Models of Computation

Written Exam on February 10, 2014

Exercise 1 (6)

Given the IMP command:

$$\mathrm{w} = \textbf{while } x \neq 0 \textbf{ do } (x := x - 1;\ y := y + 1)$$

prove using Scott computational induction that, no matter what memories σ e σ' are, we have:

$$\mathcal{C}[\![w]\!]\sigma = \sigma' \quad \Rightarrow \quad \sigma x \geq 0 \ \wedge \ \sigma' = \sigma\left[{}^{\sigma x + \sigma y}/_{y},\ {}^{0}/_{x}\right].$$

Exercise 2 (8)

Consider the set $\{\omega \times \omega\}$ of pairs of natural numbrs with the lexicographic ordering $\sqsubseteq$ defined as

$$(n_1, m_1) \sqsubseteq (n_2, m_2) \text{ if } n_1 < n_2 \text{ or if } n_1 = n_2 \text{ and } m_1 \leq m_2.$$

Prove that $\sqsubseteq$ is a partial ordering with bottom. Then show that the chain $\{(0,k)\}_{k=0,1\ldots}$ has lowest upper bound, but also exhibit a chain without lub. Moreover, consider the set $\{[N] \times \omega\}$ con $[N] = \{n | n \leq N\}$, with the same ordering, and then show, also in this case, a chain without lub . Finally, prove that $\{[N] \times (\omega \cup \{\infty\})\}$ with the same ordering, where $x \leq \infty$, is complete with bottom, and show a monotone, non continuous function on it.

Exercise 3 (6)

Redefine the operational semantics of multiplication in HOFL as follows:

$$\frac{t_1 \rightarrow n_1 \quad t_2 \rightarrow n_2}{t_1 \times t_2 \rightarrow n_1 \underline{\times} n_2} \qquad \frac{t_1 \rightarrow 0}{t_1 \times t_2 \rightarrow 0}.$$

Prove by structural induction that also with the new definition multiplication is deterministic, namely $t_1 \times t_2 \rightarrow n$ and $t_1 \times t_2 \rightarrow n'$ implies $n = n'$. However observe that now for the (well typed) term $t = 0 \times rec\ x.x$ it does not hold that $t \rightarrow c$ implies $[\![t]\!]\rho = [\![c]\!]\rho$.

Finally, redefine also the denotational semantics of multiplication and prove that $t_1 \times t_2 \rightarrow c$ implies $[\![t_1 \times t_2]\!]\rho = [\![c]\!]\rho$.

Exercise 4 (5)

Consider the CCS agents:

$$p = rec\ x.abx + bax \qquad q = ((rec\ x.a\bar{c}x) | rec\ x.bcx) \backslash c,$$

compute all the states reachable from them in the **weak** transition system ($\Rightarrow$) and, using the iterative computation method for the fixpoint, partition the reachable states in the weak bisimilarity classes, observing that p and q turn out to be bisimilar. Finally, find for every class a formula of Hennessy Milner logic which holds for the elements of the given class and not for any other element.

Exercise 5 (5)

A given DTMC consists of states $S = \{p_i\}_{i=0,\ldots,n} \cup \{q_i\}_{i=1,\ldots,n-1}$, $n \geq 2$ and of transitions:

$$p_i \xrightarrow{a_i} p_{i+1} \ \ i = 1, \ldots, n-1 \qquad p_i \xrightarrow{1-a_i} p_i \ \ i = 0, \ldots, n \qquad p_0 \xrightarrow{a_0/2} p_1 \qquad p_0 \xrightarrow{a_0/2} q_1 \qquad p_n \xrightarrow{a_n} p_0$$

$$q_i \xrightarrow{a_i} q_{i+1} \ \ i = 1, \ldots, n-2 \qquad q_i \xrightarrow{1-a_i} q_i \ \ i = 1, \ldots, n-1 \qquad q_{n-1} \xrightarrow{a_{n-1}} p_n.$$

Draw the DTMC, show for which parameter values the chain is ergodic, and prove that the relation $p_i \equiv q_i$ $i = 1 \ldots n-1$ is a bisimulation. Finally, find the steady state probabilities of all the states of the reduced DTMC.

Correzione Scritto del 10/02/2014 (1)

Esercizio

$$P(\varphi) \overset{def}{\iff} \varphi\sigma=\sigma' \Rightarrow \sigma x \geq 0 \wedge \sigma'=\sigma[{}^{\sigma x+\sigma y}/_y, {}^0/_x].$$

- P è inclusivo:

$\forall i\ \varphi_i \sigma=\sigma' \Rightarrow \sigma x \geq 0 \wedge \sigma'=\sigma[{}^{\sigma x+\sigma y}/_y, {}^0/_x]$ (A)

implica

$\underbrace{(\sqcup \varphi_i)\sigma=\sigma'}_{B} \Rightarrow \sigma x \geq 0 \wedge \sigma'=\sigma[{}^{\sigma x+\sigma y}/_y, {}^0/_x]$

Assumiamo (A) e (B). Ma se $(\sqcup \varphi_i)\sigma=\sigma'$ allora $\exists k.\ \varphi_k\sigma=\sigma'$. Quindi da (A) posso dedurre la tesi.

Utilizziamo l'induzione di Scott:

$$\frac{P(\perp_{\Sigma\to\Sigma_\perp}) \wedge \forall\varphi.\ P(\varphi) \Rightarrow P(\Gamma\varphi)}{P(fix\ \Gamma)}$$

$\mathscr{C}[\![w]\!] = fix\ \Gamma$

$\Gamma\varphi\sigma = \sigma x \neq 0 \to \varphi\sigma[{}^{\sigma x-1}/_x, {}^{\sigma y+1}/_y], \sigma$

- $P(\perp_{\Sigma\to\Sigma_\perp}) \overset{def}{\iff} \perp\sigma=\sigma' \Rightarrow \ldots$ vero essendo la premessa falsa

- Assumiamo $P(\varphi)$

$$P(\varphi) \overset{def}{\iff} \varphi\sigma''=\sigma' \Rightarrow \sigma'' x \geq 0 \wedge \sigma'=\sigma''[{}^{\sigma'' x+\sigma'' y}/_y, {}^0/_x]$$

Dimostriamo $P(\sigma\varphi)$ (2)

$$P(\sigma\varphi) \stackrel{def}{=} \sigma x \neq 0 \rightarrow \varphi\, \sigma\left[{}^{\sigma x-1}/_x,\ {}^{\sigma y+1}/_y\right], \sigma = \sigma'$$

$$\Rightarrow \sigma x \geq 0 \ \wedge\ \sigma' = \sigma\left[{}^{\sigma x+\sigma y}/_y,\ {}^{0}/_x\right]$$

Assumiamo la premesse

$$\sigma x \neq 0 \rightarrow \varphi\, \sigma\left[{}^{\sigma x-1}/_x,\ {}^{\sigma y+1}/_y\right], \sigma = \sigma'$$

Resta da dimostrare $\sigma x \stackrel{?}{\geq} 0$ (A) e $\sigma' \stackrel{?}{=} \sigma\left[{}^{\sigma x+\sigma y}/_y,\ {}^{0}/_x\right]$ (B)

<u>Caso $\sigma x = 0$</u> $\sigma = \sigma'$ (A) e (B) ovvi

<u>Caso $\sigma x \neq 0$</u> $\varphi\, \sigma\left[{}^{\sigma x-1}/_x,\ {}^{\sigma y+1}/_y\right] = \sigma'$

si applica l'ipotesi induttiva con

$$\sigma'' = \sigma\left[{}^{\sigma x-1}/_x,\ {}^{\sigma y+1}/_y\right]$$

abbiamo

$\sigma'' x \geq 0$ cioè $\sigma x - 1 \geq 0$ $\sigma x \geq 1 \geq 0$ CVD (A)

$$\sigma' = \sigma\left[{}^{\sigma x-1}/_x,\ {}^{\sigma y+1}/_y\right]\left[{}^{\sigma x-1+\sigma y+1}/_y,\ {}^{0}/_x\right]$$

$$= \sigma\left[{}^{\sigma x+\sigma y}/_y,\ {}^{0}/_x\right] \quad \text{CVD (B)}$$

Esercizio 2

Proprietà riflessiva: $(n,m) \sqsubseteq (n,m)$ ovvio

Proprietà antisimmetrica: $(n_1,m_1) \sqsubseteq (n_2,m_2)$ e $(n_2,m_2) \sqsubseteq (n_1,m_1)$
$\Rightarrow n_1 = n_2$ e $m_1 = m_2$ ovvio

Proprietà transitiva: $(n_1,m_1) \sqsubseteq (n_2,m_2) \sqsubseteq (n_3,m_3) \Rightarrow (n_1,m_1) \sqsubseteq (n_3,m_3)$
ovvio

Bottom: $(0,0) \sqsubseteq (n,m)$ ovvio

$(0,0) \sqsubseteq (0,1) \sqsubseteq \dots$ ha come lub $(1,0)$, minimo di $\{(n,m)\}_{1 \le n,\, m \in \omega}$
che l'insieme dei maggioranti

$(0,0) \sqsubseteq (1,0) \sqsubseteq (2,0) \sqsubseteq \dots$ la catena $\{(n,0)\}_{n=0,1,\dots}$ non ha maggioranti,
quindi nemmeno lub.

In $\{[N] \times \omega\}$ la catena $(N,0) \sqsubseteq (N,1) \sqsubseteq \dots \{(N,m)\}_{m=0,1,\dots}$
non ha maggioranti.

In $\{[N] \times (\omega \cup \{\infty\})\}$ in una catena $\{(n_i,m_i)\}_{i=1,2}$ infinita
$\{n_i\}_{i=1,2,\dots}$ è finita. Sia $\bar{n}$ il suo limite.

Allora $(\bar{n},\infty)$ è il suo lub, dato che per ogni N deve
esistere $(\bar{n},N')$ nella catena con $N' > N$ essendo la catena
infinita. Il bottom è lo stesso, $(0,0)$.

La funzione $f(n,m) = 0$ e $f(n,\infty) = 0$
f è monotona e non continua:
Per la catena $\{(0,n)\}_{n=0,1,\dots}$ vale $f\left(\bigsqcup_{n\in\omega}(0,n)\right) = f(0,\infty) = \infty$

mentre $\bigsqcup_{n\in\omega} f(0,n) = \bigsqcup_{n\in\omega} 0 = 0$

Esercizio 3

$$P(t_1 \times t_2) \stackrel{def}{=} \quad t_1 \times t_2 \to c \quad t_1 \times t_2 \to c' \Rightarrow c = c'$$

Caso A

$$t_1 \times t_2 \to c \xleftarrow{c = c_1 \times c_2} t_1 \to c_1 \quad t_2 \to c_2 \quad \dots$$

$$t_1 \times t_2 \to c \xleftarrow{c' = c_1' \times c_2'} t_1 \to c_1' \quad t_2 \to c_2' \quad \dots$$

Abbiamo $c_1 = c_1'$ e $c_2 = c_2'$ per ipotesi induttiva

quindi $c = c_1 \times c_2 = c_1' \times c_2' = c'$

Caso B

$$t_1 \times t_2 \to c \xleftarrow{c = 0} t_1 \to 0$$

$$t_1 \times t_2 \to c' \xleftarrow{c' = 0} t_1 \to 0$$

immediato

Caso C

$$t_1 \times t_2 \to c \xleftarrow{c = c_1 \times c_2} t_1 \to c_1 \quad t_2 \to c_2$$

$$t_1 \times t_2 \to c' \xleftarrow{c' = 0} t_1 \to 0$$

Per ipotesi induttiva $c_1 = 0$, quindi $c = 0 \times c_2 = 0 = c'$.

Caso D

simmetrico

(5)

$$t = \underbrace{0}_{int} \times \underbrace{rec\, x.\, x}_{int} \; : int$$

$$\underbrace{int}\;\underbrace{int}$$

$$0 \times rec\, x. x \to c \xleftarrow{c=0} 0 \to 0 \leftarrow \square$$

$$[\![0 \times rec\, x.x]\!] \rho = \lfloor 0 \rfloor \underline{\times}_\perp fix\, \lambda d. d = \lfloor 0 \rfloor \underline{\times}_\perp \perp_{N_\perp} = \perp_{N_\perp}$$

$$[\![0]\!] \rho = \lfloor 0 \rfloor \neq \perp_{N_\perp}$$

$$[\![t_1 \times t_2]\!] \rho = Cond\left([\![t_1]\!], \lfloor 0 \rfloor, [\![t_1]\!] \rho \underline{\times}_\perp [\![t_2]\!] \rho \right)$$

$$P(t_1 \times t_2 \to c) \stackrel{def}{=} [\![t_1 \times t_2]\!] \rho = [\![c]\!] \rho$$

prime regola

$$\frac{t_1 \to n_1 \quad t_2 \to n_2}{t_1 \times t_2 \to n_1 \underline{\times} n_2} \qquad c = n_1 \underline{\times} n_2$$

$$[\![t_1 \times t_2]\!] \rho = Cond\left([\![t_1]\!] \rho, \lfloor 0 \rfloor, [\![t_1]\!] \rho \underline{\times}_\perp [\![t_2]\!] \rho \right)$$

$$= Cond\left(\lfloor n_1 \rfloor, \lfloor 0 \rfloor, \lfloor n_1 \underline{\times} n_2 \rfloor \right) = \lfloor n_1 \underline{\times} n_2 \rfloor$$

sia che $n_1 = 0$ che $n_1 \neq 0$ CVD

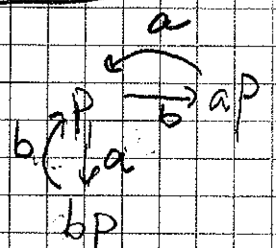
Seconda regola

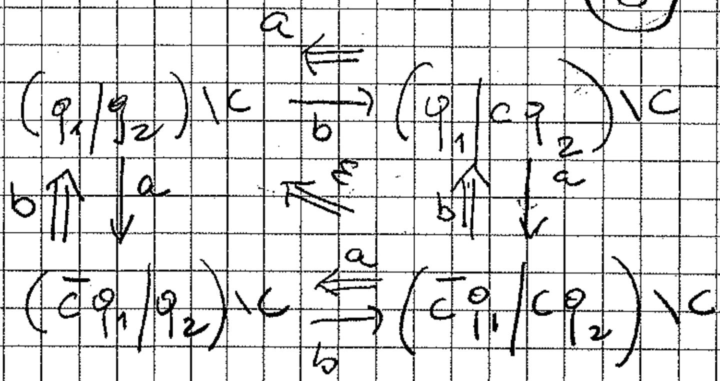
$$\frac{t_1 \to 0}{t_1 \times t_2 \to 0} \qquad c = 0$$

$$[\![t_1 \times t_2]\!] \rho = Cond\left(\lfloor 0 \rfloor, \lfloor 0 \rfloor, [\![t_1]\!] \rho \underline{\times}_\perp [\![t_2]\!] \rho \right) = \lfloor 0 \rfloor$$

CVD

Esercizio 4 (6)





anche $\stackrel{\varepsilon}{\Rightarrow}$ su ogni stato

$q = (q_1 | q_2)\backslash c$

$q_1 = \mathrm{rec}\, x.\, a \bar{c} x$

$q_2 = \mathrm{rec}\, x.\, b c x$

$R_0 = \{\{ p, bp, ap, q, (\bar{c} q_1 | q_2)\backslash c, (q_1 | c q_2)\backslash c, (\bar{c} q_1 | c q_2)\backslash c \}\}$

$R_1 = \{\{ p, q, (\bar{c} q_1 | c q_2)\backslash c \}, \{ ap, (q_1 | c q_2)\backslash c \}, \{ bp, (\bar{c} q_1 | q_2)\backslash c \}\}$

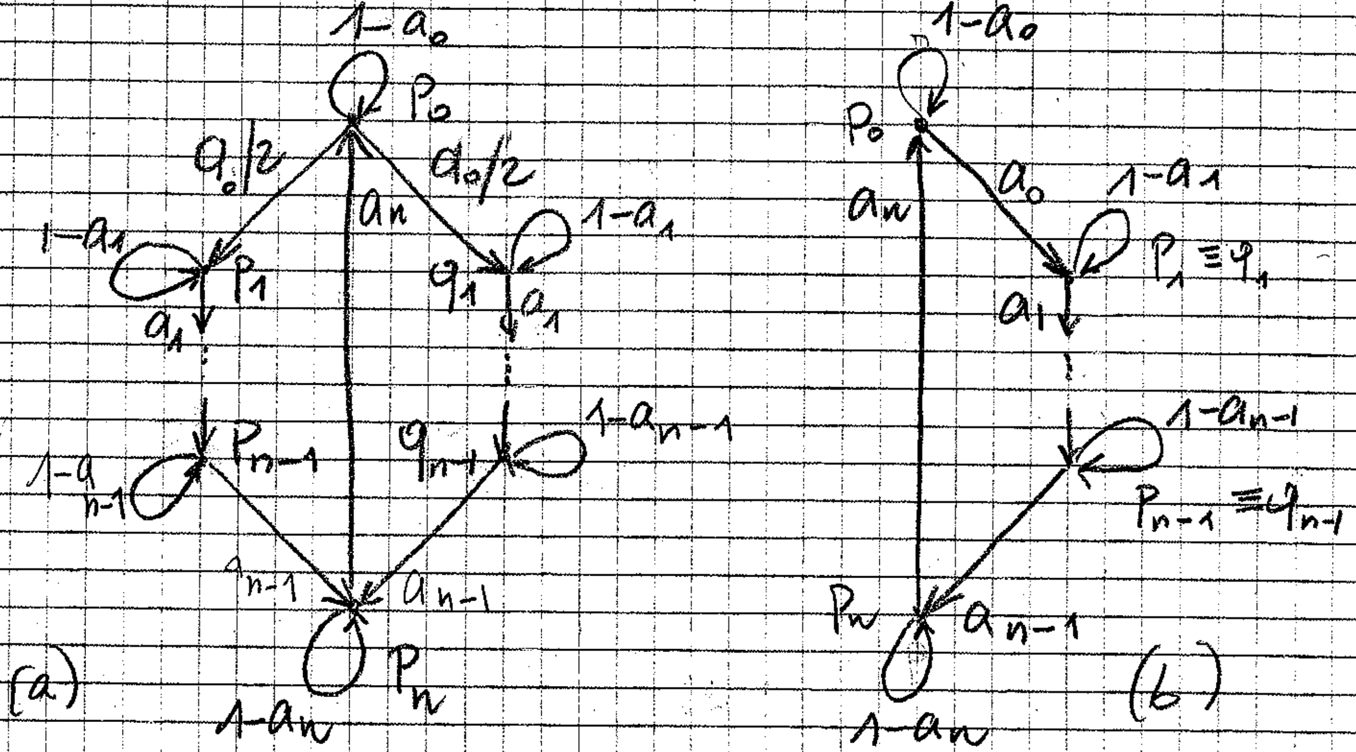
$R_2 = R_1$

$p, q, (\bar{c} q_1 | c q_2)\backslash c \models \langle a \rangle \mathrm{true} \wedge \langle b \rangle \mathrm{true}$

$ap, (q_1 | c q_2)\backslash c \models \langle a \rangle \mathrm{true} \wedge [b]\, \mathrm{false}$

$bp, (\bar{c} q_1 | q_2)\backslash c \models \langle b \rangle \mathrm{true} \wedge [a]\, \mathrm{false}$

Exercise 5



- To be ergodic we must have $a_i \neq 0$, $i = 0, \dots, n$ to guarantee reachability and it shold exist i with $a_i \neq 1$, otherwise all paths are of length $n+1$
- We have $\alpha(p_i)\{p_{i+1}, q_{i+1}\} = \alpha(q_i)\{p_{i+1}, q_{i+1}\} = a_i$ $\quad i = 1, \dots, n-1$

  Thus $\{\{p_0\}, \{p_1, q_1\}, \dots \{p_{n-1}, q_{n-1}\}, \{p_n\}\}$ is a bisimulation
- The reduced LTS is in Fig (b) above.

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The steady state equations are

$$a_n P_n + (1-a_0) P_0 = P_0 \qquad a_n P_n = a_0 P_0$$

$$a_{i-1} P_{i-1} + (1-a_i) P_i = P_i \qquad a_{i-1} P_{i-1} = a_i P_i \qquad i=1,\dots,n$$

$$\sum_{i=0,\dots,n} P_i = 1$$

$$P_0 + \frac{a_0}{a_1} P_0 + \dots + \frac{a_0}{a_n} P_0 = 1 \qquad P_0 = \frac{1}{1+\frac{a_0}{a_1}+\dots+\frac{a_0}{a_n}}$$

$$P_0 = \frac{1}{a_0} \frac{1}{\frac{1}{a_0}+\frac{1}{a_1}+\dots+\frac{1}{a_n}}$$

$$\boxed{P_i = \frac{1}{a_i} \frac{1}{\frac{1}{a_0}+\frac{1}{a_1}+\dots+\frac{1}{a_n}}} \qquad i=0,\dots,n$$