# Models of Computation

Endterm Exam on May 29, 2012

## Exercise 1 (7)

Given any monotone function $f : N_\perp \to N_\perp$, prove that $f \perp_{N\perp} = f(f \perp_{N\perp})$.
Now, given the closed, well typed HOFL term $rec\, x.t : int$, prove that its denotational semantics is either $\perp_{N\perp}$ or $[\![t]\!]\rho[d/x]$, independent of $d$.

## Exercise 2 (8)

Prove in CCS by rule induction that $P(p \xrightarrow{\mu} q) \stackrel{def}{=} \Phi(p) \xrightarrow{\Phi(\mu)} \Phi(q)$ for every permutation $\Phi$ of free names (restricted names are bound and $\alpha$-convertible, namely they are not free). Notice that, since permutations form a group, we also have that $\Phi(p) \xrightarrow{\Phi(\mu)} \Phi(q)$ implies $p \xrightarrow{\mu} q$. Finally, prove that $p[\Phi] \simeq \Phi(p)$, where $\_[\Phi]$ is a CCS construct, while $\Phi(\_)$ is a permutation operation.

## Exercise 3 (7)

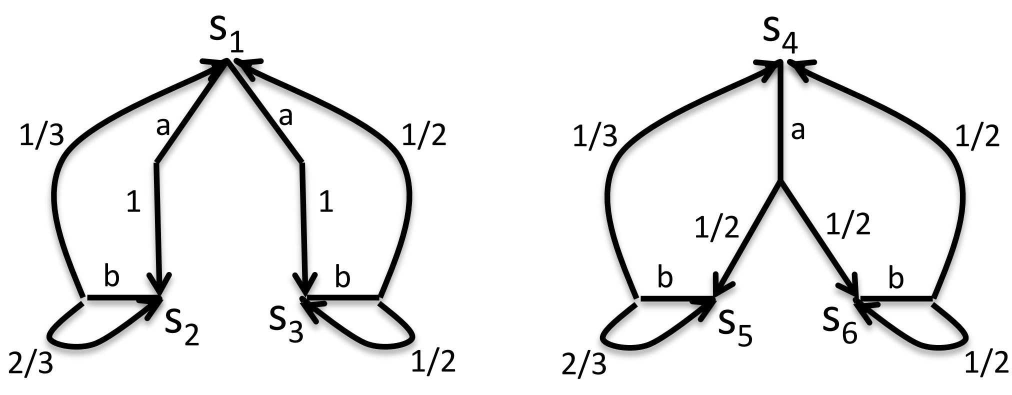
Prove that in the $\pi$-calculus $!p \stackrel{\bullet}{\sim}_L p|!p$. Consider then the alternative definition:

$$\frac{p \xrightarrow{\alpha} q}{!'p \xrightarrow{\alpha} !'p \mid q}$$

and prove in detail that for $p = ((y)\overline{x}y.nil) + x(z).nil$ we have $!'p \stackrel{\bullet}{\not\sim}_E p|!'p$.

## Exercise 4 (8)

Define formally the notion of bisimulation/bisimilarity for simple Segala automata. Then apply the partition refinement algorithm to the automaton below to check which are the bisimilar states.



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## Exercise 1

Monotonicity of $f : [N_\perp \to N_\perp]$ is defined as

$x \sqsubseteq_{N_\perp} y$ implies $f(x) \sqsubseteq_{N_\perp} f(y)$

Thus $f(\perp) = n \neq \perp$ implies $\forall x. f(x) = n$

Now, if $f(\perp) = \perp$ then $f(f(\perp)) = f(\perp) = \perp$

while if $f(\perp) = n$, then $\forall x. f(x) = n$,

in particular $f(n) = f(f(\perp)) = f(\perp) = n$.

Let us compute $[\![ \mathrm{rec}\, x. t ]\!] \rho = fix\, \lambda d. [\![ t ]\!] \rho[d/x]$

$d_0 = \perp_{N_\perp}$ $\quad d_1 = [\![ t ]\!] \rho[\perp/x]$ $\quad$ if $d_1 = \perp$ then we are done

Otherwise it means that function $\lambda d. [\![ t ]\!] \rho[d/x]$

which is monotone, is constant,

namely $(\lambda d. [\![ t ]\!] \rho[d/x])\, d_0 = (\lambda d. [\![ t ]\!] \rho[d/x])\, d_1 = d_1$,

thus $d_1$ is the fixpoint and it does not depend on $d$.

As a check, notice that $d_0 \sqsubseteq d_1 \sqsubseteq \cdots$ is a chain

in $N_\perp$, and thus it cannot be longer than 2.

## Exercise 2

- $\underline{P}(\mu p \xrightarrow{\mu} p) \stackrel{def}{=} \phi(\mu p) \xrightarrow{\phi(\mu)} \phi(p)$

obvious being $\phi(\mu p) = \phi(\mu)\,\phi(p)$

- $$\frac{p \xrightarrow{\mu} q}{p\backslash\alpha \xrightarrow{\mu} q\backslash\alpha}\ \mu \neq \alpha, \overline{\alpha}$$

$\underline{P}(p\backslash\alpha \xrightarrow{\mu} q\backslash\alpha) \stackrel{def}{=} \phi(p\backslash\alpha) \xrightarrow{\phi(\mu)} \phi(q\backslash\alpha)$

$(\phi(p))\backslash\alpha \underset{?}{\xrightarrow{\phi(\mu)}} (\phi(q))\backslash\alpha$

Notice that $\phi$ does not change $\alpha$, ie. $\phi(\alpha) = \alpha$, since $\phi$ changes only free names.
Thus $\mu \neq \alpha, \overline{\alpha}$ implies $\phi(\mu) \neq \alpha, \alpha'$.
Applying the rule on the inductive hipothesis
$P(p \xrightarrow{\mu} q) \stackrel{def}{=} \phi(p) \xrightarrow{\phi(\mu)} \phi(q)$ we get the thesis QED

- $$\frac{p \xrightarrow{\mu} q}{p+r \xrightarrow{\mu} q}$$ $\underline{P}(p+r \xrightarrow{\mu} q) \stackrel{def}{=} \phi(p+r) \xrightarrow{\phi(\mu)} \phi(q)$

$= \phi(p)+\phi(r) \xrightarrow{\phi(\mu)} \phi(q)$

obvious assuming the hipothesis $\phi(p) \xrightarrow{\phi(\mu)} \phi(q)$

- $$\frac{p \xrightarrow{\mu} q}{p|r \xrightarrow{\mu} q|r}$$ $\underline{P}(p|r \xrightarrow{\mu} q|r) \stackrel{def}{=} \phi(p|r) \xrightarrow{\phi(\mu)} \phi(q|r)$

$= \phi(p) | \phi(r) \xrightarrow{\phi(\mu)} \phi(q) | \phi(r)$

obvious assuming $\phi(p) \xrightarrow{\phi(\mu)} \phi(q)$

- $$\frac{p \xrightarrow{\mu} q}{p[\phi] \xrightarrow{\phi(\mu)} q[\phi]}$$ $P(p[\phi] \xrightarrow{\phi(\mu)} q[\phi]) \stackrel{def}{=} \phi'(p[\phi]) \xrightarrow{\phi'(\phi(\mu))} \phi'(q[\phi])$ ?

obvious assuming $p[\phi' \circ \phi] \xrightarrow{\phi'(\phi(\mu))} q[\phi' \circ \phi]$ ?

$\phi'' = \phi' \circ \phi$

- $$\frac{p_1 \xrightarrow{\lambda} q_1 \quad p_2 \xrightarrow{\bar{\lambda}} q_2}{p_1 | p_2 \xrightarrow{\tau} q_1 | q_2}$$

$$P(p_1|p_2 \xrightarrow{\tau} q_1|q_2) \stackrel{def}{=} \phi(p_1)|\phi(p_2) \xrightarrow{\tau} \phi(q_1)/\phi(q_2)$$

obvious assuming $\phi(p_1) \xrightarrow{\phi(\lambda)} \phi(q_1)$

$\phi(p_2) \xrightarrow{\overline{\phi(\lambda)}} \phi(q_2)$

- $$\frac{p[rec\ x.p/x] \xrightarrow{\mu} q}{rec\ x.p \xrightarrow{\mu} q}$$

$$P(rec\ x.p \xrightarrow{\mu} q) \stackrel{def}{=} rec\ x.\phi(p) \xrightarrow{\phi(\mu)} \phi(q)$$

obvious assuming $\phi(p)[rec\ x.\ \phi(p)/x] \xrightarrow{\phi(\mu)} \phi(q)$

Notice that $\phi$ is a permutation of names, thus it does not apply to $x$, which is a process variable.

— $p[\phi] \simeq \phi(p)$ consider the relation

$R = \{(p[\phi], \phi(p)) \mid p \text{ a process}\}$

R is a bisimulation:

$p[\phi] \xrightarrow{\phi(\mu)} q[\phi] \longleftarrow p \xrightarrow{\mu} q \Longrightarrow \phi(p) \xrightarrow{\phi(\mu)} \phi(q)$

for the previous lemma

Viceverse

$\phi(p) \xrightarrow{\mu'} q'$ implies $p \xrightarrow{\phi^{-1}(\mu')} \phi^{-1}(q')$ $\quad p[\phi] \xrightarrow{\mu'} \phi^{-1}(q')[\phi]$

But $q' R\ \phi^{-1}(q')[\phi]$. Infact, letting $q = \phi^{-1}(q')$ we have

$$\phi(q) = \phi(\phi^{-1}(q'))\ R\ (\phi^{-1}(q'))[\phi] = q[\phi].$$

## Exercise 3

Let $R = \{(!p,\ p|!p) \mid p \text{ a process}\}$ be a relation

We show that $R$ is a bisimulation.

$!p \xrightarrow{\alpha} p' \iff p|!p \xrightarrow{\alpha} p'$ directly reduced

i.e. all the transitions of $!p$ are also transitions of $p|!p$.

For $p = ((y)\bar{x}y.nil) + x(z).nil$ we have

$p \xrightarrow{\bar{x}(y)} nil$ and $p \xrightarrow{x(z)} nil$

$$!p \xrightarrow{\alpha} !p|q \leftarrow p \xrightarrow{\alpha} q \xleftarrow[\alpha=\bar{x}(y)]{q=nil} \square$$

$$\xleftarrow[\alpha=x(z)]{q=nil} \square$$

Instead:

$$p|!p \xrightarrow{\alpha} q \xleftarrow[q=(y)nil|nil]{\alpha=\tau} p \xrightarrow{\bar{x}y} nil \quad !p \xrightarrow{x(y)} nil \leftarrow \square$$

Thus $!p \not\xrightarrow{\tau}$ while $p|!p \xrightarrow{\tau} (y)nil|nil$

The difference is not about an input, thus it is independent of $\sim_E$ or $\sim_L$.

# Exercise 4

Simple Segala automata:

$\alpha : S \to \mathcal{P}(L \times \mathcal{D}(S))$

$(\ell, d) \in \alpha(s) \iff s \xrightarrow{\ell} d$

R bisimulation: $s_1 \phi(R) s_2 = s_1 \xrightarrow{\ell} d_1 \Rightarrow s_2 \xrightarrow{\ell} d_2$
$R = \phi(R)$ $\quad \forall I_R.\ d_1(I_R) = d_2(I_2)$ and vice versa

$\simeq$ bisimilarity = maximal bisimulation

$s_1 \xrightarrow{a} d_1 \quad d_1(s_2) = 1$
$s_1 \xrightarrow{a} d_1' \quad d_1'(s_3) = 1$
$s_2 \xrightarrow{b} d_2 \quad d_2(s_1) = 1/3 \quad d_2(s_2) = 2/3$
$s_3 \xrightarrow{b} d_3 \quad d_3(s_1) = 1/2 \quad d_3(s_3) = 1/2$
$S_4 \xrightarrow{a} d_4 \quad d_4(s_5) = 1/2 \quad d_4(s_6) = 1/2$
$S_5 \xrightarrow{b} d_5 \quad d_5(s_4) = 1/3 \quad d_5(s_5) = 2/3$
$S_6 \xrightarrow{b} d_6 \quad d_6(s_4) = 1/2 \quad d_6(s_6) = 1/2$

$R_0 = \{s_1, s_2, s_3, s_4, s_5, s_6\}$
$R_1 = \{\{s_1, s_4\}\{s_2, s_3, s_5, s_6\}\}$
$R_2 = \{\{s_1, s_4\}\{s_2, s_5\}\{s_3, s_6\}\}$ notice that $d_2\{s_1, s_4\} = 1/3$, $d_3(s_1, s_4) = 1/2$
$R_3 = \{\{s_1\}\{s_4\}\{s_2, s_5\}\{s_3, s_6\}\}$ " $d_1\{s_2, s_5\} = 1$ $\quad d_4\{s_2, s_5\} = 1/2$
$R_4 = \{\{s_1\}\{s_4\}\{s_2\}\{s_5\}\{s_3\}\{s_6\}\}$ " $d_2\{s_1\} = 1/3$ $\quad d_5\{s_1\} = 0$
$= \simeq$ $\quad$ " $d_3\{s_1\} = 1/2$ $\quad d_6\{s_1\} = 0$