Models of Computation

Written Examination on July 16, 2013

(First part: Exercises 1 and 2, 90 minutes Second part: Exercises 3, 4 and 5, 90 minutes)

(Previous TSD students: Exercises 1, 2 and 3, 120 minutes)

Exercise 1 (8)

Extend *IMP* with a *recursion* command $c ::= \ldots \mid X \mid \mathbf{rec}\, X.c$, redefining memories as $\Sigma' = Loc \to (N \cup (\Sigma \to \Sigma_\perp))$. The denotational semantics of the new command is defined as follows;

$$\mathcal{C}[\![X]\!]\sigma = (\sigma(X))\sigma \quad \mathcal{C}[\![\mathbf{rec}\, X.c]\!] = \mathbf{fix}\, \lambda\varphi.\lambda\sigma.\mathcal{C}[\![c]\!]\sigma[\varphi/X].$$

(i) Define the inference rule for (closed) commands $\mathbf{rec}\, X.c$. (Hint: analogous to the operational semantics of recursion in HOFL); (ii) state, without proof, the substitution lemma for extended IMP; (iii) prove the completeness of the denotational semantics of the new construct; and (iv) given the **repeat** command with denotational semantics:

$$\mathcal{C}[\![\mathbf{repeat}\ c\ \mathbf{until}\ b]\!] = \mathbf{fix}\ \Gamma \quad \Gamma\varphi\sigma = \mathbf{B}[\![b]\!]^*(\mathbf{C}[\![c]\!]\sigma) \to\ \varphi^*(\mathbf{C}[\![c]\!]\sigma),\ \mathbf{C}[\![c]\!]\sigma$$

define an equivalent command using the **rec** and **if then else** constructs and prove that it has the same denotational semantics.

Exercise 2 (7)

Consider the inference system R corresponding to the rules of the grammar (with $V = \{a, b\}$):

$$A ::= aAa|B \quad B ::= bB|b$$

where the well formed formulas are of the form $\alpha \in A$ (resp. $\alpha \in B$), with the meaning that string α belongs to the language generated by the nonterminal A (resp. B). Write down explicitly the rules in R.

Prove by rule induction - in one direction - and by mathematical induction on n,m - in the other direction - that the strings L_A generated by A (resp. L_B by B) are all the strings of the form $a^n b^m a^n$ (resp b^m), $m, n \in \omega, m \geq 1$. Finally prove by induction on derivations that the lenght of the derivation of $a^n b^m a^n \in A$ is $n + m + 1$ (resp. the derivation of $b^m \in B$ is m).

Exercise 3 (5)

Consider the closed HOFL term $t' = \mathbf{rec}\, x.t : int$. Prove that if $[\![\lambda x.t]\!]\rho$ is not a constant, then $[\![t']\!]\rho = \perp_{N\perp}$ (Hint: $[\![\lambda x.t]\!]$ must be monotone, thus if it not a constant function its value on $\perp_{N\perp}$ must be $\perp_{N\perp}$). Give examples of t such that $[\![\lambda x.t]\!]\rho$, with $x \in fv(t)$, is a constant.

Exercise 4 (5)

Prove that in the π-calculus if $p \xrightarrow{\alpha} q$ then also $p\rho \xrightarrow{\alpha\rho} q\rho$ for any substitution ρ on the free names of p which does not capture the bound names of α. Also, show that $p\rho \xrightarrow{\alpha\rho} q\rho$ implies $p \xrightarrow{\alpha} q$ for any *injective* substitution ρ, while $p\rho \xrightarrow{\alpha\rho} q\rho$ does not imply $p \xrightarrow{\alpha} q$ when ρ is a noninjective substitution. (Hint: consider the counterexample which proves that ground bisimilarity is not a congruence.)

Exercise 5 (5)

A probabilistic system has four possible states, *idle*, *working slow*, *working fast* and *failure*. From *idle*, the system can move with probability p to *working slow* and with probability $1 - p$ to *working fast*, where $0 \leq p \leq 1$ is a parameter. From *working fast*, the system returns to *idle* with 80% probability and otherwise moves to *failure*, while from *working slow*, the system returns to *idle* with 95% probability and otherwise moves to *failure*. Finally, from *failure* the system moves with 50% probabiity to *idle*, or otherwise remains in the same state. Define the DTMC corresponding to the description above, observe that the chain is ergodic, and compute the stationary probabilities in terms of the parameter p. Finally, assuming that the gain for state *working fast* is 2 and for *working slow* is 1 (0 for the other states), compute the total gain in terms of p and determine the best value of p.

Written Examination MOD – July 16, 2013

Exercise 1

(i) $$\frac{\langle c[\mathrm{rec}\, X. c/X], \sigma \rangle \to \sigma'}{\langle \mathrm{rec}\, X. c, \sigma \rangle \to \sigma'}$$

$$\mathcal{C}[\![\mathrm{rec}\, X. c]\!] = fix\, \lambda\varphi. \lambda\sigma. \mathcal{C}[\![c]\!]\sigma[\varphi/X]$$
$$\mathcal{C}[\![X]\!]\sigma = \sigma(X)$$

(iii) $$P(\langle \mathrm{rec}\, X. c, \sigma \rangle \to \sigma') \stackrel{?}{=} \mathcal{C}[\![\mathrm{rec}\, X. c]\!]\sigma = \sigma'$$

$\mathcal{C}[\![c[\mathrm{rec}\, X. c/X]]\!]\sigma = \sigma'$ by the premise

$\mathcal{C}[\![\mathrm{rec}\, X. c]\!] = \lambda\sigma. \mathcal{C}[\![c]\!]\sigma[\mathcal{C}[\![\mathrm{rec}\, X. c]\!]/X]$ by the substitution lemma
$= \lambda\sigma. \mathcal{C}[\![c[\mathrm{rec}\, X. c]]\!]$

$\mathcal{C}[\![\mathrm{rec}\, X. c]\!]\sigma = \mathcal{C}[\![c[\mathrm{rec}\, X. c]]\!]\sigma = \sigma'$ QED

(ii) Substitution Lemma

Instead of $[\![t[t'/x]]\!]\rho = [\![t]\!]\rho[[\![t']\!]\rho/x]$

we have $\mathcal{C}[\![c[c'/X]]\!]\sigma = \mathcal{C}[\![c]\!]\sigma[\mathcal{C}[\![c']\!]\sigma/X]$

(iv) $\text{rec}\, X.\, c;\ \text{if } b \text{ then } X \text{ else skip} \equiv c' \quad X \notin fn(c)$

$\mathcal{C}[\![c']\!] = fix\ \Gamma' \quad \Gamma' \varphi \sigma = \mathcal{C}[\![c; \text{if } b \text{ then } X \text{ else skip}]\!]\sigma[\varphi/X]$

$= \mathcal{C}[\![\text{if } b \text{ then } X \text{ else skip}]\!]^*(\mathcal{C}[\![c]\!]\sigma[\varphi/X])$

$= (\lambda\sigma'.\ \mathcal{B}[\![b]\!]\sigma' \to (\sigma'(X))\sigma', \sigma')^*((\mathcal{C}[\![c]\!]\sigma)[\varphi/X])$

$= \mathcal{B}[\![b]\!]^*(\mathcal{C}[\![c]\!]\sigma) \to \varphi^*(\mathcal{C}[\![c]\!]\sigma), \mathcal{C}[\![c]\!]\sigma$

Notice that $\varphi^*(\mathcal{C}[\![c]\!]\sigma)[\varphi/X] = \varphi^*(\mathcal{C}[\![c]\!]\sigma)$ since $\varphi : \Sigma \to \Sigma^*$

$\mathcal{C}[\![\text{repeat } c \text{ until } b]\!] = fix\ \Gamma$

$\Gamma \varphi \sigma = \mathcal{B}[\![b]\!]^*(\mathcal{C}[\![c]\!]\sigma) \to \varphi^*(\mathcal{C}[\![c]\!]\sigma), \mathcal{C}[\![c]\!]\sigma$

Since $\Gamma' = \Gamma$ then $fix\ \Gamma' = fix\ \Gamma$ QED.

Exercise 2

$$\frac{\alpha \in A}{a\alpha a \in A} \qquad \frac{\alpha \in B}{\alpha \in A} \qquad \frac{\alpha \in B}{b\alpha \in B} \qquad b \in B$$

$P(\alpha \in A) \stackrel{def}{=} \exists n,m.\ \alpha = a^n b^m a^n$

$P(\alpha \in B) \stackrel{def}{=} \exists m.\ \alpha = b^m$

$\frac{\alpha \in A}{a\alpha a \in A}$ $\quad \alpha = a^n b^m a^n$ $\quad a\alpha a = a^{n+1} b^m a^{n+1}$ QED.

$\frac{\alpha \in B}{\alpha \in A}$ $\quad \alpha = b^m$ $\quad \alpha = a^0 b^m a^0$ QED.

$\frac{\alpha \in B}{b\alpha \in B}$ $\quad \alpha = b^m$ $\quad b\alpha = b^{m+1}$

$b \in B$ $\quad b = b^1$ QED

$L_A = \{a^n b^m a^n \mid n \geq 0\ m \geq 1\}$ $\quad L_B = \{b^m \mid m \geq 1\}$

- $b^1 \in B$ is a theorem? $\overline{b \in B}$ QED.
- Assume $b^n \in B$. $b^{n+1} \in B$? $\frac{b^n \in B}{b b^n \in B}$ is a rule QED.
- Assume $b^n \in B$. $a^0 b^n a^0 \in A$? $\frac{b^n \in B}{a^0 b^n a^0 \in A}$ QED.
- Assume $a^n b^m a^n \in A$. $a^{n+1} b^m a^n \in A$? $\frac{a^n b^m a^n \in A}{a^{n+1} b^m a^{n+1} \in A}$ QED.

Induction on derivations

$$P(d/a^n b^m a^n \in L) \stackrel{def}{=} n+m+1$$

$$\frac{a^n b^m a^n}{a^{n+1} b^m a^{n+1}} \qquad P(d/a^n b^m a^n / a^{n+1} b^m a^{n+1}) =$$
$$P(d/a^n b^m a^n)+1 = (m+n+1)+1$$
$$= m+(n+1)+1 \quad \text{QED.}$$

$$\frac{b^m \in B}{a^0 b^m a^0 \in A} \qquad P(d/b^m/a^0 b^m a^0) =$$
$$P(d/b^m)+1 = m+1 = (m+0)+1 \quad \text{QED}$$

$$\frac{b^m \in B}{b^{m+1} \in B} \qquad P(d/b^m/b^{m+1}) =$$
$$P(d/b^m)+1 = m+1 \quad \text{QED}$$

$$b \in B \qquad P(d/b) = 1 \quad \text{QED}$$

Exercise 3

- If $[\![\lambda x.t]\!]\rho$ is not a constant, then

$[\![\lambda x.t]\!]\rho \perp_{N_\perp} = \perp_{N_\perp}$

In fact, $[\![\lambda x.t]\!]\rho$ is a monotone (continuos) function, and $[\![\lambda x.t]\!]\rho \lfloor n \rfloor = \lfloor h \rfloor$

$[\![\lambda x.t]\!]\rho \lfloor m \rfloor = \lfloor K \rfloor$

then $[\![\lambda x.t]\!]\rho \perp_{N_\perp} = x$ with $x \sqsubseteq \lfloor h \rfloor$ and $x \sqsubseteq \lfloor K \rfloor$

i.e. $x = \perp_{N_\perp}$

- $[\![rec\, x.t]\!]\rho = fix\, \lambda d.\, [\![t]\!]\rho[d/x]$

$d_0 = \perp_{N_\perp}$

$d_1 = \lambda d.\, [\![t]\!]\rho[d/x] = [\![\lambda x.t]\!]\rho \perp_{N_\perp} = \perp_{N_\perp}$

QED.

- Case analysis:

op has a natural extension $\Rightarrow$ on bottom evaluates to $\perp$

if t_0 then t_1 else t_2 has a natural extension to t_0 but not to t_1 and t_2 $\lambda x.$ if 0 then 0 else x is constant and symmetrically

$(t_1\, t_2)$ has a natural extension on t_1 but not on t_2

$((\lambda x.3)\, x)$ is constant

$\lambda x.t'$ cannot have type int

6

Exercise 4

- $p \xrightarrow{\alpha} q \Rightarrow p\rho \xrightarrow{\alpha\rho} q\rho$

Given a proof of $p \xrightarrow{\alpha} q$, it is possible to derive a proof of $p\rho \xrightarrow{\alpha\rho} q\rho$ by applying substitution ρ on the free names of P.
Notice that bound names of α cannot be free names of p and thus are not involved in the substitution, but substitutions capturing them must be excluded

- $p\rho \xrightarrow{\alpha\rho} q\rho \Rightarrow p \xrightarrow{\alpha} q$

If ρ is injective then we have $\rho \circ \rho^{-1} = id$.
Thus we can apply the previous result with ρ^{-1}.

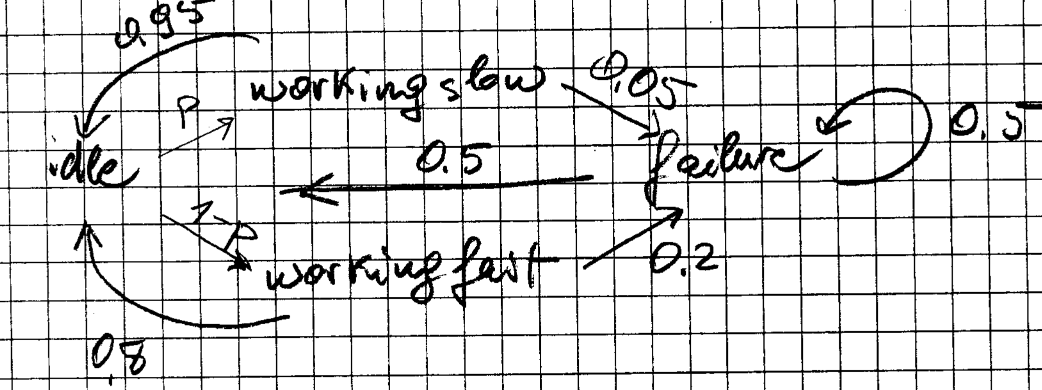
- If ρ is not injective, we have the counterexample:

given $p = (x'(y).nil) \mid \bar{x}x.nil$

we have $p[x/x'] \xrightarrow{\tau} nil \mid nil$

while $p \not\xrightarrow{\tau}$.

Exercise 5



$$\left|\begin{array}{l} i = 0.95s + 0.8f + 0.5K \\ s = p\,i \\ f = (1-p)\,i \\ K = 0.05s + 0.2f + 0.5K \end{array}\right.$$

$$i + s + f + K = 1$$

$$K = 0.1s + 0.4f = \left(0.1p + 0.4(1-p)\right) i = \left(0.4 - 0.3p\right) i$$

$$\left(1 + \cancel{p} + 1 - \cancel{p} + 0.4 - 0.3p\right) i = 1$$

$$i = \frac{1}{2.4 - 0.3p} \qquad s = \frac{p}{2.4 - 0.3p}$$

$$f = \frac{1-p}{2.4 - 0.3p} \qquad K = \frac{0.4 - 0.3p}{2.4 - 0.3p}$$

$$\text{Gain} = \frac{p + 2(1-p)}{2.4 - 0.3p} = \frac{2-p}{2.4 - 0.3p}$$

$p = 0$ (fast) $\quad \text{Gain} = \frac{2}{2.4} = 0.833$

$p = 1$ (slow) $\quad \text{Gain} = \frac{1}{2.1} = 0.476$

Derivative of Gain: $\dfrac{-1}{2.4 - 0.3p} - \dfrac{2-p}{(2.4-0.3p)^2} \cdot (-0.3)$

$= \dfrac{-1.8}{(2.4 - 0.3p)^2}$ always negative: max gain for $p = 0$ fast