Models of Computation

Written Exam on January 28, 2013

(MOD students: Exercises 1-5, 180 minutes   Previous TSD students: Exercises 1-3, 120 minutes)

Exercise 1 (7)

Modify the denotational semantics of **while** as follows:

$$\mathcal{C}[\![\textbf{while } b \textbf{ do } c]\!] = \mathit{fix}\ \Gamma \qquad \Gamma\varphi\sigma = \mathcal{B}[\![b]\!]\sigma \ \rightarrow \ \varphi\sigma,\ \sigma.$$

Prove that $\mathcal{C}[\![c]\!]\sigma = \sigma' \ \Rightarrow \ \langle c\ ,\sigma\rangle \rightarrow \sigma'$ and show with a counterexample that $\langle c\ ,\sigma\rangle \rightarrow \sigma' \ \Rightarrow \ \mathcal{C}[\![c]\!]\sigma = \sigma'$ does not hold.

Exercise 2 (7)

Consider the set PI of partial injective functions from ω to ω, with the ordering $\sqsubseteq$ explained in the course (inclusion of sets of pairs, namely $f \sqsubseteq g$ if $Gr(f) \subseteq Gr(g)$, with $Gr(h) = \{< x, y > \mid h(x) = y\}$). If we identify a function f with its *graph* $Gr(f)$, we have that f partial injective means $< x, y >, < x, y' > \in f \Rightarrow y = y'$ e $< x, y >, < x', y > \in f \Rightarrow x = x'$. Prove that $(PI, \sqsubseteq)$ is a complete partial ordering. Finally, prove that function $F : PI \rightarrow PI$ with $F(f) = \{< 2x, y > \mid < x, y > \in f\}$ is monotone continuous.
(Hint: Consider F as computed by the immediate consequences operator $\hat{R}$, with R consisting only of the rule $< x, y > / < 2x, y >$.)

Exercise 3 (6)

Consider the following equation:

$$[\![t]\!]\rho = \lfloor([\![first(t)]\!]\rho\ ,\ [\![snd(t)]\!]\rho)\rfloor.$$

Determine the type of t and the semantic domains all the subterms of the equation belong to, checking that the left and the right hand side of the equation belong to the same type. Then show with a counterexample that the equation does not hold. Finally, give necessary and sufficient conditions under which it holds.

Exercise 4 (5)

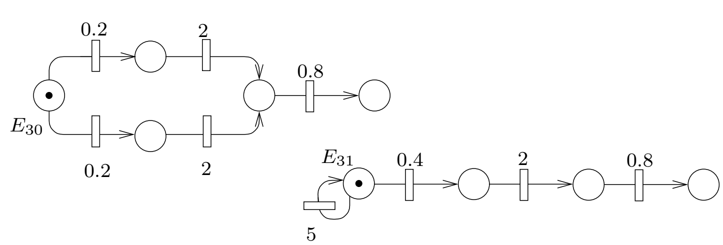
The *terminating traces* semantics (TT) of a **CCS** agent p, is the set of action sequences p can perform arriving at a *quiescent* state:

$$[\![p]\!] = \{\mu_1\mu_2 \dots \mu_n \mid p \xrightarrow{\mu_1}\xrightarrow{\mu_2} \dots \xrightarrow{\mu_n} q \ \wedge \ q \not\rightarrow\}.$$

Prove that two bisimilar agents have the same TT semantics. Finally, considering the two agents $p = \alpha.(\beta.nil + \gamma.nil)$ and $q = \alpha.\beta.nil + \alpha.\gamma.nil)$, and the context $C[\ \_\ ] = (\ \_\ |\overline{\alpha}.\overline{\beta}.nil)\backslash\alpha\backslash\beta\backslash\gamma$, prove that the TT semantics is not a congruence.

Exercise 5 (5)

In the notes, the CTMC notion of bisimilarity has been defined for *unlabeled* TSs, while PEPA TS is labeled. Extend the definition of bisimilarity to the labeled version.



Define two PEPA processes for the two TS in the figure above (assume all the transitions as decorated by the same label and disregard the self loop on E_{31}: why?) and compute iteratively the bisimilarity relation as the fixpoint of function Φ.

Correzione Scritto

Esercizio 1

(i) $P(w) \stackrel{def}{=} \mathcal{C}[\![w]\!]\sigma = \sigma' \Rightarrow \langle w,\sigma\rangle \to \sigma'$

Possiamo calcolare direttamente $\mathcal{C}[\![w]\!]\sigma$

$$\varphi_0 = \bot$$
$$\varphi_1 = \Gamma\varphi_0 = \lambda\sigma.\, \mathcal{B}[\![b]\!]\sigma \to \bot, \sigma$$
$$\varphi_2 = \Gamma\varphi_1 = \lambda\sigma.\, \mathcal{B}[\![b]\!]\sigma \to (\lambda\sigma'.\, \mathcal{B}[\![b]\!]\sigma' \to \bot, \sigma')\sigma, \sigma$$
$$= \lambda\sigma.\, \mathcal{B}[\![b]\!]\sigma \to (\mathcal{B}[\![b]\!]\sigma \to \bot, \sigma), \sigma$$
$$= \lambda\sigma.\, \mathcal{B}[\![b]\!]\sigma \to \bot, \sigma = \varphi_1 = fix\,\Gamma$$

$$\mathcal{C}[\![w]\!]\sigma = \mathcal{B}[\![b]\!]\sigma \to \bot, \sigma$$

Quindi $\mathcal{C}[\![w]\!]\sigma = \sigma' \Rightarrow (\mathcal{B}[\![b]\!]\sigma = false \wedge \sigma = \sigma')$

Quindi le premesse della regola

$$\frac{\langle b,\sigma\rangle \to false}{\langle w,\sigma\rangle \to \sigma}$$

sono soddisfatte e $\sigma = \sigma'$. CVD

(ii) while $x \neq 0$ do $x := x-1 = w$

Abbiamo

$$\langle w, \sigma[1/x]\rangle \to \sigma' \leftarrow \langle x := x-1, \sigma[1/x]\rangle \to \sigma'',\ \langle w, \sigma''\rangle \to \sigma' \leftarrow$$
$$\leftarrow \langle w, \sigma[1/x][1-1/x]\rangle \to \sigma' \leftarrow$$
$$\leftarrow \langle w, \sigma[0/x]\rangle \to \sigma' \xleftarrow{\sigma' = \sigma[0/x]} \Box$$

mentre

$$\mathcal{C}[\![w]\!]\sigma[1/x] = \mathcal{B}[\![x \neq 0]\!]\sigma[1/x] \to \bot, \sigma[1/x] = \bot$$

Esercizio 2

Proprietà riflessiva: $f \sqsubseteq f$ ovvio essendo $f \subseteq f$

Proprietà antisimmetrica: $f \sqsubseteq g, g \sqsubseteq f$ ovvio essendo
$f \subseteq g$ e $g \subseteq f \Rightarrow g = f$

Proprietà transitiva: $f \sqsubseteq g, g \sqsubseteq h \Rightarrow f \sqsubseteq h$ ovvio essendo
$f \subseteq g$ $g \subseteq h \Rightarrow f \subseteq h$

Completezza

Data una catena

$$f_0 \subseteq f_1 \subseteq f_2 \subseteq \dots \subseteq f_i \subseteq \dots$$

con $\langle x,y \rangle, \langle x,y' \rangle \in f_i \Rightarrow y = y'$ e $\langle x,y \rangle, \langle x',y \rangle \in f_i \Rightarrow x = x'$

Bisogna dimostrare che anche il lub è parziale iniettiva:

$$\langle x,y \rangle, \langle x,y' \rangle \in \bigcup_{i \in \omega} f_i \Rightarrow y = y' \quad e \quad \langle x,y \rangle, \langle x',y \rangle \in \bigcup_{i \in \omega} f_i \Rightarrow x = x'$$

Infatti se $\langle x,y \rangle, \langle x,y' \rangle \in \bigcup_{i \in \omega} f_i$ allora esiste un k
con $\langle x,y \rangle, \langle x,y' \rangle \in f_k$.
Quindi $y = y'$ essendo f_k parziale iniettiva.

Similmente per $\langle x,y \rangle, \langle x',y \rangle \in \bigcup_{i \in \omega} f_i \Rightarrow x = x'$

Da ultimo $\hat{R} : PI \to PI$ con $R = \left\{ \frac{\langle x,y \rangle}{\langle 2x,y \rangle} \right\}$

è continuo avendo tutte le regole un
numero finito di premesse.

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Es 3

$$\underbrace{[\![t]\!]\rho}_{\substack{\tau_1 * \tau_2 \\ \in \left((V_{\tau_1})_\bot \times (V_{\tau_2})_\bot\right)_\bot}} = \underbrace{\overbrace{\lfloor \big(\underbrace{\underbrace{[\![\mathit{first}(\overbrace{t}^{\tau_1 * \tau_2})]\!]\rho}_{\tau_1}}_{(V_{\tau_1})_\bot}, \underbrace{\underbrace{[\![\mathit{snd}(\overbrace{t}^{\tau_1 * \tau_2})]\!]\rho}_{\tau_2}}_{(V_{\tau_2})_\bot} \big) \rfloor}^{(V_{\tau_1})_\bot \times (V_{\tau_2})_\bot}}_{\left((V_{\tau_1})_\bot \times (V_{\tau_2})_\bot\right)_\bot}$$

Sia $t = \mathrm{rec}\ x : \tau_1 * \tau_2 .\ x$

Quindi $[\![t]\!]\rho = \mathit{fix}\ \lambda x. x$

$$x^0 = \bot_{(V_{\tau_1 * \tau_2})_\bot}$$

$$x^1 = (\lambda x. x)\bot = \bot = [\![t]\!]\rho$$

$$\lfloor ([\![\mathit{first}(t)]\!]\rho, [\![\mathit{snd}(t)]\!]\rho) \rfloor =$$

$$\lfloor ((\mathrm{let}\ v \Leftarrow [\![t]\!]\rho .\ \pi_1 v), (\mathrm{let}\ v' \Leftarrow [\![t]\!]\rho .\ \pi_2 v')) \rfloor =$$

$$\lfloor ((\mathrm{let}\ v \Leftarrow \bot .\ \pi_1 v), (\mathrm{let}\ v' \Leftarrow \bot .\ \pi_2 v')) \rfloor =$$

$$\lfloor (\bot_{(V_{\tau_1})_\bot}, \bot_{(V_{\tau_2})_\bot}) \rfloor \neq \bot_{\left((V_{\tau_1})_\bot \times (V_{\tau_2})_\bot\right)_\bot}$$

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5

Assumiamo la condizione $[\![t]\!]\rho = \lfloor(v_1, v_2)\rfloor$

cioè $[\![t]\!]\rho \neq \perp_{(V_{\tau_1 * \tau_2})_\perp}$ cioè $t \to c$.

Allora dalle derivazioni precedenti abbiamo

$\lfloor([\![\mathrm{fst}(t)]\!]\rho, [\![\mathrm{snd}(t)]\!]\rho)\rfloor =$

$= \lfloor((\mathrm{let}\ v \Leftarrow \lfloor(v_1, v_2)\rfloor.\ \pi_1 v), \mathrm{let}\ v' \Leftarrow \lfloor(v_1, v_2)\rfloor.\ \pi_2 v')\rfloor$

$= \lfloor((\pi_1(v_1, v_2), \pi_2(v_1, v_2))\rfloor = \lfloor(v_1, v_2)\rfloor = [\![t]\!]\rho$ CVD.

Esercizio 3

- Per assurdo: $[\![p]\!] \neq [\![q]\!] \Rightarrow p \not\approx q$

Assumiamo $[\![p]\!] \neq [\![q]\!]$. Quindi ad es.

$\mu_1 \cdots \mu_n \in [\![p]\!]$ e $\mu_1 \cdots \mu_n \notin [\![q]\!]$

Per induzione matematica (sob positivi) su n:

$(*)\ P(n) \stackrel{def}{=} \forall \mu_1 \cdots \mu_n.\ \mu_1 \cdots \mu_n \in [\![p]\!]$ e $\mu_1 \cdots \mu_n \notin [\![q]\!] \Rightarrow p \not\approx q$

Per $n=1$

$P(1) \stackrel{def}{=} \forall \mu.\ \mu \in [\![p]\!]$ e $\mu \notin [\![q]\!] \stackrel{?}{\Rightarrow} p \not\approx q$

$\stackrel{def}{=} \forall \mu.\ p \xrightarrow{\mu} p'$ e $p' \not\to$ e $q \stackrel{\mu}{\not\to} q', q' \not\to$

Allora Alice sceglie $p \xrightarrow{\mu} p'$, e Bob invece non può andare in uno stato quiescente con μ.

Quindi o Alice vince subito, se $q \stackrel{\mu}{\not\to}$, oppure $q \xrightarrow{\mu} q'$ non quiescente. Allora Alice sceglie $q' \xrightarrow{\mu'} q''$ e Bob non può rispondere con $p' \not\to$

Passo induttivo

Vale $(*)$ e bisogna dimostrare

$P(n+1) \stackrel{def}{=} \forall \mu_1 \cdots \mu_{n+1}.\ \mu_1 \cdots \mu_{n+1} \in [\![p]\!]$ e $\mu_1 \cdots \mu_{n+1} \notin [\![q]\!] \Rightarrow p \not\approx q$

$\stackrel{def}{=} \forall \mu_1 \cdots \mu_{n+1}\ p \xrightarrow{\mu_1} \cdots \xrightarrow{\mu_{n+1}} p'$ e $p' \not\to$ e $q \xrightarrow{\mu_1} \cancel{\cdots} \xrightarrow{\mu_{n+1}} q', q' \not\to$

Allora Alice sceglie $p \xrightarrow{\mu_1} p''$ con $p'' \xrightarrow{\mu_2} \cdots \xrightarrow{\mu_{n+1}} p'$ e $p' \not\rightarrow$

Bob invece non può $q \xrightarrow{\mu_1} q''$ con $q'' \xrightarrow{\mu_2} \cdots \xrightarrow{\mu_{n+1}} q'$ e $q' \not\rightarrow$

Quindi o Alice vince subito, se $q \not\xrightarrow{\mu_1}$, oppure $q \xrightarrow{\mu_1} q''$ con $q'' \cancel{\xrightarrow{\mu_2} \cdots \xrightarrow{\mu_{n+1}}} q'$ e $q' \not\rightarrow$. Comunque $p'' \not\simeq q''$, dato che Alice ha una strategia vincente per l'ipotesi induttiva (*):

$$p'' \xrightarrow{\mu_2} \cdots \xrightarrow{\mu_{n+1}} p' \text{ e } p' \not\rightarrow \text{ e } q'' \cancel{\xrightarrow{\mu_2} \cdots \xrightarrow{\mu_{n+1}}} q' \text{ e } q' \not\rightarrow \Rightarrow p'' \not\simeq q''$$

con traccia lunga n.

⑤ $p = \alpha.(\beta.nil + \gamma.nil) \qquad q = \alpha.\beta.nil + \alpha.\gamma.nil$

$$[\![p]\!] = \{\alpha\beta, \alpha\gamma\} = [\![q]\!]$$

$$C[p] = (\alpha.(\beta.nil + \gamma.nil) \mid \bar{\alpha}.\bar{\beta}.nil) \setminus \alpha \setminus \beta \setminus \gamma$$
$$C[q] = ((\alpha.\beta.nil + \alpha.\gamma.nil) \mid \bar{\alpha}.\bar{\beta}.nil) \setminus \alpha \setminus \beta \setminus \gamma$$

$$[\![C[p]]\!] = \{\tau\tau\} \qquad [\![C[q]]\!] = \{\tau, \tau\tau\}$$

Essendo $[\![p]\!] = [\![q]\!]$ ma $[\![C[p]]\!] \neq [\![C[q]]\!]$

$[\![-]\!]$ non è una congruenza.

8

Exercise ~~4~~ 5

PEPA $\quad S_1 \xrightarrow{(\ell, r)} S_2 \qquad r_\ell(S_1) = \sum \alpha S_1 \ell S_2$

apparent rate

$\alpha : S \to \mathcal{L} \to S \to \mathbb{R}^{>0}$

$\alpha S_1 \ell S_2 = \sum \{ r \mid S_1 \xrightarrow{(\ell, r)} S_2 \}$

$S_1 \, \phi(R) \, S_2 \Rightarrow \forall \ell, I \in R. \ \alpha S_1 \ell I = \alpha S_2 \ell I$

$E_{30} = (0.2).(2).(0.8).stop + (0.2).(2).(0.8).stop$

$E_{31} = (0.4).(2).(0.8).stop$

$P = (0.8)\, stop$

$q = (2).P$ $\qquad$ label the same for all transitions

$E_{30} = (0.2)q + (0.2)q$

$E_{31} = (0.4)q$

$R_0 = \{\{E_{30}, E_{31}, q, P, stop\}\}$

$R_1 = \phi(R_0) = \{\{E_{30}, E_{31}\}, \{q\}, \{P\}, \{stop\}\}$ based on the apparent rate!

$R_2 = R_1$

Self loop are not meaningful: both the probability of the event and its negation contribute to the sojourn time.