Models of Computation

Written Exam on July 11, 2012

(First part: Exercises 1 and 2 Second part: Exercises 3 and 4)

Exercise 1 (7 punti)

Extend IMP with the arithmetic expression $sqrt(a)$ for the square root, defined by the following rule:

$$\frac{\langle a, \sigma\rangle \to n \times n}{\langle sqrt(a), \sigma\rangle \to n}.$$

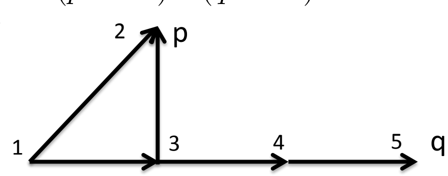
Determinism of operational semantics (namely $\langle c, \sigma\rangle \to \sigma'$ and $\langle c, \sigma\rangle \to \sigma''$ implies $\sigma' = \sigma''$) still holds? If not, exhibit a counterexample. Now add to the premise of the above rule the condition $n \geq 0$. Modify the denotational semantics of IMP arithmetic expressions to take into account expressions with no value. Finally, give the denotational semantics of the new construct and prove the equivalence of its operational and denotational semantics.

Exercise 2 (8)

Decreasing functions $f : \omega \to \omega \cup \infty$ ($n < m \Rightarrow f(n) \geq f(m)$) are ordered pointwise as usual ($f \sqsubseteq g \Leftrightarrow \forall x. f(x) \leq g(x)$). Assume also that $f \sqsubset g$ iff $f \sqsubseteq g$ but not $f = g$. Prove that $\sqsubset$ is a well founded relation (even starting from $\forall n. f_\infty(n) = \infty$ there are no infinite descending chains!), and that $\sqsubseteq$ is a complete partial ordering with bottom.
(Hint: for wellfoundedness, partition functions into classes: $C_\infty = \{f_\infty\}$; and $C_{h,k}$, for all $h, k \in \omega$. Class $C_{h,k}$ contains all functions f with: (i) $f(n) = \infty$ if $0 \leq n < h$ and $f(n) \neq \infty$ if $h \leq n$; and (ii) $\exists m.\ k < f(n)$ if $n < m$ and $f(n) = k$ if $m \leq n$. Show that this is actually a partition. Show also that for every f in a class, there are none or finitely many g in the same class with $g \sqsubset f$. Also if $g \sqsubset f$ and $g \in C_{h',k'} \neq C_{h,k}$, then $h' \leq h$ and $k' \leq k$, but of course with either $h' < h$ or $k' < k$ or both. Conclude the proof.)

Exercise 3 (7)

Given the μ-calculus formula $\Phi = \nu x.(p \vee \Box x) \wedge (q \vee \Diamond x)$ write its denotational semantics $[\![\Phi]\!]\rho$ and evaluate it on the LTS below



Exercise 4 (8)

A philosopher has three states: either a: (s)he thinks; or b: (s)he keeps a fork; or c: (s)he keeps both forks and eats. A fork has two states: up and down. A philosopher and a fork syncronize with action *get* when the philosopher moves from a to b or from b to c and the fork from down to up, while action *rel* is used similarly for releasing the forks. Actions *get* and *rel*, for both philosophers and forks, have the same rate λ. The system consists of two philosophers and two forks. Define a PEPA program representing the system and derive the corresponding CTMC (with actions). To compute the correct rates, write down explicitly the proof for the transition where the process consisting of the two philosophers synchronizes (in four different ways) on *get* with the process consisting of the two forks. Finally, find the bisimilar states and draw the minimal CTMC.

(1)

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Exercise 1

- Determinism does not hold. Here we have a counterexample

$$\langle x := sqrt(4) \rangle \to \sigma' \xleftarrow{\sigma' = \sigma[n/x]} \langle sqrt(4), \sigma \rangle \to n$$

$$\xleftarrow{n=2} \square \qquad \xleftarrow{n=-2} \square$$

- $\mathcal{A} : Aexp \to \Sigma \to N_\perp$

$\mathcal{B} : Bexp \to \Sigma \to T_\perp$

$$\mathcal{A}[\![sqrt(a)]\!]\sigma = \underline{sqrt}(\mathcal{A}[\![a]\!]\sigma) \text{ where}$$

$$\underline{sqrt}(m) = \text{case } m$$
$$n \times n : n \geq 0$$
$$\text{otherwise}: \perp_{N_\perp}$$

- Operational → Denotational

$$\frac{\langle a, \sigma \rangle \to n \times n \quad n \geq 0}{\langle sqrt(a), \sigma \rangle \to n} \qquad P(\langle sqrt(a), \sigma \rangle \to n) \stackrel{def}{=} \mathcal{A}[\![sqrt(a)]\!]\sigma \stackrel{?}{=} n$$

$$\mathcal{A}[\![a]\!]\sigma = n \times n$$

$$\mathcal{A}[\![sqrt(a)]\!]\sigma = n \geq 0 \quad \underline{sqrt(n \times n)} = n$$

QED.

- Denotational → Operational

$$P(sqrt(a)) = \mathcal{A}[\![sqrt(a)]\!]\sigma = n \stackrel{?}{\Rightarrow} \langle sqrt(a), \sigma \rangle \to n$$

$$\mathcal{A}[\![sqrt(a)]\!]\sigma = n \quad \underline{sqrt}(\underbrace{\mathcal{A}[\![a]\!]\sigma}_{m}) = n \quad m \text{ cannot be } \perp, n \geq 0$$

$$m = n \times n \quad n \geq 0 \quad \langle a, \sigma \rangle \to n \times n \quad \langle sqrt(a), \sigma \rangle \to n$$

QED.

Exercise 2

Given $f \in C_{h,k}$, let $n \leq m \Rightarrow f(n) > k$ and $n > m \Rightarrow f(n) = k$.
Functions $g \in C_{h,k}$ with $g \sqsubseteq f$ are thus constrained as follows: $n < h \Rightarrow g(n) = \infty$ $h \leq n < m \Rightarrow g(n) \leq f(n)$ and $n > m \Rightarrow g(n) = k$. Thus g can take a finite set of possible values for a finite set of arguments. Thus there is a finite number of such functions.
If $g \sqsubseteq f$ but $g \in C_{h',k'}$, then either $g(h) \neq \infty$, and thus $h' < h$, or the limit value is smaller, i.e. $k' < k$.
Thus any descending chain can stay for a finite number of steps in any $C_{h',k'}$ with $h' \leq h$ and $k' \leq k$, i.e. it must be finite.
Starting with f_∞, then $g \sqsubset f_\infty$ must be in some class $C_{h,k}$, and the same proof applies.

Ordering $\sqsubseteq$ is complete, $\left(\bigsqcup_{i \in \omega} f_i\right)(n) = \bigsqcup_{i \in \omega} f_i(n)$

As for HOFL functions, continuity is obvious: any function smaller than $\bigsqcup_{i \in \omega} f_i$ would not be the lub for some argument.
Clearly $f(n) = 0$ is the bottom.

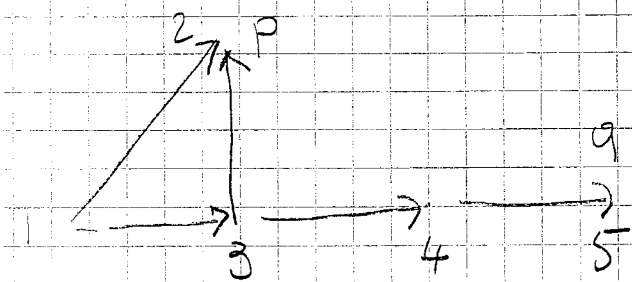
To prove that classes C_∞ and $C_{h,k}$ actually partition decreasing functions, give a function f, it can have an initial segment, or all the arguments, where it evaluates to ∞. This determines if $f \in C_\infty$ or $f \in C_{h,k}$, with some h. Then if $f \in f_\infty$ it can decrease only a finite number of times. This determines k in $C_{h,k}$.

③

Exercise 3

$$[\![\nu x. (p \vee \Box x) \wedge (q \vee \Diamond x)]\!] \rho =$$

$$= \mathrm{Fix}\ \lambda S. \left(\rho p \cup \{ v \mid \forall v'.\ v \to v',\ v' \in S \} \right)$$
$$\cap \left(\rho q \cup \{ v \mid \exists v'\ v \to v'.\ v' \in S \} \right)$$



$$S_0 = \{1, 2, 3, 4, 5\}$$

$$S_1 = \left(\{2\} \cup \{1,2,3,4,5\} \right) \cap \left(\{5\} \cup \{1,3,4\} \right) = \{1,3,4,5\}$$

$$S_2 = \left(\{2\} \cup \{4,5\} \right) \cap \left(\{5\} \cup \{1,3,4\} \right) = \{4,5\}$$

$$S_3 = \left(\{2\} \cup \{4,5\} \right) \cap \left(\{5\} \cup \{3,4\} \right) = \{4,5\} = \mathrm{Fix}$$

Exercise 4

$a = (get, \lambda).b$

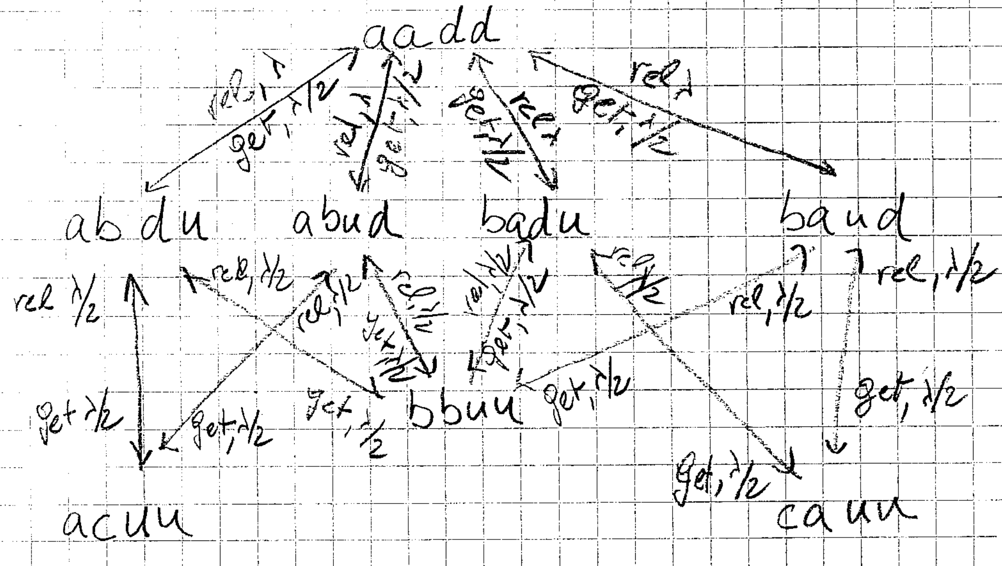
$b = (get, \lambda).c + (rel, \lambda).a$

$c = (rel, \lambda).b$

$down = (get, \lambda).up$

$up = (rel, \lambda).down$

$System = (a \bowtie a) \bowtie_{get, rel} (down \bowtie down)$



$r_{get}(aa) = 2\lambda \qquad r_{get}(dd) = 2\lambda$

$$\frac{aa \xrightarrow{get, \lambda} ab \qquad dd \xrightarrow{get, \lambda} du}{aadd \xrightarrow{get, \lambda/2} abdu} \qquad \frac{\lambda}{2\lambda} \cdot \frac{\lambda}{2\lambda} \cdot \min(2\lambda, 2\lambda) = \frac{\lambda}{2}$$

Bisimilarity

R_0 : all equivalent

R_1 : $\{\{aadd\}, \{abdu, abud, badu, baud\}, \{bbuu\}, \{acuu, cauu\}\}$

$R_2 = R_1$