Models of Computation

Written Exam on January 10, 2012

(MOD students: Exercises 1-5, 180 minutes   Previous TSD students: Exercises 1-3, 120 minutes)

Exercise 1 (7)

Consider the command IMP **while** b **do** c **for** n **times**, where n is a natural number (not a location), with the following operational semantics:

$$\frac{< b, \sigma > \rightarrow F}{< \textbf{while } b \textbf{ do } c \textbf{ for } n+1 \textbf{ times}, \sigma > \rightarrow \sigma}$$

$$\frac{< b, \sigma > \rightarrow T \quad < c, \sigma > \rightarrow \sigma'' \quad < \textbf{while } b \textbf{ do } c \textbf{ for } n \textbf{ times}, \sigma'' > \rightarrow \sigma'}{< \textbf{while } b \textbf{ do } c \textbf{ for } n+1 \textbf{ times}, \sigma > \rightarrow \sigma'.}$$

Notice that there are no inference rules for **while** b **do** c **for** 0 **times**. Prove the equivalence between operational and denotational semantics also for the new construct, with
$\mathcal{C}[\![\textbf{while } b \textbf{ do } c \textbf{ for } n \textbf{ times}]\!] = \Gamma^n \bot$, where as usual $\Gamma = \lambda\varphi.\lambda\sigma.\mathcal{B}[\![b]\!]\sigma \rightarrow \varphi^*(\mathcal{C}[\![c]\!]\sigma), \sigma$.

Exercise 2 (6)

Given the complete partial ordering with bottom $(D, \sqsubseteq)$, consider the structure $< \mathcal{S}, \subseteq >$, where $\mathcal{S}$ is the class of *convex* subsets S of D, namely such that $d_1 \sqsubseteq d_2 \sqsubseteq d_3$ and d_1, $d_3 \in S$ implies $d_2 \in S$, and where $\subseteq$ is set inclusion.
Prove that $< \mathcal{S}, \subseteq >$ is *(i)* a partial ordering, *(ii)* complete, and *(iii)* with bottom.

Exercise 3 (7)

Consider the HOFL term $t = rec\ x.((\lambda y.\textbf{if}\ y\ \textbf{then}\ 0\ \textbf{else}\ 0)\ x)$ and determine its type. Then compute its operational and denotational semantics, checking their equivalence. Finally, find a term $t' : int$, with x free in t', such that $[\![rec\ x.t']\!]\rho$ is different than $\bot_{N_\bot}$.

Exercise 4 (4)

Consider the π-calculus, and prove that: (i) $x \notin fn(P) \Rightarrow (x)P \equiv P$, where $\equiv$ is structural equivalence (defined by the axioms in the lecture notes); and (ii) $P \sim Q \not\Rightarrow fn(P) = fn(Q)$, where $\sim$ is any of the bisimilarities defined for π-calculus processes. (Hint: for the second property, consider a process where an additional free variable occurs only in a deadlocked subprocess.)

Exercise 5 (6)

Let *non-stopping*, reactive, probabilistic labelled transition systems (PLTS) be the reactive PLTS with $\alpha : S \rightarrow L \rightarrow D(S)$ (rather than $\alpha : S \rightarrow L \rightarrow (D(S) + 1)$). Prove that all the states of a non-stopping, reactive PLTS are bisimilar. Then give the definition of bisimilarity also for *generative* PLTS. Furthermore, consider the non-stopping subclass of generative PLTS and show an example where some states are not bisimilar. Moreover, give the definition of bisimilarity also for Segala PLTS, and show that Segala bisimilarity reduces to generative PLTS bisimilarity in the deterministic case (namely when, for every state s, $\alpha(s)$ is a singleton).

Correzione Prova Scritta del 10/1/2012

Esercizio 1

$$P(\langle c,\sigma\rangle \to \sigma') = \mathcal{C}[\![c]\!]\sigma = \sigma'$$

$$\frac{\langle b,\sigma\rangle \to F}{\langle \text{while } b \text{ do } c \text{ for } n+1 \text{ times}, \sigma\rangle \to \sigma}$$

$$P(\langle \text{while } b \text{ do } c \text{ for } n+1 \text{ times}, \sigma\rangle \to \sigma) \stackrel{def}{=} \Gamma^{n+1}\bot\,\sigma \stackrel{?}{=} \sigma$$

$$\Gamma(\Gamma^n\bot)\,\sigma \stackrel{?}{=} \sigma$$

$$\mathcal{B}[\![b]\!]\sigma \to (\Gamma^n\bot)^*(\mathcal{C}[\![c]\!]\sigma), \sigma \stackrel{?}{=} \sigma$$ ovvio essendo $\mathcal{B}[\![b]\!]\sigma = F$ c.v.d.

$$\frac{\langle b,\sigma\rangle \to T \quad \langle c,\sigma\rangle \to \sigma'' \quad \langle \text{while } b \text{ do } c \text{ for } n \text{ times}, \sigma''\rangle \to \sigma'}{\langle \text{while } b \text{ do } c \text{ for } n+1 \text{ times}, \sigma\rangle \to \sigma'}$$

$$P(\langle \text{while } b \text{ do } c \text{ for } n+1 \text{ times}, \sigma\rangle \to \sigma') \stackrel{def}{=} \Gamma^{n+1}\bot\,\sigma \stackrel{?}{=} \sigma'$$

$$P(\langle \text{while } b \text{ do } c \text{ for } n+1 \text{ times}, \sigma''\rangle \to \sigma') \stackrel{def}{=} \Gamma^{n}\bot\,\sigma'' = \sigma'$$

$$P(\langle c,\sigma\rangle \to \sigma'') \stackrel{def}{=} \mathcal{C}[\![c]\!]\sigma = \sigma''$$

$$\underbrace{\mathcal{B}[\![b]\!]\sigma}_{T} \to \underbrace{(\Gamma^n\bot)^*(\underbrace{\mathcal{C}[\![c]\!]\sigma}_{\sigma''})}_{\sigma'}, \sigma \stackrel{?}{=} \sigma'$$

c.v.d.

Non serve l'induzione matematica.

$P(\text{while } b \text{ do } c \text{ for } n \text{ times}) \overset{def}{=} \mathcal{C}[\![\text{while } b \text{ do } c \text{ for } n \text{ times}]\!]\sigma = \sigma' \overset{?}{\Rightarrow} \langle \text{while } b \text{ do } c \text{ for } n \text{ times}, \sigma\rangle \to \sigma'$

Per induzione matematica. Base $n=0$

$\mathcal{C}[\![\text{while } b \text{ do } c \text{ for } n \text{ times}]\!]\sigma = \Gamma^0 \bot \sigma = \bot\sigma = \bot$

la premessa è falsa quindi l'implicazione è vera

Passo induttivo per $n>0$. Assumiamo la premessa

$\mathcal{B}[\![b]\!]\sigma \to (\Gamma^{n-1}\bot)^*(\mathcal{C}[\![c]\!]\sigma), \sigma = \sigma'$

Caso $\mathcal{B}[\![b]\!]\sigma = F$, cioè $\langle b,\sigma\rangle \to F$, cioè $\sigma = \sigma'$

$\langle \text{while } b \text{ do } c \text{ for } n \text{ times}, \sigma\rangle \to \sigma'$ per la prima regola.

Caso $\mathcal{B}[\![b]\!]\sigma = T$ cioè $\langle b,\sigma\rangle \to T$

$(\Gamma^{n-1}\bot)\underbrace{(\mathcal{C}[\![c]\!]\sigma)}_{\sigma''} = \sigma'$ quindi $\langle c,\sigma\rangle \to \sigma''$ per ip. indutt. strutturale

L'ipotesi induttiva è

$\mathcal{C}[\![\text{while } b \text{ do } c \text{ for } n-1 \text{ times}]\!]\sigma'' = \sigma' \to \langle \text{while } b \text{ do } c \text{ for } n-1 \text{ times}, \sigma''\rangle \to \sigma'$

cioè

$(\Gamma^{n-1}\bot)\sigma'' = \sigma' \to \langle \text{while } b \text{ do } c \text{ for } n-1 \text{ times}, \sigma''\rangle \to \sigma'$

Essendo vera la premessa vale anche $\langle \text{while } b \text{ do } c \text{ for } n-1 \text{ times}, \sigma''\rangle \to \sigma'$

Quindi si può applicare la 2ª regola di inferenza alle formule sottoli

CVD.

Esercizio 2

La relazione $\langle \mathcal{S}, \subseteq \rangle$ è un ordinamento parziale.

Infatti le proprietà:

$x \subseteq x$ riflessiva

$x \subseteq y,\ y \subseteq x \Rightarrow x = y$ antisimmetrica

$x \subseteq y,\ y \subseteq z \Rightarrow x \subseteq z$ transitiva

valgono tra gli elementi di $\mathcal{S}$, che sono insiemi.

L'ordinamento parziale $\langle \mathcal{S}, \subseteq \rangle$ è dotato di bottom:

infatti l'insieme vuoto è convesso, essendo la premessa falsa.

Per la completezza, va dimostrato che, data una catena di insiemi convessi

$S_0 \subseteq S_1 \subseteq \ldots$

anche il lub $\bigcup_{i \in \omega} S_i$ è convesso.

Cioè $d_1, d_3 \in \bigcup_{i \in \omega} S_i$ e $d_1 \sqsubseteq d_2 \sqsubseteq d_3 \Rightarrow d_2 \in \bigcup_{i \in \omega} S_i$

Infatti se $d_1, d_3 \in \bigcup_{i \in \omega} S_i$ allora esiste K con $d_1, d_3 \in S_K$

Ma $d_2 \in S_K$, essendo S_K convesso. Quindi $d_2 \in \bigcup_{i \in \omega} S_i$

CVD.

Esercizio 3

$$t = \underbrace{rec\,x^{int}.\,\underbrace{(\underbrace{(\lambda y.\ \underbrace{if\ \underbrace{y}_{int}\ then\ \underbrace{0}_{int}\ else\ \underbrace{0}_{int}}_{int})}_{int \to int}\ \underbrace{x}_{int})}_{int}}_{int}$$

$t \to c \Leftarrow ((\lambda y.\ if\ y\ then\ 0\ else\ 0)\ t) \to c$

$\Leftarrow if\ t\ then\ 0\ else\ 0 \to c$ $[\Leftarrow t \to 0 \quad 0 \to c$

$\overset{c=0}{\Leftarrow} t \to 0 \quad \ldots \Leftarrow if\ t\ then\ 0\ else\ 0 \to 0 \Leftarrow$

$\Leftarrow t \to 0$ e così indefinitamente$]$ oppure

$[\Leftarrow t \to n,\ n \neq 0 \quad 0 \to c$

$\overset{c=0}{\Leftarrow} t \to n,\ n \neq 0 \quad \ldots \Leftarrow if\ t\ then\ 0\ else\ 0 \to n \quad n \neq 0$

impossibile$]$

Non esiste forma canonica

(5)

$[\![rec\, x. ((\lambda y.\, if\ y\ then\ 0\ else\ 0)\, x)]\!] \rho =$

$fix\ \lambda d.\ [\![((\lambda y.\, if\ y\ then\ 0\ else\ 0)\, x)]\!] \rho[^d/_x] =$

$fix\ \lambda d.\ let\ \varphi \Leftarrow \lfloor \lambda d'.\, Cond(d', \lfloor 0 \rfloor, \lfloor 0 \rfloor) \rfloor .\ \varphi\, d =$

$fix\ \lambda d.\ Cond(d, \lfloor 0 \rfloor, \lfloor 0 \rfloor)$

$d_0 = \perp_{N_\perp}$

$d_1 = Cond(\perp_{N_\perp}, \lfloor 0 \rfloor, \lfloor 0 \rfloor) = \perp_{N_\perp} = fix$

La sem. denotazionale è $\perp$, quindi equivalente

$[\![rec\, x.\ if\ 0\ then\ 0\ else\ x]\!] \rho =$

$= fix\ \lambda d.\ Cond(\lfloor 0 \rfloor, \lfloor 0 \rfloor, d) =$

$= fix\ \lambda d.\ \lfloor 0 \rfloor$

$d_0 = \perp_{N_\perp}$

$d_1 = \lfloor 0 \rfloor = fix$

Exercise 4

(i) $x \notin fn(P) \overset{?}{\Longrightarrow} (x)P \equiv P$

$$(x)P \equiv (x)(P \mid nil) \equiv$$
$$\equiv P \mid (x)\, nil \equiv P \mid nil \equiv P$$

(ii) $P = (x)\bar{x}y.nil \quad Q = nil$

$fn(P) = \{y\} \quad fn(Q) = \phi$

$P \sim Q$ since both $P \not\rightarrow$ and $Q \not\rightarrow$

Exercise 5

For non-stopping reactive PLTS we have:

$\alpha : S \to L \to D(S)$ with $\forall s. \forall \ell. \ \alpha s \ell S = 1$
since D is a probability distribution over S

A bisimulation is any equivalence relation R on S with

$s_1 R s_2 \implies \forall \ell. \forall I. \ \alpha s_1 \ell I = \alpha s_2 \ell I$ I equiv. class of R

Thus the relation with just one equivalence class, S, is a bisimulation. Thus all the states are bisimilar.

For generative PLTS $\alpha : S \to (D(L \times S) + 1)$
a bisimulation is any equivalence relation R on S with

$s_1 R s_2 \implies$ either $\alpha s_1 = *$ and $\alpha s_2 = *$
or $\forall \ell. \forall I. \ \sum_{s \in I} \alpha s_1(\ell, s) = \sum_{s \in I} \alpha s_2(\ell, s)$

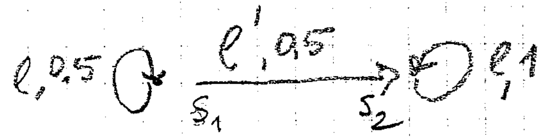
The condition on D is here

$\forall s. \ \alpha(s) \in D(L \times S) \implies \sum_{(\ell, s') \in L \times S} \alpha s(\ell, s') = 1$

For non-stopping generative PLTS we have

$\alpha : S \to D(L \times S)$

Consider the non-stopping system:



with

$\alpha s_1(l, s_1) = 0.5 \quad \alpha s_1(l, s_2) = 0.5$ otherwise 0

$\alpha s_2(l, s_2) = 1$ otherwise 0

The two states are not bisimilar:

$\sum_{s \in \{s_1, s_2\}} \alpha s_1(l, s) = 0.5 \qquad \sum_{s \in \{s_1, s_2\}} \alpha s_2(l, s) = 1$

$\sum_{s \in \{s_1, s_2\}} \alpha s_1(l', s) = 0.5 \qquad \sum_{s \in \{s_1, s_2\}} \alpha s_2(l', s) = 0$

For Segala PLTS we have:

$\alpha : S \to \mathcal{P}(D(L \times S))$

with the following definition of bisimulation

$s_1 R s_2$ implies $s_1 \to d_1 \Rightarrow s_2 \to d_2$

with $\forall l \, \forall I . \sum_{s \in I} d_1(l, s) = \sum_{s \in I} d_2(l, s)$ I equiv class of R

and viceversa

In the deterministic case $\alpha : S \to D(L \times S)$

and $d_1 = \alpha s_1 \quad d_2 = \alpha s_2$

Thus bisimulation reduces to the non-stopping generative case.