Models of Computation

Endterm Exam on May 29, 2013

Exercise 1 (9)

If $C(t)$ means that t is a canonical term, in HOFL we have $C(n)$, $C(\lambda x.t)$ and $C((t_1, t_2))$. Modify the definition of canonical term letting $D(n)$, $D(\lambda x.t)$ and $\frac{D(t_1)\ D(t_2)}{D((t_1,t_2))}$. Then modify the operational semantics $t \rightarrow d$, where d denotes a D-canonical term, by adding a clause for $(t_1, t_2) \rightarrow d$; and the denotational semantics, if needed, to accommodate it to the new definition. Also, prove that $t \rightarrow d \Rightarrow [\![t]\!]\rho = [\![d]\!]\rho$. Furthermore, find two terms t_1 and t_2 such that $[\![t_1]\!]\rho = [\![t_2]\!]\rho$, $t_1 \rightarrow c_1$, $t_2 \rightarrow c_2$, $t_1 \rightarrow d_1$, $t_2 \rightarrow d_2$, and $d_1 = d_2$ but $c_1 \neq c_2$. Finally, find t'_1 and t'_2 such that $[\![t'_1]\!]\rho = [\![t'_2]\!]\rho$ and $t'_1 \rightarrow d'_1$, $t'_2 \rightarrow d'_2$ but still $d'_1 \neq d'_2$.

Exercise 2 (12)

Find two families of CCS terms $\{p_i\}_{i\in\omega}$ and $\{q_i\}_{i\in\omega}$ and prove that $p_{i-1} \simeq_{i-1} q_{i-1}$ but $p_{i-1} \not\simeq_i q_{i-1}$, with $i = 1, 2, \ldots$, where $\simeq_i$ is the *ith* approximation of $\simeq$. (Hint: take $p_2 = \alpha.\alpha.\beta.nil$ and $q_2 = \alpha.\alpha.\gamma.nil$.) Also, show a family $\{\Phi_i\}_{i\in\omega}$ of HML formulas such that Φ_i distinguishes p_{i-1} from q_{i-1} but not p_i from q_i. Finally, find a μ-calculus formula which distinguishes p_i from q_i for all i.

Exercise 3 (9)

A *DT Markow star* is a DTMC consisting of a *center* state s_0 and of a family $\{s_i\}_{i=1,\ldots,n}$ of *external* states. The transitions are as follows, for $i = 1, \ldots, n$:

$$s_0 \xrightarrow{a_i} s_i \qquad s_0 \xrightarrow{a} s_0 \qquad s_i \xrightarrow{b_i} s_0 \qquad s_i \xrightarrow{1-b_i} s_i$$

where $a + \sum_{i=1,\ldots,n} a_i = 1$. Prove for which values of the parameters Markov stars are ergodic, and find the steady state probabilities of all the states. Finally, show for which values of the parameters it is possible to construct a bisimulation with two equivalence classes (lumpings), the first a singleton containing the center and the second all the external states. Then define the corresponding DTMC.

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EX1) * Add to the operational semantics the clause:

$$\frac{t_1 \to d_1 \quad t_2 \to d_2}{(t_1, t_2) \to (d_1, d_2)} \quad (a)$$

notice that if d_1 and d_2 are canonical, (d_1, d_2) is canonical according to the new definition

* The denotational semantics is not changed

* The property $t \to d \Rightarrow [\![t]\!]\rho = [\![d]\!]\rho$ is proved by rule induction. Thus to extend the proof is enough to consider the new rule (a):

$$P((t_1,t_2) \to (d_1,d_2)) \stackrel{def}{=} [\![(t_1,t_2)]\!]\rho = [\![(d_1,d_2)]\!]\rho$$

$$[\![(t_1,t_2)]\!]\rho = \lfloor ([\![t_1]\!]\rho, [\![t_2]\!]\rho) \rfloor = \lfloor ([\![d_1]\!]\rho, [\![d_2]\!]\rho) \rfloor$$

by the inductive hyp.

$$= [\![(d_1,d_2)]\!]\rho \quad \text{QED.}$$

* $t_1 = (1+0, 1)$ $t_2 = (1,1)$ We have

$$(1+0,1) \to d \xleftarrow{d=(d_1,d_2)} 1+0 \to d_1 \ \ 1 \to d_2 \xleftarrow{d_1=1\ d_2=1} \square$$

$$(1,1) \to d \xleftarrow{d=(1,1)} \square$$

in the old semantics, t_1 and t_2 are canonical, and different

* $t_1' = \lambda x.1+0$ $t_2' = \lambda x.1$ are canonical and different in both the new and the old semantics

Exercise 2

$p_i = \underbrace{\alpha \dots \alpha}_{i} \beta\, nil \qquad q_i = \underbrace{\alpha \dots \alpha}_{i} \gamma\, nil \qquad i = 0, 1, \dots$

The reachable states are $P = \{p_i\}_{i \in \omega} \cup \{q_i\}_{i \in \omega} \cup \{nil\}$

The approximations of $\simeq$ are:

$\simeq_0 = \{\{P\}\}$ one equivalence class

$\simeq_1 = \{\{nil\}, \{\beta nil\}, \{\gamma nil\}, \{p_i, q_i\}_{i=1,\dots}\}$

$\simeq_i = \{\{nil\}, \{p_0\}, \dots \{p_{i-1}\}, \{q_0\}, \dots, \{q_{i-1}\}, \{p_j, q_j\}_{j=i,\dots}\} \quad i = 1, \dots$

It is easy to see that $\Phi(\simeq_i) = \simeq_{i+1}$.

In fact $p_i \xrightarrow{\alpha} p_{i-1}$ and $q_i \xrightarrow{\alpha} q_{i-1}$, p_{i-1} and q_{i-1} being in different classes while no other class is broken.

Notice that, as required, we have $p_{i-1} \simeq_{i-1} q_{i-1}$ but $p_{i-1} \not\simeq_i q_{i-1}$.

HML formulas are:

$$\phi_0 = true \qquad \phi_{i+1} = \Diamond\alpha\, \phi_i \vee \Diamond\beta\, true$$

It is easy to see that

$$p_j \models \phi_i \quad \text{for all } i, j \in \omega.$$

$$q_j \not\models \phi_i \quad \text{for } j < i \qquad q_j \models \phi_i \quad \text{for } j \geq i$$

Thus ϕ_i distinguishes p_j and q_j for all $j < i$

(3)

The μ-calculus formula is

$$\Psi = \mu x. \langle\alpha\rangle x \vee \langle\beta\rangle true$$

Its semantics is as follows:

$$[\![\Psi]\!]\rho = fix\ \lambda S. \{\sigma \mid \exists\sigma'. \sigma \xrightarrow{\alpha} \sigma' \wedge \sigma' \in S\} \cup \{\sigma \mid \exists\sigma'. \sigma \xrightarrow{\beta} \sigma'\}$$

On our reachable states the iterations are as follows:

$$S_0 = \emptyset \quad S_1 = \{\beta nil\} \quad S_2 = \{\beta nil, \alpha\beta nil\}$$

$$S_n = \{\beta nil, \alpha\beta nil, \ldots, \underbrace{\alpha \ldots \alpha}_{n-1}\beta nil\}$$

$$lub\{S_n\} = \{p_i\}_{i \in \omega} = [\![\Psi]\!]\rho$$

Thus we have that every p_i is distinguished from every q_i

Exercise 3

To be ergodic, we must have $a_i, b_i \neq 0 \quad i=1,\dots,n$
otherwise s_i is not properly connected.
Furthermore, we must have $a \neq 0$ or $b_i \neq 1$ for some i,
otherwise we have only paths of even length.

The steady state equations are

$$\begin{cases} a_0 s_0 + b_1 s_1 + \dots + b_n s_n = s_0 \\ a_i s_0 + (1-b_i) s_i = s_i \qquad a_i s_0 = b_i s_i \qquad s_i = \dfrac{a_i}{b_i} s_0 \\ s_0 + s_1 + \dots + s_n = 1 \end{cases}$$

$$s_0 + \frac{a_1}{b_1} s_0 + \dots + \frac{a_n}{b_n} s_0 = 1$$

$$s_0 = \frac{1}{1 + \frac{a_1}{b_1} + \dots + \frac{a_n}{b_n}} \qquad s_i = \frac{\frac{a_i}{b_i}}{1 + \frac{a_1}{b_1} + \dots + \frac{a_n}{b_n}}$$

$\{\{s_0\}, \{s_i\}_{i=1,\dots,n}\}$ is a bisimulation if

$$\alpha(s_i)(\{s_i\}_{i=1,\dots,n}) = \alpha(s_j)(\{s_i\}_{i=1,\dots,n})$$

$$\alpha(s_i)(s_i) = \alpha(s_j)(s_j) \qquad \alpha(s_i)(s_0) = \alpha(s_j)(s_0)$$

$$1 - b_i = 1 - b_j \qquad \boxed{b_i = b_j = b}$$

The "minimal" DTMC is: