}) = n\lambda$.

Also $R_2 = R_1 = fix$.

The minimal process B is just A, where the composition operator $|_\phi$ is considered associative.