Models of Computation

Midterm Exam on April 05, 2013

Exercise 1 (12)

Extend IMP with: (i) the new syntactic category S (with variables s) of *expressions with side effects*, shortly *elat*, having $\langle s, \sigma \rangle \to (\sigma', n)$ as well formed formulas for the operational semantics and semantic domain $\Sigma \to (\Sigma \times N)_\perp$; and (ii) the new productions

$$s ::= \{a\} \mid c\,;s \mid s_0 + s_1 \qquad\qquad c ::= x := s$$

with the following informal semantics: given σ, for $s ::= \{a\}$ the meaning is to return σ and the value produced by a; for $s ::= c\,;s$, c is executed first and then s, returning the value of s; for $s ::= s_0 + s_1$, s_0 is executed first, then s_1, and finally the sum of the two values is returned. For $c ::= x := s$, the meaning is first to evaluate s and then to execute the assignment, employing the memory modified by s as memory, and the value of s as value to assign to x.

The denotational semantics of the latter production is as follows:

$$\begin{array}{rl} \mathcal{C}[\![x := s]\!]\sigma = & \textbf{case } \mathcal{S}[\![s]\!]\sigma \textbf{ of} : \\ & \perp_{(\Sigma \times N)_\perp} : \perp_{\Sigma_\perp} \\ & (\sigma', n) \quad : \sigma'[n/x]. \end{array}$$

Define the operational and the (remaining) denotational semantics of the new constructs and show the operational-denotational equivalence. Finally, prove if the following properties hold:

$$(s_0 + s_1) + s_2 \equiv s_0 + (s_1 + s_2) \qquad (c_0; s_0) + (c_1; s_1) \equiv (c_0; c_1); (s_0 + s_1)$$

and, if not, give a counterexample.

Exercise 2 (9)

Consider the logic inference system R corresponding to the rules of the context-sensitive grammar:

$$(i)\ S ::= \lambda \quad (ii)\ S ::= aSbc \quad (iii)\ xcby ::= xbcy$$

where the well formed formulas (wwf) are of the form $x \in L$, and where x is a string on $\{a, b, c\}$. Also λ is the empty string. In rule schema *(iii)*, x and y stay for any string on $\{a, b, c\}$.

Write down explicitly the rules in R.

Prove by rule induction that if $z \in L$ is a theorem, then z is of the form $a^n w_n$, $n \in \omega$, where w_n contains no a's, exactly n b's and n c's (formally $w_n|_a = 0, w_n|_b = w_n|_c = n$). Also, show that if rule *(iii)* does not apply to theorem $z \in L$, then z is of the form $a^n b^n c^n$. Conversely, prove by mathematical induction that for all n, $a^n b^n c^n \in L$ is a theorem.

Exercise 3 (9)

Find the type of the following HOFL program:

$$I = rec\ f.\lambda n.\textbf{if } n \textbf{ then } 0 \textbf{ else if } n - 1 \textbf{ then } 1 \textbf{ else } 2 \times (f\ (n-2)) - (f\ (n-1)) + 3.$$

Then, using symbolic goal reduction, prove by (strong!) mathematical induction on $k, k \geq 0$, that $P(k) \stackrel{def}{=} t \to k$ implies $(I\ t) \to k$.

Finally, observe that I is in fact a definition by well-founded recursion, and write explicitely function $F(b, h) \in C$, with $b \in B$ and $h :<^{-1} \{b\} \to C$ and the equation $I(b) = F(b, I \upharpoonright <^{-1} \{b\})$.

Midterm Exam on April 5, 2013

Exercise 1

Operational semantics

$$\frac{\langle a, \sigma \rangle \to n}{\langle \{a\}, \sigma \rangle \to (\sigma, n)}$$

$$\frac{\langle c, \sigma \rangle \to \sigma'' \quad \langle s, \sigma'' \rangle \to (\sigma', n)}{\langle c; s, \sigma \rangle \to (\sigma', n)}$$

$$\frac{\langle s_0, \sigma \rangle \to (\sigma'', n_0) \quad \langle s_1, \sigma'' \rangle \to (\sigma', n_1)}{\langle s_0 + s_1, \sigma \rangle \to (\sigma', n_0 + n_1)}$$

$$\frac{\langle s, \sigma \rangle \to (\sigma', n)}{\langle x := s, \sigma \rangle \to \sigma'[n/x]}$$

Denotational semantics

$$\mathcal{S}[\![\{a\}]\!]\sigma = (\sigma, \mathcal{A}[\![a]\!]\sigma)$$

$$\mathcal{S}[\![c; s]\!]\sigma = \mathcal{S}[\![s]\!]^*(\mathcal{C}[\![c]\!]\sigma)$$

$$\begin{aligned}\mathcal{S}[\![s_0 + s_1]\!]\sigma = \ & \text{case } \mathcal{S}[\![s_0]\!]\sigma \text{ of} \\ & \perp_{(\Sigma \times N)_\perp} : \perp_{(\Sigma \times N)_\perp} \\ & (\sigma'', n_0) : \text{case } \mathcal{S}[\![s_1]\!]\sigma'' \text{ of} \\ & \qquad \perp_{(\Sigma \times N)_\perp} : \perp_{(\Sigma \times N)_\perp} \\ & \qquad (\sigma', n_1) : (\sigma', n_0 + n_1)\end{aligned}$$

Operational semantics → denotational semantics

$$\frac{\langle a,\sigma\rangle \to n}{\langle \{a\},\sigma\rangle \to (\sigma,n)}$$ $P(\langle \{a\},\sigma\rangle \to (\sigma,n)) \overset{def}{=} \mathcal{S}[\![\{a\}]\!]\sigma = (\sigma,n)$

obvious with $\mathcal{A}[\![a]\!]\sigma = n$

$$\frac{\langle c,\sigma\rangle \to \sigma'' \quad \langle s,\sigma''\rangle \to (\sigma',n)}{\langle c;s,\sigma\rangle \to (\sigma',n)}$$

$P(\langle c;s,\sigma\rangle \to (\sigma',n)) \overset{def}{=} \mathcal{C}[\![c;s]\!]\sigma = (\sigma',n)$

$\mathcal{C}[\![c;s]\!]\sigma = \mathcal{S}[\![s]\!]^*\underbrace{(\mathcal{C}[\![c]\!]\sigma)}_{\substack{\sigma'' \\ \text{for the first hypothesis}}} = \underbrace{(\sigma',n)}_{\text{second hypothesis}}$ * can be eliminated

$$\frac{\langle s_0,\sigma\rangle \to (\sigma'',n_0) \quad \langle s_1,\sigma''\rangle \to (\sigma',n_1)}{\langle s_0+s_1,\sigma\rangle \to (\sigma',n_0+n_1)}$$

$\mathcal{S}[\![s_0,s_1]\!] = (\sigma',n_0+n_1)$ where $\underbrace{\mathcal{S}[\![s_0]\!]\sigma = (\sigma'',n_0)}_{\text{first hypothesis}}$ and $\underbrace{\mathcal{S}[\![s_1]\!]\sigma'' = (\sigma',n_1)}_{\text{second hypothesis}}$

$$\frac{\langle s,\sigma\rangle \to (\sigma',n)}{\langle x:=s,\sigma\rangle \to \sigma'[n/x]}$$

$\mathcal{C}[\![x:=s]\!]\sigma = \sigma'[n/x]$ where $\underbrace{\mathcal{S}[\![s]\!]\sigma = (\sigma',n)}_{\text{hypothesis}}$

Denotational semantics → operational semantics

$P(\{a\}) \overset{def}{=} \mathcal{S}[\![\{a\}]\!]\sigma = (\sigma',n) \Rightarrow \langle \{a\},\sigma\rangle \to (\sigma',n)$

obvious with $\sigma'=\sigma$ and $\mathcal{A}[\![a]\!]\sigma = n$, namely $\langle a,\sigma\rangle \to n$

$P(c;s) = \mathcal{S}[\![c;s]\!]\sigma = (\sigma',n) \Rightarrow \langle c;s,\sigma\rangle \to (\sigma',n)$

$\mathcal{S}[\![s]\!]^*\underbrace{(\mathcal{C}[\![c]\!]\sigma)}_{\sigma''} = (\sigma',n')$ thus $\langle c,\sigma\rangle \to \sigma''$ * eliminated

and $\langle s,\sigma''\rangle \to (\sigma',n)$ the rule can be applied

$P(s_0+s_1) \stackrel{def}{=} \mathcal{S}[\![s_0+s_1]\!]\sigma = (\sigma', n) \Rightarrow \langle s_0+s_1, \sigma\rangle \to (\sigma', n)$

$\mathcal{S}[\![s_0+s_1]\!] = (\sigma', n)$ where $\underbrace{\mathcal{S}[\![s_0]\!]\sigma = (\sigma'', n_0)}_{\langle s_0,\sigma\rangle \to (\sigma'', n_0)}$, $\underbrace{\mathcal{S}[\![s_1]\!]\sigma'' = (\sigma', n_1)}_{\langle s_1,\sigma''\rangle\to(\sigma',n_1)}$, $n = n_0+n_1$

Thus the rule can be applied

$P(x:=s) \stackrel{def}{=} \mathcal{C}[\![x:=s]\!]\sigma=\sigma' \Rightarrow \langle x:=s,\sigma\rangle\to\sigma'$

$\mathcal{C}[\![x:=s]\!]\sigma = \sigma''[n/x]$ where $\underbrace{\mathcal{S}[\![s]\!]\sigma = (\sigma'', n)}_{\langle s,\sigma\rangle\to(\sigma'',n)}$

Thus the rule can be applied with $\sigma'=\sigma''[n/x]$

The equivalence $(s_0+s_1)+s_2 \equiv s_0+(s_1+s_2)$ does hold.

In fact, both els are computed by equivalent proofs

$$\frac{\dfrac{\langle s_0,\sigma\rangle\to(\sigma'',n_0) \quad \langle s_1,\sigma''\rangle\to(\sigma''',n_1)}{\langle s_0+s_1,\sigma\rangle \to (\sigma''',n_0+n_1)} \quad \langle s_2,\sigma'''\rangle\to(\sigma',n_2)}{\langle (s_0+s_1)+s_2,\sigma\rangle \to (\sigma', n_0+n_1+n_2)}$$

$$\frac{\langle s_0,\sigma\rangle\to(\sigma'',n_0) \quad \dfrac{\langle s_1,\sigma''\rangle\to(\sigma''',n_1) \quad \langle s_2,\sigma'''\rangle\to(\sigma',n_2)}{\langle s_1+s_2,\sigma''\rangle\to(\sigma',n_1+n_2)}}{\langle s_0+(s_1+s_2),\sigma\rangle\to(\sigma', n_0+n_1+n_3)}$$

The equivalence $(c_0;s_0)+(c;s_1) \equiv (c_0;c_1);(s_0+s_1)$ does not hold.

A counterexample is

$\langle \sigma, (x:=0;\{x\})+(x:=1;\{0\})\rangle \to (\sigma[1/x], 0)$

$\langle \sigma, (x:=0;x:=1);\{x\}+\{0\}\rangle \to (\sigma[1/x], 1)$

Exercise 2

Inference system R:

$$\lambda \in L \qquad \frac{x \in L}{axbc \in L} \qquad \frac{xcby \in L}{xbcy \in L} \quad x,y \in \{a,b,c\}^*$$

$P(z \in L) \overset{def}{=} \exists n, w_n .\ z = a^n w_n \quad w_n|a = 0 \quad w_n|b = n \quad w_n|c = n$

$\dfrac{-}{\lambda \in L}$ $\quad n=0 \quad w_n = \lambda \quad$ obvious

$\dfrac{x \in L}{axbc \in L}$ $\quad x = a^n w_n \qquad axbc = a^{n+1} w_n bc$

$n' = n+1 \quad w'_{n+1} = w_n bc \quad$ CVD

$\dfrac{xcby \in L}{xbcy \in L}$ $\quad xcby = a^n w_n \quad x = a^n w \quad wcby = w_n$

$n' = n \quad w'_n = wbcy$

$x \in L \quad x = a^n w_n$

- w_n contains n b's and n c's, and does not contain cb

$\rightarrow w_n = b^n c^n$

$z = a^n b^n c^n \quad n = 0, 1, \ldots \quad z \in L$?

Lemma

$\forall n.\ a^n (bc)^n \in L$. Mathematical induction, obvious

Then apply the last rule as much as possible.

When the proof terminates, we have $a^n b^n c^n \in L$ as shown

The proof always terminates. If we write

$(bc)^n = b_1 c_1 b_2 c_2 \cdots b_n c_n$ we need to apply the last rule exactly $1 + 2 + \cdots = \dfrac{n(n-1)}{2}$ times.

5

Exercise 3

$I = rec\ \underbrace{f}_{int \to int}.\ \lambda \underbrace{n}_{int}.\ if\ n\ then\ 0\ else\ if\ n-1\ then\ 1\ else\ 2 \times (f\ \underbrace{(n-2)}_{int}) - (f(n-1)) + 3$

Mathematical induction

Strong! $P(0),\ P(1)$ $P(K,\ P(K+1) \Rightarrow P(K+2)$

$P(K) \overset{def}{=} t \to K \Rightarrow (I\ t) \to K$

$P(0) \overset{def}{=} t \to 0 \Rightarrow (I\ t) \overset{?}{\to} 0$ $t \to 0$

$if\ t\ then\ 0\ else\ if\ t-1\ then\ 1\ else\ 2 \times (I\ (t-2)) - (I(t-1)) + 3 \to n$

$\leftarrow t \to 0 \quad 0 \to c \quad \xleftarrow{c=0} \square$ CVD

$P(1) \overset{def}{=} t \to 1 \Rightarrow (I\ t) \to 1$ $t \to 1$ $(I\ t) \overset{?}{\to} 1$

$\leftarrow t \to n,\ n \neq 0,$ $if\ t-1\ then\ 1\ else\ 2 \times (I\ (t-2)) - (I(t-1)) + 3 \to c$

$\leftarrow t-1 \to 0 \quad 1 \to c \quad \xleftarrow[1 \to 1]{t \to 1} 1 \to c \xleftarrow{c=1} \square$ CVD

$P(K) \overset{def}{=} t \to K \Rightarrow (I\ t) \to K$ assumption

$P(K+1) \overset{def}{=} t \to K+1 \Rightarrow (I\ t) \to K+1)$ assumption

$P(K+2) \overset{def}{=} t \to K+2 \Rightarrow (I\ t) \to K+2$ to prove

$t \to K+2 \quad K \geq 0 \quad (I\ t) \overset{?}{\to} K+2$

$\leftarrow 2 \times (I\ (t-2)) - (I(t-1)) + 3 \to c$ $c \overset{?}{=} K+2$

$\xleftarrow{c = 2 \times c_1 - c_2 + 3} (I\ (t-2)) \to c_1 \quad (I\ (t-1)) \to c_2$

$t-2 \to c_3 \quad \xleftarrow{c_3 = c_4 - 2} t \to c_4 \quad \xleftarrow{c_4 = K+2} \square \quad c_3 = K \quad (I\ (t-2)) \to K \quad c_1 = K$

$t-1 \to c_5 \quad \xleftarrow{c_5 = c_4 - 1} \square \quad c_5 = K+1 \quad (I(t-1)) \to K+1 \quad c_2 = K+1$

$c = 2 \times K - (K+1) + 3 = K+2$ CVD.

Well Founded recursion

$B, C = \omega$

$$F(b,h) = \text{case } b \text{ of} \quad \begin{array}{l} 0 : 0 \\ 1 : 1 \\ k+2 : 2 \times h(k) - h(k+1) + 3 \end{array}$$

$$I(b) = F(b, I \restriction <^{-1}\{b\}) = \text{case } b \text{ of} \quad \begin{array}{l} 0 : 0 \\ 1 : 1 \\ k+2 : 2 \times I(k) - I(k+1) + 3 \end{array}$$

Or alternatively

$$I(0) = 0$$
$$I(1) = 1$$
$$I(k+2) = 2 \times I(k) - I(k+1) + 3$$

Being identity $I(k) = k$ a solution, it's the only solution.