Models of Computation

Written Exam on June 6, 2012

(First part: Exercises 1 and 2 Second part: Exercises 3 and 4)

Exercise 1 (8)

We modify the definition of *memory* σ in IMP as a *partial* function from locations to integers. The update function $\_[\_] : \Sigma \times \Sigma \to \Sigma$ is computed as follows: $\sigma[\sigma'](x) =$ if $\sigma'(x)$ is defined, then $\sigma'(x)$, else if $\sigma(x)$ is defined then $\sigma(x)$, else undefined.

The syntax of IMP is modified as follows. A new syntactic category Mass of *multiple assigments*, with notation m, is introduced, with two constructs, ordinary (single) assignment $x := a$ and multiple assignment composition $m|m$. Then multiple assigments are commands: $c ::= \{m\} \mid \ldots$, while single assignments are not directly commands anymore. The denotational semantics is as follows:

$$\mathcal{M}[\![x := a]\!]\sigma = (\mathcal{A}[\![a]\!]\sigma/x) \quad \mathcal{M}[\![m_1|m_2]\!]\sigma = (\mathcal{M}[\![m_1]\!]\sigma)[\mathcal{M}[\![m_2]\!]\sigma] \quad \mathcal{C}[\![\{m\}]\!]\sigma = \sigma[\mathcal{M}[\![m]\!]\sigma]$$

where (n/x) is a memory with $(n/x)(x) = n$ and undefined elsewhere.

Prove that: (i) multiple assignment composition is associative, namely

$$\mathcal{M}[\![(m_1|m_2)|m_3]\!]\sigma = \mathcal{M}[\![m_1|(m_2|m_3)]\!]\sigma;$$

(ii) the denotational semantics of a single assignment $x := a$ in the ordinary version of IMP coincides with $\mathcal{C}[\![\{x := a\}]\!]$; (iii) $\mathcal{C}[\![\{x := 1\}; \{x := 2|y := x\}]\!]\sigma \neq \mathcal{C}[\![\{x := 1\}; \{x := 2\}; \{y := x\}]\!]\sigma$. Finally, (iv) give the operational semantics of the new constructs and (v) prove their equivalence with the denotational semantics.

Exercise 2 (8)

We consider strings $\alpha, \beta \in V^*$ with $V = \{1, 2, \ldots, b\}$, which are intended to represent natural numbers in base b with digits without zeroes. A relation $<$ is defined on them:

$$\frac{|\alpha| < |\beta|}{\alpha < \beta} \qquad \frac{|\alpha| = |\beta| \wedge n < m}{n\alpha < m\beta} \qquad \frac{|\alpha| = |\beta| \wedge \alpha < \beta}{n\alpha < n\beta}.$$

Prove that: (i) $<$ is a transitive, well founded relation on V^*; (ii) $\leq\, = \,<\, \cup \{(n, n)\}$ is a non complete partial ordering with bottom; and (iii) it is a *total* ordering. Then, (iv) show that total ordering $(V^*, <)$ is isomorphic to $(\omega, <)$ via monotone function $[\![\_]\!]$ defined as $[\![\lambda]\!] = 0$ and $[\![n\alpha]\!] = n + b[\![\alpha]\!]$ (where λ is the string of lenght zero), namely $[\![\_]\!]$ is bijective and $\alpha < \beta \to [\![\alpha]\!] < [\![\beta]\!]$.

Exercise 3 (8)

Consider the HOFL program:

$$t = rec\ f.\lambda n.\textbf{if}\ n\ \textbf{then}\ 0\ \textbf{else if}\ n-1\ \textbf{then}\ 1\ \textbf{else}\ 2 \times (f\ n-1) - (f\ n-2),$$

Compute the corresponding $\Gamma : [N_\perp \to N_\perp]_\perp \to [N_\perp \to N_\perp]_\perp$ and prove that $\lfloor f^* \rfloor : [N_\perp \to N_\perp]_\perp$ - where $f(n) = \lfloor n \rfloor$ - is a fixpoint of Γ. Observing that t is in fact a definition by well-founded recursion, conclude that $\lfloor f^* \rfloor$ is the unique fixpoint.

Exercise 4 (6)

Determine the processes reachable from $p = ((rec\ x.\alpha x)|(rec\ x.(\overline{\alpha}.x|\beta.nil)))\backslash\alpha$ and from $q = rec\ x.\beta x$ and prove that p e q are not strong bisimilar, but that they are weak bisimilar. For simplicity, assume the structural axioms $p|q = q|p$, $(p|q)|r = p|(q|r)$ e $p|nil = p$.

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Exercise 1

(i) $\mathcal{M}\mathcal{C}[\![(m_1|m_2)|m_3]\!]\sigma = (\sigma_1[\sigma_2])[\sigma_3] = \sigma$

$\mathcal{M}\mathcal{C}[\![m_1|(m_2|m_3)]\!]\sigma = \sigma_1[\sigma_2[\sigma_3]] = \sigma'$

where $\sigma_1 = \mathcal{M}\mathcal{C}[\![m_1]\!]\sigma \quad \sigma_2 = \mathcal{M}\mathcal{C}[\![m_2]\!]\sigma \quad \sigma_3 = \mathcal{M}\mathcal{C}[\![m_3]\!]\sigma$

We have $\sigma=\sigma'$, since in both cases they contain assignments occurring once, while for assignments occurring more than once the resulting assignment is the one in the memory with highest index.

(ii) $\mathcal{C}_{old}[\![x:=a]\!]\sigma = \sigma\left[{}^{\mathcal{A}[\![a]\!]\sigma}/_{x}\right]$

$\mathcal{C}[\![\{x:=a\}]\!]\sigma = \sigma[\mathcal{M}\mathcal{C}[\![x:=a]\!]\sigma] = \sigma\left[\left({}^{\mathcal{A}[\![a]\!]\sigma}/_{x}\right)\right]$

(iii) $\mathcal{C}[\![\{x:=1\};\{x:=2|y:=x\}]\!]\sigma = \sigma'\left[{}^{2}/_{x}, {}^{\sigma'(x)}/_{y}\right]$ with $\sigma' = \sigma\left[{}^{1}/_{x}\right]$

$= \sigma\left[{}^{1}/_{x}\right]\left[{}^{2}/_{x}, {}^{1}/_{y}\right] = \sigma\left[{}^{2}/_{x}, {}^{1}/_{y}\right]$

$\mathcal{C}[\![\{x:=1\};\{x:=2\};\{y:=x\}]\!]\sigma = \sigma''\left[{}^{\sigma''(x)}/_{y}\right]$ with

$\sigma'' = \sigma'\left[{}^{2}/_{x}\right] \quad \sigma' = \sigma\left[{}^{1}/_{x}\right]$

$= \sigma\left[{}^{2}/_{x}\right]\left[{}^{2}/_{y}\right]$

ok

(IV) $$\frac{\langle a,\sigma\rangle \to n}{\langle x:=a,\sigma\rangle \to (n/x)}$$

$$\frac{\langle m_1,\sigma\rangle \to \sigma_1 \quad \langle m_2,\sigma\rangle \to \sigma_2}{\langle m_1 | m_2,\sigma\rangle \to \sigma_1[\sigma_2]}$$

$$\frac{\langle m,\sigma\rangle \to \sigma'}{\langle \{m\},\sigma\rangle \to \sigma[\sigma']}$$

(V) <u>Operational $\to$ Denotational</u>

- $$\frac{\langle a,\sigma\rangle \to n}{\langle x:=a,\sigma\rangle \to n/x} \qquad \mathcal{A}[\![a]\!]\sigma = n$$

$$\mathcal{M}[\![x:=a]\!]\sigma = (n/x)$$
$$= (\mathcal{A}[\![a]\!]\sigma/x) \quad \text{QED.}$$

- $$\frac{\langle m_1,\sigma\rangle \to \sigma_1 \quad \langle m_2,\sigma\rangle \to \sigma_2}{\langle m_1|m_2,\sigma\rangle \to \sigma_1[\sigma_2]} \qquad \mathcal{M}[\![m_1]\!]\sigma = \sigma_1 \quad \mathcal{M}[\![m_2]\!]\sigma = \sigma_2$$

$$\mathcal{M}[\![m_1 | m_2]\!]\sigma = \mathcal{M}[\![m_1]\!]\sigma\,[\mathcal{M}[\![m_2]\!]\sigma] = \sigma_1[\sigma_2] \quad \text{QED.}$$

- $$\frac{\langle m,\sigma\rangle \to \sigma'}{\langle \{m\},\sigma\rangle \to \sigma[\sigma']} \qquad \mathcal{M}[\![m]\!]\sigma = \sigma'$$

$$\mathcal{B}[\![\{m\}]\!]\sigma = \sigma[\mathcal{M}[\![m]\!]\sigma] = \sigma[\sigma'] \quad \text{QED.}$$

Denotational → Operational

- $P(x := a) \overset{def}{=} \mathcal{MC}[\![x := a]\!]\sigma = \sigma' \overset{?}{\Longrightarrow} \langle x := a, \sigma \rangle \to \sigma'$

$(\mathcal{A}[\![a]\!]\sigma/x) = \sigma' \quad \langle a, \sigma \rangle \to n \quad (n/x) = \sigma'$ QED

- $P(m_1 | m_2) \overset{def}{=} \mathcal{MC}[\![m_1 | m_2]\!]\sigma = \sigma' \overset{?}{\Longrightarrow} \langle m_1 | m_2, \sigma \rangle \to \sigma'$

$\underbrace{\mathcal{MC}[\![m_1]\!]\sigma}_{\sigma_1}[\underbrace{\mathcal{MC}[\![m_2]\!]\sigma}_{\sigma_2}] = \sigma' \quad \langle m_1, \sigma \rangle \to \sigma_1 \quad \langle m_2, \sigma \rangle \to \sigma''$

$\langle m_1 | m_2, \sigma \rangle \; \sigma_1[\sigma_2]$ QED

- $P(\{m\}) = \mathcal{C}[\![\{m\}]\!]\sigma = \sigma' \overset{?}{\longrightarrow} \langle \{m\}, \sigma \rangle \to \sigma'$

$\sigma[\underbrace{\mathcal{MC}[\![m]\!]\sigma}_{\sigma''}] = \sigma' \quad \langle m, \sigma \rangle \to \sigma'' \quad \langle \{m\}, \sigma \rangle \to \sigma[\sigma'']$ QED

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Exercise 2

(i) $<$ is transitive: $\alpha < \beta < \gamma \Rightarrow \alpha < \gamma$

If either $|\alpha| < |\beta|$ or $|\beta| < |\gamma|$ then also $|\alpha| < |\gamma|$.

If $|\alpha| = |\beta| = |\gamma| = 0$ then the hypothesis is impossible

If $|\alpha| = |\beta| = |\gamma| = k$ we assume transitivity holds

If $|a_1\alpha| = |a_2\beta| = |a_3\gamma| = k+1$ and $a_1 < a_2$ or $a_2 < a_3$

then transitivity holds

If $|a\alpha| = |a\beta| = |a\gamma| = k+1$ then transitivity hold

for the math induction hypothesis

$<$ is well founded: it is acyclic. Since it is transitive it is enough to notice $\alpha < \alpha$ cannot be proved, no rule applies

If $|\alpha| = k$ the strings β with $\beta < \alpha$ can be at most $\sum_{n=0}^{k} b^n$.

(ii) we already proved transitivity and antisymmetry (acyclicity). Reflexivity is added by $\{(n,n)\}$.

Minimal element is the empty string.

$1 < 11 < 111 \ldots$ is a chain with no lub.

(iii) $\leq$ is a total ordering. For α, β only three possibility are possible: either $\alpha = \beta$ or $\alpha < \beta$ or $\beta < \alpha$. It is possible to see it by applying the inference rules symbolically in a goal oriented way.

Only one of them applies

(iv) $[\![\lambda]\!] = 0 \quad [\![n\alpha]\!] = n + b[\![\alpha]\!]$

This function can be inverted. The second equation correspond to take the quotient $[\![\alpha]\!]$ of $[\![n\alpha]\!]$ and b, with a remainder $1 \le r \le b$ instead of $0 \le r < b$ as usual. $[\![\delta[\![n]\!]]\!] = n$

Examining any of the above two functions, one observes monotonicity, e.g. quotient & remainder are monotone functions

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Exercise 3

$[\![f]\!]\rho = \mathit{fix}\ \Gamma$ $\qquad f : int \to int$

$\Gamma = \lambda f. \lfloor \lambda d.\ Cond(d, \lfloor 0 \rfloor, Cond(d =_\perp \lfloor 1 \rfloor, \lfloor 1 \rfloor,$

$\lfloor 2 \rfloor \times (let\ \varphi \Leftarrow f. \varphi(d =_\perp \lfloor 1 \rfloor) - \quad \lfloor f^* \rfloor$

$let\ \varphi \Leftarrow f. \varphi(d =_\perp \lfloor 2 \rfloor)))\rfloor \qquad f(n) = \lfloor n \rfloor$

$\Gamma \lfloor \lambda n. \lfloor n \rfloor \rfloor =$

$\lfloor \lambda d.\ Cond(d, \lfloor 0 \rfloor, Cond(d =_\perp \lfloor 1 \rfloor, \lfloor 1 \rfloor),$

$\lfloor 2 \rfloor \times_\perp (d =_\perp \lfloor 1 \rfloor) =_\perp (d =_\perp \lfloor 2 \rfloor))\rfloor$

But this is just $\lfloor f^* \rfloor$:

$let\ \varphi \Leftarrow \lfloor f^* \rfloor. \varphi \perp_{N_\perp} = \perp_{N_\perp}$ $\qquad$ ok on $\perp$

$let\ \varphi \Leftarrow \lfloor f^* \rfloor. \varphi \lfloor 0 \rfloor = \lfloor 0 \rfloor$ $\qquad$ ok on 0

$let\ \varphi \Leftarrow \lfloor f^* \rfloor. \varphi \lfloor n \rfloor = \lfloor 2 \rfloor \times_\perp (\lfloor n \rfloor =_\perp \lfloor 1 \rfloor) - (\lfloor n \rfloor - \lfloor 2 \rfloor)$

$= \lfloor 2(n-1) - (n-2) \rfloor = \lfloor n \rfloor$

ok car $\lfloor n \rfloor$

Definition by mathematical recursion:

$f(0) = 0 \quad f(1) = 1 \quad f(n+2) = 2 f(n+1) - f(n)$

It is defined for all n.

In the domain $\lfloor N_\perp \to N_\perp \rfloor_\perp$ it is a maximal element

minimal fixpoint & maximal element ⇒ unique fixpoint.

Exercise 4

$\mathrm{rec}\, x.\, \alpha x = p'$ $\qquad \mathrm{rec}(x.\, \bar{\alpha} x) \mid \beta\, nil = p''$

$$\underbrace{\beta\, nil \mid \beta\, nil \mid \dots \mid \beta\, nil}_{n \text{ volte}} = r_n$$

$$\begin{tikzcd}
P = (p' \mid p'')\backslash\alpha \arrow[r, shift left, "\beta"] \arrow[d, shift right, "\tau"'] & (p' \mid \bar{\alpha}.p'')\backslash\alpha \arrow[l, shift left, "\tau"] \\
(p' \mid p'' \mid r_1)\backslash\alpha \arrow[r, shift left, "\beta"] \arrow[u, shift right, "\beta"'] \arrow[d, dotted] & (p' \mid \bar{\alpha} p'' \mid r_1)\backslash\alpha \arrow[l, shift left, "\tau"] \arrow[u, "\beta"] \\
(p' \mid p'' \mid r_n)\backslash\alpha \arrow[r, shift left, "\beta"] \arrow[u, "\beta"] & (p' \mid \bar{\alpha} p'' \mid r_2)\backslash\alpha \arrow[l, shift left, "\tau"] \arrow[u, "\beta"]
\end{tikzcd}$$

$P_n = (p' \mid p'' \mid r_n)\backslash\alpha \qquad \overline{P}_n = (p' \mid \bar{\alpha} p'' \mid r_n) \qquad n = 0, 1, \dots$

$P_n \xrightarrow{\beta} \overline{P}_n \qquad \overline{P}_n \xrightarrow{\tau} P_n \qquad\qquad p = P_0$

$P_n \xrightarrow{\tau} P_{n+1}$

$P_{n+1} \xrightarrow{\beta} P_n \qquad \overline{P}_{n+1} \xrightarrow{\beta} \overline{P}_n$

$q \xrightarrow{\beta} q$

$p \not\sim q$ in fact $p \xrightarrow{\tau} (p' \mid p'' \mid r_1)$ and $q \not\xrightarrow{\tau}$

$p \approx q$ in fact from every state we have $\overset{\varepsilon}{\Longrightarrow}$ e $\overset{\beta}{\Longrightarrow}$