Varieties of residuated lattices with an MV-retract
and an investigation into state theory

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To Franco Montagna, in his inspiring memory.
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Abstract

In the shade of the algebraic study of substructural logics, we deal with the variety of residuated lattices. In particular, in the first part of this thesis we will introduce new ways of constructing bounded (commutative, integral) residuated lattices, generalizing in this context, among other things, the notion of disconnected and connected rotation. The class of structures that can be described by this approach is rather large. Some of the generated varieties are well-known algebraic semantics of substructural logics, such as product algebras, Gödel algebras, the variety generated by perfect MV-algebras, NM-algebras, $n$-potent BL-algebras, Stonean residuated lattices, pseudocomplemented MTL-algebras. Furthermore, starting from any variety of (commutative, integral) residuated lattices, via our construction we generate and characterize new varieties of bounded (commutative, integral) residuated lattices, that will hence correspond to new substructural logics. Such varieties will result in having a retraction testified by a term into an MV-algebra, or, as a special case and starting point of our investigation, a Boolean algebra.

Moreover, we will give a categorical representation of such generated varieties by means of categories whose objects are triples made of an MV-algebra (or, as a special case, a Boolean algebra), a residuated lattice and an operator intuitively representing the algebraic join of MV-elements (or Boolean elements) and the elements of the residuated lattice. As a corollary, we will obtain categorical equivalences between the classes of involutive algebras generated by our construction from one side, and the ones generated by liftings of residuated lattices from the other side.

Subsequently, we focus on prime filters for the algebras with a Boolean retract. First, we directly exhibit a (weak) Boolean product representation for the algebras our varieties, and then we study the posets of prime lattice filters. For the latter, given any algebra in our varieties, we show an order isomorphism between its poset of prime lattice filters and a structure constructed from the ultrafilters of the Boolean skeleton and the prime lattice filters of the radical of the algebra that aims at dualizing our triple construction.

The second part of the thesis is concerned with an investigation into the theory of states for some of the structures studied in the first part. In particular, our first main contribution with respect to state theory
consists in introducing and studying states for product logic. We axiomatize a notion of state that results in characterizing Lebesgue integrals of truth-functions of product logic formulas with respect to regular Borel probability measures. We prove that the relation between our states and regular Borel probability measures is one-one. Moreover, and interestingly, we prove that every state belongs to the convex closure of product logic valuations.

Then, we will use the algebraic decomposition theorems proved in the first part of the thesis to define states on some interesting subvarieties of MTL-algebras. In particular, we will define a notion of hyperreal-valued state (or hyperstate) of perfect MV-algebras, for which Mundici’s notion trivializes to only one possible state taking just Boolean values, 0 and 1. Such a notion will be generalized to the variety generated by involutive perfect MTL-algebras, which is the variety of involutive MTL-algebras satisfying Di Nola and Lettieri equation $2x^2 = (2x)^2$. In order to do so, we will also define a notion of state of GMTL-algebras (unbounded MTL-algebras). For doing so, we will go through the intuition that any lattice-ordered monoid has an homomorphism into an ℓ-group, using Grothendieck well-known construction.
Contents

Introduction 9

1 Preliminary algebraic notions 18
  1.1 Basic notions of Universal Algebra ................. 18
  1.2 Residuated lattices and hoops ................. 23
  1.3 Basic notions of Category Theory ................. 33

I Varieties of residuated lattices with an MV-retraction term 36

2 Disconnected $\delta$-rotations of residuated lattices 37
  2.1 Disconnected $\delta$-rotation ..................... 37
  2.2 rDL-algebras ..................................... 43
  2.3 RL-triplets and rDL-algebras .................. 52
  2.4 Categorical equivalences ......................... 62
  2.5 A general construction to build rDL-algebras .......... 70
  2.6 Chen and Grätzer’s triple construction ............ 77

3 Boolean products and dual triple construction 83
  3.1 rDL-algebras and (weak) Boolean products .......... 84
  3.2 Filter pairs ........................................ 86
  3.3 Prime filters fixed by Boolean ultrafilters .......... 89
  3.4 An order isomorphism for the poset of prime $\ell$-filters .... 94
4 Generalized $\delta$-rotations of residuated lattices 101
4.1 Generalized $\delta$-rotation .................................. 101
4.2 Varieties generated by generalized $\delta$-rotations .......... 108
4.3 Quadruple-representation for MVR$_n$-algebras ................. 115
4.4 Categorical equivalences for MVR$_n$-algebras .............. 125

II Theory of states 134

5 States of residuated structures 135
5.1 Preliminaries on states: a probability theory for many-valued logics ......................................................... 135

6 States of the free $n$-generated product algebra 143
6.1 Product algebras and product functions ...................... 143
6.2 States of free product algebras ................................ 148
6.3 Integral representation .......................................... 155
6.4 The state space and its extremal points .................... 165

7 Towards a notion of states for sIDL-algebras 171
7.1 From $\ell$-monoids to $\ell$-groups and hoops .................. 171
7.2 States of GMTL-algebras ...................................... 174
7.3 States of sIDL-algebras and their representation .......... 177

Conclusions and future work 181

Bibliography 185
Introduction

Mathematical Logic studies correct reasoning. It deals with propositions and the relation of consequence among them. Different logics diverge in their definition of sentences and notions of consequence. But the idea around which every logic evolves is the concept of truth, and the way it spreads through the propositions. Classical logic considers truth as an absolute. However, as there is not a unique way of interpreting reality, or the concept of truth itself, a variety of different logics naturally arises, and during the 20th century the study of non-classical logics has become extensive.

Substructural logics and residuated lattices

Substructural logics, for which we refer to [66], are the background of our investigation and include most of the interesting nonclassical logics. Among them, intuitionistic logic [27], many-valued logics [68, 72], fuzzy logics [74, 53], relevance logics [9, 10], linear logic [69], besides including classical logic as a limit case. When formulated as a Gentzen-style system, in which instead of the axiomatic approach of Hilbert-style calculi we have a system of deduction rules, classical logic is characterized by the structural rules of contraction, weakening and exchange. Substructural logics were introduced as logics which lack some (or none as a special case) of these three classical structural rules. The general study of substructural logics, meant as axiomatic extensions of the Full Lambek Calculus system FL, in which no structural rule holds, has hence an interesting comparative potential. In particular, semantical methods provide a powerful tool for analyzing substructural logics from a uniform perspective. The key observation is that these logics all share the residuation
property. Logically speaking, it means that a formula $\gamma$ follows from formulas $\alpha$ and $\beta$ if and only if the implication $\alpha \rightarrow \gamma$ follows from $\beta$. In algebraic models, this can be expressed with the following:

\[ x \cdot y \leq z \text{ iff } y \leq x \rightarrow z, \]

which is known as the law of residuation in ordered structures. Indeed, it turns out that the equivalent algebraic semantics of substructural logics, in the sense of Blok and Pigozzi [20], is the variety of residuated lattices with an extra constant 0, i.e. FL-algebras, and the lattice of axiomatic extensions of FL is dually isomorphic to the lattice of subvarieties of FL-algebras (see [66, Theorem 2.29]). Residuated lattices, as we shall see in detail, are ordered structures together with a monoidal operation that has a residuum whose relation with the order is given by the residuation law. They were first introduced by Ward and Dilworth in 1939 [113], as the main tool in the abstract study of ideal theory in rings. Under a more general definition, residuated lattices have been deeply studied in recent years: examples of such structures include Boolean algebras, Heyting algebras, MV-algebras, and lattice-ordered groups.

Historically, the use of algebra in logic starts with Boole, who in [24, 25] creates what we recognize as Boolean algebras. However, it is Lindenbaum and Tarski’s construction, which is now known as a Lindenbaum-Tarski algebra [110] (or just Lindenbaum algebra), that explains precisely the connection between propositions and algebraic terms. Algebraic investigations into logic were deepened after the rise of nonclassical logics, in order to give them semantics (as for instance Heyting algebras for intuitionistic logic, or MV-algebras for Lukasiewicz logic). In particular, in 1989 Blok and Pigozzi summarized their work on the algebraizability of logics in [20], where they actually gave birth to what is now called abstract algebraic logic, making the algebraic study fundamental for the understanding of non-classical logics.

**Fuzzy logics**

As we have observed, residuated lattices encompass the algebraic semantics of many substructural logics. In particular, they include the algebraic semantics
of fuzzy logics ([74],[53]), which are the setting that mostly inspired the present work and we are therefore going to present in full generality.

Fuzzy logics, inside of the wider context of many-valued logics, are a well-established framework arising to deal with vagueness, or fuzziness. They can indeed be seen as a way of tackling sorites paradoxes where a sufficient number of applications of a legitimate deduction rule (modus ponens) leads from reasonably true premises, to a clearly false conclusion. For example, consider a man sitting on the shore of the Atlantic ocean, removing water one cup at a time. The removal of a single cup of water does not prevent the Atlantic from being legitimately called an ocean. However, being its water in a finite amount, iterating this process will, at some point, result in only a few cups of water left. Would anyone call such a puddle an ocean? Reasonably not. The solution offered by many-valued logics is to give sentences intermediate degrees of truth. In our example, intuitively, the Atlantic becomes less and less an ocean after the removal of each cup.

In particular, in his monograph [74] of 1998, Hájek establishes the theoretical ground for what is referred to as Mathematical Fuzzy Logic. His approach consists in fixing the real unit interval $[0,1]$ as standard domain to evaluate atomic formulas, while the evaluation of compound sentences only depends on the chosen operation which provides the semantics for the so called strong conjunction connective. His general approach to fuzzy logics is grounded on the observation that, if strong conjunction is interpreted by a continuous $t$-norm, then any other connective of a logic has a natural standard interpretation. A $t$-norm is a binary operation $*$ on $[0,1]$ satisfying the following conditions:

(i) $*$ is commutative and associative, i.e. for all $x, y, z \in [0,1]$:

\[
x * y = y * x,
\]

\[
(x * y) * z = x * (y * z).
\]

(ii) $*$ is non-decreasing in both arguments:

\[
x_1 \leq x_2 \text{ implies } x_1 * y \leq x_2 * y,
\]

\[
y_1 \leq y_2 \text{ implies } x * y_1 \leq x * y_2.
\]
(iii) \( 1 \ast x = x \) and \( 0 \ast x = 0 \) for all \( x \in [0,1] \).

Among continuous t-norms, the so called Lukasiewicz, Gödel and product t-norms play a fundamental role:

1. **Lukasiewicz t-norm**: \( x \ast y = \max(0, x + y - 1) \),

2. **Gödel t-norm**: \( x \ast y = \min(x, y) \),

3. **Product t-norm**: \( x \ast y = x \cdot y \) (product of reals).

Indeed, Mostert-Shields’ Theorem shows that a t-norm is continuous if and only if it can be built from the previous three ones by the construction of ordinal sum [82]. These three operations determine the three different algebraizable propositional logics (bringing the same names as their associated t-norms), whose equivalent algebraic semantics are the varieties of MV, Gödel and product algebras respectively. Hájek proves that given a continuous t-norm \( \ast \) there is a unique operation \( x \Rightarrow y \) satisfying the residuation condition with respect to \( \ast \), that is chosen as the semantics for implication. However in [53], published in 2001, Esteva and Godo proved that it is sufficient and necessary for the t-norm to be left-continuous in order to have a (unique) residuum. Thus, they introduced the monoidal t-norm based logic MTL, that was proved to be in fact complete with respect to the semantics given by all left-continuous t-norms by Jenei and Montagna in [80]. The algebraic semantics of MTL is given by the variety of MTL-algebras, that are *prelinear* bounded, integral, commutative residuated lattices.

**State theory**

As stated earlier, fuzzy logics constitute an approach to manage vagueness, a particular aspect of incomplete information. A different facet of partial knowledge is represented by *uncertainty*. Indeed, we may need to reason about sentences, or *events*, that are unknown at the present, but will assume a determined truth-value in the future. A well-known way to study reasoning under uncertainty is to examine it from the measure-theoretic point of view, for example using probability theory as a way to treat *degrees of belief*.
In the second part of this thesis we will hence approach a formal theory able to deal with the uncertainty of vagueness: the theory of states, as meant to represent the probability theory of many-valued events. Namely, real-life inspired events that when verified may take intermediate truth values, such as “Tomorrow there will be traffic”, “We will have a cold winter”. The first generalization of probability theory to the nonclassical settings of t-norm based fuzzy logics in the sense of Hájek, is due to Mundici who, in 1995, introduced the notion of state for the class of MV-algebras with the aim of capturing the notion of average degree of truth for a proposition in Lukasiewicz logic [94]. Such functions suitably generalize the classical notion of finitely additive probability measures on Boolean algebras, in addition to corresponding to convex combinations of Lukasiewicz valuations. MV-algebraic states have been deeply studied in recent years, as they enjoy several important properties and characterizations (see [60] for a survey). One of the most important results in that framework is Kroupa-Panti theorem [95, §10], a representation result showing that every state of an MV-algebra is the Lebesgue integral with respect to a regular Borel probability measure. Many attempts for defining suitable notions of state in different structures have been made (see again [60, §8] for a survey). In particular, in [7], the authors provide a definition of state for the Lindenbaum algebra of Gödel logic that corresponds to the integration of the $n$-place truth-functions corresponding to Gödel formulas, with respect to Borel probability measures on the real unit cube $[0,1]^n$. Moreover, such states are shown to correspond to convex combinations of finitely many truth-value assignments.

**Outline**

In the shade of the algebraic study of substructural logics, this thesis deals with the variety of residuated lattices, which, encompassing a multitude of different structures, is fairly complicated to study and understand. Thus the structural investigation of interesting subvarieties is an appealing problem to address.

In particular, in the first part of this thesis we will introduce new ways of
constructing bounded (commutative, integral) residuated lattices, generalizing
in this context, among other things, the notion of disconnected and connected
rotation introduced by Jenei [79]. The class of structures that can be de-
scribed by this approach is rather large. Some of the generated varieties are
well-known algebraic semantics of substructural logics. Among them, prod-
uct algebras, Gödel algebras, the variety generated by perfect MV-algebras,
NM-algebras, n-potent BL-algebras, Stonean residuated lattices, pseudocom-
plemented MTL-algebras. Furthermore, starting from any variety of (com-
mutative, integral) residuated lattices, via our construction we generate and
characterize new varieties of bounded (commutative, integral) residuated lat-
tices, that will hence correspond to new substructural logics. Interestingly,
such varieties will result in having a retraction, testified by a term, into an
MV-algebra, or, as a special case and starting point of our investigation, a
Boolean algebra. This approach indeed goes in the direction started by Cig-
noli and Torrens who in [42, 43] study the subvarieties of bounded residuated
lattices admitting a Boolean retraction term. Our work [35] is a first ap-
proach aiming at understanding some classes of algebras that have instead
an MV-retract, characterizing the directly indecomposable elements and the
generated varieties. Moreover, we will give a categorical representation of such
generated varieties by means of categories whose objects are triples made of
an MV-algebra (or, as a special case, a Boolean algebra), a residuated lattice
and an operator intuitively representing the algebraic join of MV-elements
(or Boolean elements) and the elements of the residuated lattice. As a corol-
lary, we will obtain categorical equivalences between the classes of involutive
algebras generated by our construction from one side, and those generated
by liftings of residuated lattices from the other side (e.g. the variety gen-
erated by perfect MV-algebras is categorically equivalent to the variety of
product algebras, the variety generated by perfect IMTL-algebras is categor-
ically equivalent to the variety of pseudocomplemented MTL-algebras and so
on). The aforementioned approach indeed generalizes, and is inspired by, the
method we used in [92] for characterizing the variety of product algebras, and
that we extended in [5] to a large subvariety of MTL-algebras with a Boolean
retraction term. Since the Boolean case of our construction lies in a quite
established setting, it deserved to be presented separately from the other, though more general one. Indeed, it also allows a more refined description of the generated varieties and interesting results concerning the space of prime filters. In particular, we first directly exhibit a (weak) Boolean product representation for the algebras of our varieties, and then we study the posets of prime lattice filters. For the latter, given any algebra in our varieties, we show an order isomorphism between its poset of prime lattice filters and a structure constructed from the ultrafilters of the Boolean skeleton and the prime lattice filters of the radical of the algebra that aims at dualizing our triple construction [64].

The second part of the thesis is concerned with an investigation into the theory of states for some of the structures studied in the first part. In particular, our first main contribution with respect to state theory consists in introducing and studying states for product logic, the remaining fundamental fuzzy logic for which such a notion is still lacking [59]. Focusing on the Lindenbaum algebra of product logic with $n$ variables, i.e. the free $n$-generated product algebra, and using its functional representation, we axiomatize a notion of state that results in characterizing Lebesgue integrals of truth-functions of product logic formulas with respect to regular Borel probability measures on $[0,1]^n$. We prove indeed that the relation between our states and regular Borel probability measures is one-one. Moreover, and interestingly, we prove that every state belongs to the convex closure of product logic valuations.

Then, taking inspiration from how Mundici’s well known categorical equivalence between MV-algebras and unital $\ell$-groups lifts to states of corresponding structures (states of $\ell$-groups where indeed introduced by Goodearl in [71]), we will use the algebraic decomposition theorems proved in the first part of the thesis to define states on some interesting subvarieties of MTL-algebras [61]. In particular, we will define a notion of hyperreal-valued state (or hyperstate) of perfect MV-algebras, for which Mundici’s notion trivializes to only one possible state taking just Boolean values, 0 and 1. Such a notion will be generalized to the variety generated by perfect IMTL-algebras, which is the variety of involutive MTL-algebras satisfying Di Nola and Lettieri equation $2x^2 = (2x)^2$. In order to do so, we will also define a notion of
state of GMTL-algebras (unbounded MTL-algebras). For doing so, we will go through the intuition that any lattice-ordered monoid has a homomorphism into an \( \ell \)-group, using Grothendieck’s well-known construction.

More precisely, the thesis is organized as follows.

- Chapter 1, where we recall preliminary algebraic notions, in order to make the manuscript as self-contained as possible. In particular we deal with the basics of Universal Algebra, residuated lattices theory and some concepts of Category Theory.

Subsequently, we have Part I, which consists of:

- Chapter 2, where we introduce and study the notion of disconnected \( \delta \)-rotations of residuated lattices, where \( \delta \) is a particular nucleus operator, that result in generating varieties of bounded residuated lattices with a Boolean retraction term. We study the triple-representation for these algebras and prove categorical equivalences results. In order to capture the whole variety of algebras characterized by our construction, which we named rDL, we introduce a category of quadruples and prove a categorical equivalence. In the end we also make a comparison with an analogue triple-construction developed by Chen and Grätzer for Stone algebras.

- Chapter 3, in which we focus on filters of rDL-algebras. Firstly, we show how a (weak) Boolean product representation can be proven directly with our construction. Secondly, we prove an order isomorphism for the poset of prime lattice filters, going in the direction of a dual triple construction.

- Chapter 4, where the results of Chapter 2 are generalized, and we introduce generalized \( \delta \) rotations. We study the generated varieties that admit an MV-retraction term, and prove categorical equivalences with respect to categories of quadruples.

Then, we have Part II outlined as follows:
- In Chapter 5, we give an overview of the theory of states of many-valued events, starting from states of $\ell$-groups, going through states of MV-algebras and of other residuated structures.

- In Chapter 6, we present our work on states of the free $n$-generated product algebra.

- In Chapter 7, via the extension of Grothendieck construction to ordered structures, we develop a theory of states for GMTL-algebras and then for so-called sIDL-algebras.
Chapter 1

Preliminary algebraic notions

We are now going to introduce the general notions needed in order to make this manuscript as self-contained as feasible, and establish the notation that will be used throughout the thesis. The more specific concepts useful for each chapter will be introduced when needed.

1.1 Basic notions of Universal Algebra

Assuming familiarity with basic set-theoretical concepts, in this section we will recall the basic notions of Universal Algebra. We shall refer to [29] for a more detailed exposition.

Given a (non-empty) set \( A \), a \( n \)-ary operation on \( A \) is any function \( f \) from \( A^n \) to \( A \); the map \( \sigma(f) = n \), that associates to a function symbol a natural number, is called the \textit{arity} of \( f \). The image of \((a_1, \ldots, a_n)\) under an \( n \)-ary operation \( f \) is denoted by \( f(a_1, \ldots, a_n) \). An algebraic \textit{type} is a pair \( \mathcal{F} = (F, \sigma) \) of a set of function symbols \( F \) together with an arity map \( \sigma : F \to \mathbb{N} \).

An \textit{algebra} of type \( \mathcal{F} \) is a pair \( \mathbf{A} = (A, \{f^A\}_{f \in F}) \) made of a domain set \( A \) and a family \( \{f^A\}_{f \in F} \) of operations \( f^A : A^{\sigma(f)} \to A \). We will refer to them as the \textit{fundamental} operations of \( \mathbf{A} \). The underlying set \( A \) is often called the \textit{universe} of the algebra. The superscripts of the operations will usually be omitted in the text, and we will often write the type of the algebra as the sequence \( \{\sigma(f^A)\}_{f \in F} \).

By a \textit{subalgebra} of \( \mathbf{A} \) we mean an algebra \( \mathbf{B} = (B, \{f^B\}_{f \in F}) \) where
Chapter 1. Preliminary algebraic notions

$B \subseteq A$, by $f^A \restriction_B$ we mean the restriction of $f^A$ to $B$, and $B$ is closed under the operations of $A$, i.e. $f^A(b_1, b_2, \ldots, b_{\sigma(f^A)}) \in B$, for all $b_1, \ldots, b_{\sigma(f^A)} \in B$. If $\mathcal{F}$ is a type and $G \subseteq F$, the $G$-reduct of an algebra of type $\mathcal{F}$, $A = (A, \{f^A\}_{f \in F})$, is the algebra $A^G$ with underlying set $A$ and operations $(f^A)_{f \in G}$.

Suppose that $A$ and $B$ are two algebras of the same type $\mathcal{F}$. A mapping $h : A \to B$ is called a homomorphism from $A$ to $B$ if

$$h(f^A(a_1, \ldots, a_n)) = f^B(h(a_1), \ldots, h(a_n))$$

for each $f$ of arity $n$ in $F$ and every $a_1, \ldots, a_n \in A$. Moreover, $h$ is an epimorphism if it is surjective, monomorphism (or embedding) if it is injective, and an isomorphism if it is bijective. Notice that injective maps may be called one-one, but being in one-one correspondence means that there is a bijection. Whenever there is an isomorphism $h : A \to B$ we will write $A \cong B$.

A congruence relation $\theta$ on an algebra $A$ of type $\mathcal{F}$ is an equivalence relation that is compatible with the operations of $A$, meaning that given any $f \in F$, if $a_1 \theta b_1, \ldots, a_{\sigma(f)} \theta b_{\sigma(f)}$ for some $a_1, \ldots, a_{\sigma(f)}, b_1, \ldots, b_{\sigma(f)} \in A$, then $f(a_1, \ldots, a_{\sigma(f)}) \theta f(b_1, \ldots, b_{\sigma(f)})$. The congruence generated by a set $X$ of pairs of elements of $A$ is the least congruence relation $\Theta(X)$ containing $X$. The congruence generated by a single element is called principal. The quotient algebra of $A$ by $\theta$, written $A/\theta$, is the algebra whose universe is the set of equivalence classes $[a]_{\theta} = \{x \in A : a \theta x\}$, with $a \in A$, and whose fundamental operations satisfy

$$f^A/\theta(a_1/\theta, \ldots, a_{\sigma(f)}/\theta) = f^A(a_1, \ldots, a_{\sigma(f)})/\theta$$

where $a_1, \ldots, a_{\sigma(f)} \in A$ and $f \in F$.

If $\{A_i : i \in I\}$ is a family of algebras of the same type, we define the direct product algebra $\prod_{i \in I} A_i$, with universe the Cartesian product of the universes $A_i$, and fundamental operations defined by the following stipulation:

$$f^{\prod}(\langle a_{i1} \rangle_{i \in I}, \ldots, \langle a_{i\sigma(f)} \rangle_{i \in I}) = \langle f^{A_i}(a_{i1}, \ldots, a_{i\sigma(f)}) \rangle_{i \in I},$$

for all $a_{ij} \in A_i$, $i \in I$ and $j \in \{1, \ldots, \sigma(f)\}$. An algebra $A$ is said directly indecomposable if for every isomorphism $\alpha : A \to \prod_{i \in I} A_i$ there exists $i \in I$...
such that $A$ is isomorphic to $A_i$. A subdirect product of an indexed set $\{A_i\}_{i \in I}$ of algebras of type $\mathcal{F}$, is a subalgebra $A$ of the direct product $\prod_{i \in I} A_i$, such that the projection to the i-th coordinate map from $A$ to $A_i$ is onto. The embedding $\alpha : A \to \prod_{i \in I} A_i$ is said to be the subdirect representation of $A$. An algebra $A$ is said subdirectly irreducible if for every subdirect representation $\alpha : A \to \prod_{i \in I} A_i$ there exists $i \in I$ such that $A$ is isomorphic to $A_i$.

A class of algebras of the same type is called a variety if it is closed under homomorphic images, subalgebras and direct products. We shall refer to the variety generated by a class of algebras $K$ as $\mathcal{V}(K)$. Let now $\mathcal{H}(K), \mathcal{S}(K)$ and $\mathcal{P}(K)$ denote respectively the classes of homomorphic images, subalgebras and direct products of algebras in $K$, then the following holds.

**Theorem 1.1.1** (Tarski, [109]). For every class of algebras $K$, $\mathcal{V}(K) = \mathcal{HSP}(K)$.

This means that, in particular, $\mathcal{S}\mathcal{H}(K) \subseteq \mathcal{HSP}(K)$, $\mathcal{S}\mathcal{S}(K) \subseteq \mathcal{S}\mathcal{P}(K)$ and $\mathcal{S}\mathcal{H}(K) \subseteq \mathcal{S}\mathcal{P}(K)$. The following is a well-known and fundamental result by Birkhoff.

**Theorem 1.1.2** (Birkhoff). Every algebra $A$ of type $\mathcal{F}$ is isomorphic to a subdirect product of subdirectly irreducible algebras of the same type (which are homomorphic images of $A$).

This clearly implies that every member of a variety $V$ is isomorphic to a subdirect product of subdirectly irreducible members of $V$. In this sense, subdirectly irreducible algebras do form the building blocks of a variety. Moreover, another fundamental Birkhoff’s result states that varieties coincide with classes of models of equational theories (see [29], Theorem 11.9). In the following, we are going to clarify what this means.

Let $X$ be a set of variables, $\mathcal{F}$ a type and $(X \cup F)^*$ the set of all finite sequences of elements of $X \cup F$. The set $T_\mathcal{F}(X)$ of terms in $\mathcal{F}$ over $X$ is the least subset of $(X \cup F)^*$ that contains $X$ and if $f \in F$ and $t_1, t_2, \ldots, t_{\sigma(f)} \in T_\mathcal{F}(X)$, then the sequence $ft_1t_2\ldots t_{\sigma(f)} \in T_\mathcal{F}(X)$. The term algebra $T_\mathcal{F}(X)$ is the algebra with underlying set $T_\mathcal{F}(X)$, type $\mathcal{F}$ and operations $f^{T_\mathcal{F}(X)}$, for $f \in F$, defined by $f^{T_\mathcal{F}(X)}(t_1, t_2, \ldots, t_{\sigma(f)}) = ft_1t_2\ldots t_{\sigma(f)}$, for all $t_i \in T_\mathcal{F}(X)$. 


If $A$ is an algebra of type $T$, $t$ a term in $T$ over a set of variables $X$ and the variables occurring in $t \ Var(t) = \{x_1, x_2, \ldots, x_n\}$, we define the term operation $t^A$ of $t$ inductively on the sub-terms of $t$ to be the operation defined as follows: $x_i^A$ is the $i$-th projection operation on $A^n$, and if $ft_1t_2\ldots t_{\sigma(f)}$, where $f \in F$ and $t_1, t_2, \ldots, t_{\sigma(f)} \in T_F(X)$, then $s^A$ is defined by $s^A(a_1, a_2, \ldots, a_n) = f^A(t_1^A(a_1, a_2, \ldots, a_n), t_2^A(a_1, a_2, \ldots, a_n), \ldots, t_{\sigma(f)}^A(a_1, a_2, \ldots, a_n))$. If $t_1, t_2, \ldots, t_n$ are terms of $T_F(X)$ and $n = |\ Var(t)|$, then the substitution of $t_1, t_2, \ldots, t_n$ into $t$ is the element $t^{T_F(X)}(t_1, t_2, \ldots, t_n)$. If $A$ is an algebra of type $T$ and $t$ a term in $T$, then the operation $t^A$ is called a term operation. Two algebras of possibly different types are called term equivalent if every operation of one is a term operation of the other.

An equation, or identity, of type $T$ over a set of variables $X$ is a pair of terms of $T_F(X)$. If $t, s$ are terms we write $t = s$ for the equation they define, instead of $(t, s)$. We say that an equation $t = s$ in $T$ over $X$ is valid in an algebra $A$ of type $T$, or it is satisfied by $A$, in symbols $A \models t = s$, if $t^A = s^A$. The notion of validity is extended to classes of algebras and sets of equations. A set $\mathcal{E}$ of equations in a type $T$ is said to be valid in, or satisfied by a class $K$ of algebras of type $T$, in symbols $K \models \mathcal{E}$, if every equation of $\mathcal{E}$ is valid in every algebra of $K$. Equations are preserved by subalgebras, homomorphic images and direct product. A theory of equations, or equational theory $T$ in a type $T$ is a congruence on $T_F(X)$ closed under substitutions, i.e., if $(t = s) \in T, \ Var(t) \cup \ Var(s) = \{x_1, \ldots, x_n\}$, and $t_1, \ldots, t_n \in T_F(X)$, then $(t^{T_F(X)}(t_1, \ldots, t_n) = s^{T_F(X)}(t_1, \ldots, t_n)) \in T$. It is easy to see that if $K$ is a class of algebras of type $T$, then $Th_{Eq}(K) = \{(t = s) \in T_F(X) : K \models t = s\}$ is an equational theory, called the equational theory of $K$. Given a set $\mathcal{E}$ of equations of a similarity type $T$ the equational class axiomatized by $\mathcal{E}$ is defined to be the class $Mod(\mathcal{E}) = \{A : A \models \mathcal{E}\}$ of algebras of type $T$, that satisfy all equations of $\mathcal{E}$; the set $\mathcal{E}$ is called an equational basis for $Mod(\mathcal{E})$. By previous observations, every variety is an equational class. More precisely, the following well-known theorem holds.

**Theorem 1.1.3** (Birkhoff, [13]). For every class of algebras $K$,

$$KSP(K) = Mod(Th_{Eq}(K)).$$

21
Chapter 1. Preliminary algebraic notions

Thus \( K \) is a variety iff it is the class of models of an equational theory.

Remark 1.1.4. By Theorem 1.1.2, any algebra \( A \) in a variety \( V \) is a subdirect product of subdirectly irreducible algebras in \( V, \{ A_i \}_{i \in I} \). Since subdirect products correspond to subalgebras of direct products, there is a monomorphism from \( A \) into the direct product of algebras in \( \{ A_i \}_{i \in I} \). In other words, \( A \) is isomorphic to a subalgebra of \( \prod_{i \in I} A_i \). Thus, since equations are preserved by subalgebras and direct products, if an identity holds for every subdirectly irreducible algebra in \( V \), it holds for any algebra of the variety. Moreover, since every subdirectly irreducible algebra is also by definition directly indecomposable, proving that an identity holds for every directly indecomposable algebra in \( V \) implies that it is satisfied by every algebra of the variety.

If \( K \) is a class of algebras of similarity type \( \mathcal{F} \), then the quotient algebra \( F_K(X) = \text{Th}_{\mathcal{F}}(X)/\text{Th}_{\mathcal{E}_q}(K) \) is called the free algebra for \( K \) over \( X \). Given a class of algebras \( K \) of type \( \mathcal{F} \), an algebra \( F_K(X) \in K \) is said to be free in \( K \) over \( X \) if it generated by \( X \) and it has the following universal mapping property: every map from \( X \) to an algebra \( A \) of \( K \) can be extended, in a unique way, to a homomorphism from \( F_K(X) \) to \( A \). Let \( F_1(X_2) \) and \( F_2(X_2) \) be free for \( K \) over \( X_1 \) and \( X_2 \) respectively, then if \( X_1 \) and \( X_2 \) have the same cardinality, \( F_1(X_2) \) is isomorphic to \( F_2(X_2) \). Thus, we can speak about the free algebra \( F_K(n) \) in \( K \) over \( \kappa \) generators, with \( \kappa \) any cardinal. Interestingly, every algebra in a variety is a homomorphic image of a free algebra, more precisely:

**Theorem 1.1.5** (Birkhoff, [29] Corollary 10.11). Given a class of algebras \( K \) of type \( \mathcal{F} \), and an algebra \( A \in K \), then for sufficiently large \( \kappa \), \( A \) is an homomorphic image of \( F_K(\kappa) \).

This implies that free algebras generate the variety, and hence their study is fundamental. We now state another important result that will be used in the investigations contained in this thesis. Given a variety \( V \) we shall say that an algebra \( A \) in \( V \) is generic for \( V \) if \( \mathcal{H}SP(A) = V \), i.e. if \( A \) generates the whole variety. It can be proved that the \( n \)-generated free algebra in \( V \) can be seen as an algebra of functions from \( A^n \) to \( A \) [4]. More precisely:
Chapter 1. Preliminary algebraic notions

**Theorem 1.1.6.** Let $A$ be generic for a variety $V$. Then, for each cardinal $\kappa$ the free algebra in $V$ over $\kappa$ generators is isomorphic to the subalgebra of $A^{\kappa}$ generated by the projections $\pi_\alpha : (a_1, a_2, \ldots, a_\alpha, \ldots) \mapsto a_\alpha$, for $\alpha \leq \kappa$.

1.2 Residuated lattices and hoops

For a deeper analysis of what will be presented in this section we refer the reader to [29], [66], [53], [54].

A lattice $L$ is an algebra $L = (L, \wedge, \vee)$ of type $(2, 2)$, where $\wedge, \vee$ are commutative, associative and idempotent operations for which the absorption laws hold:

$$x = x \vee (x \wedge y), \quad x = x \wedge (x \vee y).$$

Algebras $(L, \wedge)$, $(L, \vee)$ where $\wedge, \vee$ are commutative, associative and idempotent will be called respectively $\wedge, \vee$-semilattices. Lattices correspond to partially ordered sets (or posets) $P = (P, \leq)$ such that for every $x, y \in P$ both $\sup \{x, y\}$ and $\inf \{x, y\}$ exist in $P$. Indeed, given such a poset $(P, \leq)$ we can define $\wedge_P, \vee_P$ by $x \wedge_P y = \inf \{x, y\}, \ x \vee_P y = \sup \{x, y\}$. Viceversa, given a lattice $(L, \wedge, \vee)$, we can define the poset $(L, \leq_L)$ where $x \leq_L y$ iff $x \wedge y = x$ iff $x \vee y = y$.

A commutative residuated lattice is an algebra $A = (A, \wedge, \vee, \cdot, \rightarrow, 1)$ of type $(2, 2, 2, 2, 0)$ where:

(RL1) $(A, \wedge, \vee)$ is a lattice;

(RL2) $(A, \cdot, 1)$ is a commutative monoid (i.e. $\cdot$ is associative, with unit 1);

(RL3) Residuation law holds: for every $x, y, z \in A$,

$$x \cdot y \leq z \iff y \leq x \rightarrow z$$

Residuation law can equivalently be expressed by means of equations:

$$y = y \wedge (x \rightarrow ((x \cdot y) \vee z)), \ x \cdot (y \wedge (x \rightarrow z)) \vee z = z.$$
Hence, since clearly lattices and monoids are equationally definable, commutative residuated lattices form a variety. A commutative residuated lattice is:

- **pointed** if it has an extra constant $c$.
- **bounded** if it has a least element 0 and a greatest element $u$.
- **integral** if the greatest element coincides with the monoidal unit, $u = 1$.

When speaking of pointed and bounded residuated lattices, we mean that $c = 0$. Notice that via residuation, 0 is also a nullary element for the multiplication in any bounded commutative residuated lattice. We collect in the following propositions some well-known properties of residuated lattices that will be useful in this manuscript.

**Proposition 1.2.1** ([43], [22], [66]). The following equations hold in every commutative residuated lattice $L$.

1. $a \leq b$ if and only if $a \rightarrow b = 1$,
2. $1 \rightarrow a = a$,
3. $(a \rightarrow b) \rightarrow ((b \rightarrow c) \rightarrow (a \rightarrow c)) = 1$,
4. $(a \land b) \rightarrow c = (a \rightarrow c) \land (b \rightarrow c)$,
5. $(a \cdot b) \rightarrow c = a \rightarrow (b \rightarrow c)$,
6. $(a \lor b) \rightarrow c = (a \rightarrow c) \lor (b \rightarrow c)$,
7. $a \cdot (b \lor c) = (a \cdot b) \lor (a \cdot c)$,
8. $a \rightarrow (b \land c) = (a \rightarrow b) \land (a \rightarrow c)$.

If $L$ is also pointed, we can define $\neg x := x \rightarrow c$ and the following properties hold:

9. $x \leq y$ implies $\neg y \leq \neg x$,
10. $x \leq \neg \neg x$, 

24
Chapter 1. Preliminary algebraic notions

\[ x_i \rightarrow \neg y = y \rightarrow \neg x, \]

\[ x_i \rightarrow \neg y = \neg \neg x \rightarrow \neg y. \]

Proposition 1.2.2 ([84]). A residuated lattice is directly indecomposable iff its maximum Boolean subalgebra is the 2-element Boolean algebra 2.

Notation 1.2.3. (i) For the sake of a lighter notation, in all the following we will call a residuated lattice, or RL, any commutative integral residuated lattice \( H = (H, \land, \lor, \cdot, \rightarrow, 1) \). Integrality can be expressed equationally, hence residuated lattices form a variety that we will denote with RL. We will also call bounded residuated lattice, or BRL, any bounded commutative integral pointed residuated lattice \( A = (A, \land, \lor, \cdot, \rightarrow, 0, 1) \). We will denote the corresponding variety as BRL. We will often refer to linearly ordered algebras as chains.

(ii) We can define further operations and abbreviations in the following manner: \( \neg x := x \rightarrow 0 \), \( x \oplus y := \neg (\neg x \cdot \neg y) \), \( nx := x \oplus \ldots \oplus x \) (\( n \) times), \( x^n := x \cdot \ldots \cdot x \) (\( n \) times).

We shall now introduce other classes of structures that will be of interest in this thesis.

An MTL-algebra \( A = (A, \land, \lor, \cdot, \rightarrow, 0, 1) \) is a BRL that satisfies prelinearity equation:

\[ (x \rightarrow y) \lor (y \rightarrow x) = 1 \]

MTL-algebras form a variety, MTL, and constitute the equivalent algebraic semantics of monoidal t-norm based logic MTL introduced by Esteva and Godo in [53], to which we refer. A GMTL-algebra is a RL satisfying prelinearity, i.e. it is an unbounded MTL-algebra, and we denote the variety by GMTL. Prelinearity equation characterizes the classes of BRLs and RLs that are semilinear, or representable, meaning that they are a subdirect product of chains. That is to say, subdirectly irreducible members of MTL and GMTL.
are linearly ordered.

A notable subvariety of MTL-algebras is given by the class of \textit{BL-algebras}, i.e. MTL-algebras that are divisible:

\[ x \cdot (x \rightarrow y) = x \land y. \]

BL-algebras are the algebraic semantics of Hájek Basic Logic [74], its most relevant subvarieties being the one of MV-algebras, Gödel algebras and product algebras.

An \textit{MV-algebra} is a BL-algebra that satisfies double negation law:

\[ \neg \neg x = x; \]

A \textit{Gödel algebra} is a BL-algebra that satisfies contraction:

\[ x \cdot x = x; \]

A \textit{product algebra} is a BL-algebra satisfying:

\[ \neg x \lor (x \rightarrow x \cdot y) \rightarrow y = 1. \]

BL-algebras, MV-algebras, Gödel algebras and product algebras form varieties that will be denoted respectively with \textit{BL}, \textit{MV}, \textit{G} and \textit{P}. In particular, \textit{MV}, \textit{G} and \textit{P} are generated by the standard algebras on \([0,1]\), that we denote respectively with \([0,1]_L, [0,1]_G\) and \([0,1]_\Pi\), determined by the corresponding t-norms: Lukasiewicz t-norm \( x \cdot_L y = \max(0, x + y - 1) \), Gödel t-norm \( x \cdot_G y = \min(x, y) \) and product t-norm \( x \cdot_\Pi y = x \cdot y \) (product between reals). Indeed, as we recalled in the Introduction, the choice of the t-norm univocally determines the residuum, and the lattice operations can be defined from them.

Boolean algebras, the algebraic semantics of Classical Logic, can be seen as the subvariety of MTL satisfying:

\[ a \lor \neg a = 1. \]

In particular, the variety \textit{B} of Boolean algebras result to be subvariety of all MV-algebras, Gödel algebras and product algebras (of which, in particular, is
the unique proper subvariety, see [49]). Heyting algebras, whose variety we denote by \( \text{HA} \), and that are the algebraic semantics of Intuitionistic Logic, can be regarded as non-prelinear Gödel algebras.

We shall now see other significant subvarieties of \( \text{RL} \) that will be object of our investigation.

**Stonean residuated lattices** are bounded residuated lattices satisfying:

\[
\neg x \lor \neg \neg x = 1.
\]

As a subvariety of \( \text{SRL} \), we will mention **pseudo-complemented MTL-algebras**, or \( \text{SMTL} \)-algebras, that are MTL-algebras satisfying:

\[
x \land \neg x = 0,
\]

whose variety is denoted by \( \text{SMTL} \) and include as subvarieties \( \text{P} \) and the variety of Stonean Heyting algebras \( \text{SHA} \) (Heyting algebras satisfying \( \neg x \lor \neg \neg x = 1 \)), that in turn has \( \text{G} \) as a subvariety.

**Involutive residuated lattices** are residuated lattices satisfying:

\[
\neg \neg x = x,
\]

whose variety we denote by \( \text{IRL} \). In particular, **involutive MTL-algebras** are prelinear involutive RLs. They constitute a variety, \( \text{IMTL} \), which includes MV-algebras as a subvariety.

**Hoops**

We shall now see how residuated lattices can be displayed inside the theory of **hoops**. Hoops were introduced by Büchi and Owens in the 1970s, in an unpublished work “Complemented Monoids and Hoops”, inspired by the work of Bosbach on naturally ordered monoids (see for instance [26]). More recently, Blok and Pigozzi retrieved their work to study **hoops with dual normal operators** [21], which are a generalization of Boolean algebras with operators,
but the first study of the structural properties of hoops is certainly to be found in Ferreirim PhD thesis [55]. It is also worth citing her works with Pigozzi [18] and [19], and the work by Ferreirim, Aglianò and Montagna [1], that established the deep connection of hoops with many-valued logics. Another useful reference we are going to follow is [54].

A *semihoop* ([54]) is an algebra $\mathbf{H} = (H, \cdot, \rightarrow, \wedge, 1)$ of type $\langle 2, 2, 2, 0 \rangle$ such that:

(S1) $(H, \wedge, 1)$ is a $\wedge$-semilattice with upper bound 1;
(S2) $(H, \cdot, 1)$ is a commutative monoid isotonic w.r.t. the $\wedge$-semilattice order;
(S3) it is naturally ordered: for every $a, b \in H$, $a \leq b$ iff $a \rightarrow b = 1$;
(S4) $(x \cdot y) \rightarrow z = x \rightarrow (y \rightarrow z)$.

A *prelinear semihoop* is a semihoop $\mathbf{H} = (H, \cdot, \rightarrow, \wedge, 1)$ in which the following prelinearity condition holds:

(Sprel) For every $a, b, c \in H$, $(a \rightarrow b) \rightarrow c \leq ((b \rightarrow a) \rightarrow c) \rightarrow c$. 

Figure 1.1: Lattice of some subvarieties of BRL.
Chapter 1. Preliminary algebraic notions

Prelinearity implies that the $\lor$-operation is determined by the other connectives, as $a \lor b = ((a \to b) \to b) \land ((b \to a) \to a)$. Prelinear semihoops are term equivalent to GMTL-algebras, thus MTL-algebras can be defined as bounded and pointed prelinear semihoops.

A hoop is an algebra $H = (H, \cdot, \to, 1)$ of type $(2, 2, 2, 0)$ such that:

(H1) $(H, \cdot, 1)$ is a commutative monoid;

(H2) $x \to x = 1$;

(H3) $x \cdot (x \to y) = y \cdot (y \to x)$;

(H4) $(x \cdot y) \to z = x \to (y \to z)$.

We can define a natural order on a hoop $H$ in the usual way: $a \leq b$ iff $a \to b = 1$. The order defined is a $\land$-semilattice order where $x \land y = x \cdot (x \to y)$. Semihoops that further satisfy (H3) are equivalent to hoops [54].

A basic hoop is a hoop satisfying prelinearity, which is term equivalent to a GMTL-algebra satisfying (H3).

A Wajsberg hoop is a basic hoop satisfying:

$$(x \to y) \to y = (y \to x) \to x.$$ 

A cancellative hoop is a basic hoop satisfying cancellativity:

$$x \to (x \cdot y) = y.$$ 

A Gödel hoop is a basic hoop satisfying contraction:

$$x \cdot x = x.$$ 

We will denote the varieties of basic hoops, Wajsberg hoops, cancellative hoops and Gödel hoops respectively with $BH$, $WH$, $CH$, $GH$. Since they all satisfy prelinearity, we will often include the join $\lor$ in the signature and we will see them as subvarieties of GMTL, and then of $RL$. The next proposition collects some well-known results.
Chapter 1. Preliminary algebraic notions

Proposition 1.2.4 ([1], [19]). (1) Basic hoops are precisely the subreducts of BL-algebras in the language \{\cdot, \to, 1\} of hoops.

(2) Wajsberg hoops are precisely the subreducts of MV-algebras in the language of hoops.

(3) Gödel hoops are precisely the subreducts of Gödel algebras in the language of hoops.

Cancellative hoops instead, can be described in terms of lattice ordered abelian groups. A lattice-ordered abelian group is an algebra \((G, +, -, \lor, \land, 0)\) of type \(\langle 2, 2, 2, 2, 0 \rangle\) where:

i. \((G, +, -, 0)\) is an abelian group,

ii. \((G, \lor, \land)\) is a lattice,

iii. the equation \(x + (y \lor z) = (x + y) \lor (x + z)\) holds.

The negative cone \(G^-\) of a lattice ordered abelian group \(G\) is the algebra whose domain is \(\{g \in G : g \lor 0 = 0\}\) (the set of negative elements of \(G\)), equipped with the constant 0 and with the operations \(x \cdot y = x + y\) and \(x \to y = (y - x) \land 0\). Cancellative hoops are precisely the negative cones of lattice ordered abelian groups, as shown in [55].

Another important result is proven by Agliano and Montagna in [2]. In particular, they prove a fundamental decomposition theorem for BL-chains and hoops, involving the construction of ordinal sum.

Definition 1.2.5. Let \((I, \leq)\) be a totally ordered set. For all \(i \in I\) let \(A_i\) be a hoop such that for \(i \neq j\), \(A_i \cap A_j = \{1\}\). Then \(\bigoplus_{i \in I} A_i\), the ordinal sum of the family \((A_i)_{i \in I}\), is the structure whose base set is \(\bigcup_{i \in I} A_i\) and the operations
Chapter 1. Preliminary algebraic notions

are defined as follows:

\[ x \rightarrow y = \begin{cases} 
  x \rightarrow_{A_i} y & \text{if } x, y \in A_i, \\
  y & \text{if } x \in A_i \text{ and } y \in A_j \text{ with } i > j \\
  1 & \text{if } x \in A_i \setminus \{1\} \text{ and } y \in A_j \text{ with } i < j.
\]

\[ x \cdot y = \begin{cases} 
  x \cdot_{A_i} y & \text{if } x, y \in A_i, \\
  y & \text{if } x \in A_i \text{ and } y \in A_j \setminus \{1\} \text{ with } i > j \\
  x & \text{if } x \in A_i \setminus \{1\} \text{ and } y \in A_j \text{ with } i < j.
\]

More precisely, they prove the following.

**Theorem 1.2.6** (Theorem 3.7 [2]). *Every totally ordered hoop (BL-algebra) is the ordinal sum of a family of Wajsberg hoops (whose first component is a Wajsberg algebra).*

**Filters and congruences**

We will now introduce another fundamental notion for the study of residuated structures.

An *implicative filter*, or just *filter*, of a (bounded) RL (or of a hoop) \( A \) is a subset \( F \) of \( A \) such that:

(i) \( 1 \in F \),

(ii) if \( a, a \rightarrow b \in F \), then \( b \in F \).

A *lattice filter* of \( A \) is a subset \( L \) of \( A \) such that:

(i) \( 1 \in L \);

(ii) it is upwards closed, i.e. if \( x \in L \) and \( x \leq y \), then \( y \in L \);

(iii) it is closed under \( \wedge \): if \( x, y \in L \) then \( x \wedge y \in L \).

An implicative filter can be equivalently defined as a subset \( F \) such that: \( 1 \in F \), \( F \) is upwards closed and closed under products. Indeed, if these conditions hold, given \( a, a \rightarrow b \in F \), we have \( a \cdot (a \rightarrow b) \leq b \), and being \( F \) upwards closed
b ∈ F. Viceversa, if 1 ∈ F and a, a → b ∈ F implies b ∈ F, we have that F is upwards closed since if a ∈ F, a ≤ b, being a = 1 · a ≤ b iff a → b = 1, and 1 ∈ F, we get b ∈ F. Moreover, if a, b ∈ F then from a · b = a · b we get via residuation b ≤ a → (a · b), and F is upwards closed, thus a → (a · b), a ∈ F which implies a · b ∈ F.

Hence it is easy to check that all filters of a (bounded) RL (or of a hoop) A are also lattice filters, since for every x, y ∈ A, x · y ≤ x ∧ y, while the inverse does not hold in general. Moreover, each filter is the domain of a subalgebra of A. For each X ⊆ A we denote by (X) the filter generated by X (i.e., the intersection of all filters containing X). The set of filters of A, ordered by inclusion, is a bounded lattice, that we shall denote Fil(A). Fil(A) will denote the lattice of lattice filters, or ℓ-filters for short, and the ℓ-filter generated by a single element x, a principal ℓ-filter, will be denoted by [x]. The least element, or bottom, of both Fil(A) and Fil(A) is {1}, while A is the top, and given F₁, F₂ ∈ Fil(A), F₁ ∧ F₂ = F₁ ∩ F₂ and F₁ ∨ F₂ = (F₁ ∪ F₂).

A filter F is said to be:

- trivial if its only element is 1;
- proper if it is not the whole universe A;
- maximal if it is not included in any other proper filter;
- prime if given x, y ∈ A, x ∨ y ∈ F implies x ∈ F or y ∈ F.

If A is prelinear, a filter F of A is prime iff for every x, y ∈ A, x → y ∈ F or y → x ∈ F. Indeed, let F be prime. Since (x → y) ∨ (y → x) = 1, by primality of F we have that x → y ∈ F or y → x ∈ F. Viceversa, and notice that what follows holds in any RL, let x ∨ y ∈ F. By hypothesis, x → y ∈ F or y → x ∈ F. If x → y ∈ F, since x ∨ y = ((x → y) → y) ∧ ((y → x) → x), then x ∨ y ≤ (x → y) → y ∈ F. Thus also (x → y) · (x → y) → y ∈ F. Since (x → y) · ((x → y) → y) ≤ y by residuation, we have that y ∈ F. Similarly, if y → x ∈ F we get x ∈ F.

Most importantly, filters stand in bijection with congruences: given a congruence θ on a (bounded) RL (or of a hoop) A, the set \( F_θ = \{ x ∈ A : (x, 1) ∈ θ \} \)
is a filter of $A$, and given a filter $F$ on $A$, the set $\theta_F = \{(x, y) : x \leftrightarrow y \in F\}$ is a congruence on $A$. Moreover, the maps $\theta \mapsto F_\theta$ and $F \mapsto \theta_F$ are mutually inverse isomorphisms between the filter lattice and the congruence lattice of $A$.

1.3 Basic notions of Category Theory

In this section we will recall the basic notions of Category Theory that will be needed in the present manuscript. For the reader interested in a broader exposition, we refer to [89].

A category $C$ is a pair $(O, A)$, where $O$ is a collection of objects and $A$ is a collection of morphisms, such that:

1. to every morphism $f$ we associate two objects: $\text{dom}(f)$ and $\text{cod}(f)$. We shall write $f : \text{dom}(f) \to \text{cod}(f)$;
2. for every object $x$ there exists a morphism $\text{id}_x : x \to x$;
3. the set of morphisms is closed for composition: whenever there exist $f, g \in A$ with $f : a \to b$ and $g : b \to c$, the morphism $g \circ f : a \to c$ is also in $A$.
4. composition, when defined, is associative:

$$f \circ (g \circ h) = (f \circ g) \circ h$$

whenever $h : a \to b$, $g : b \to c$, $f : c \to d$;
5. if $x = \text{dom}(f) = \text{cod}(g)$, then:

$$f \circ \text{id}_x = f, \quad \text{id}_x \circ g = g.$$

Any variety of algebras $V$ generates a category, that with a slight abuse of notation we denote with the same symbol $V$, where the objects are the algebras in $V$, and the morphisms are $V$-homomorphisms. We will refer to such category as the algebraic category of $V$. Let now $C$ be any category, we define the
opposite category $C^{op}$ to be the category whose objects are the objects of $C$, and whose morphisms $f^{op}$ are in bijection with the morphisms $f$ of $C$, and are such that $\text{dom}(f^{op}) = \text{cod}(f)$ and $\text{cod}(f^{op}) = \text{dom}(f)$ (the direction is reversed). The composition $f^{op} \circ g^{op} = (g \circ f)^{op}$ is defined in $C^{op}$ when $g \circ f$ is defined in $C$.

Given $C$ and $D$ categories, a functor from $C$ to $D$ is a map $\Phi : C \to D$ that is a morphism of categories. In particular, it assigns to every object of $C$ an object of $D$, and to every morphism of $C$, $f : a \to b$, it assigns a morphism of $D$, $\Phi(f) : \Phi(a) \to \Phi(b)$. Moreover, it preserves identity and composition of morphisms:

$$\Phi(id_x) = id_{\Phi(x)}, \quad \Phi(f \circ g) = \Phi(f) \circ \Phi(g).$$

We will denote with $1_C : C \to C$ the functor that is the identity on objects and morphisms. A functor $\Phi : C \to D$ is:

- **full** when it is surjective on the set of morphisms, i.e. if there exists a morphism $g : \Phi(c) \to \Phi(c')$, then there exists a morphism $f : c \to c'$ such that $\Phi(f) = g$.

- **faithful** when it is injective on the set of morphisms, i.e. for every pair of maps $f, f' : c \to c'$, the equality $\Phi(f) = \Phi(f')$ implies $f = f'$.

- **essentially surjective**, or **dense**, if every object $d$ of $D$ is isomorphic to an object $\Phi(c)$ for some object $c$ in $C$.

A subcategory $S$ of a category $C$ is a collection of some of the objects and some of the arrows of $C$, which includes: with each arrow $f$ both the objects $\text{dom}(f)$ and $\text{cod}(f)$; with each object $s$ its identity arrow $id_s$; with each pair of composable arrows their composite. Notice that these conditions insure that $S$ is a category. Moreover the map from $S$ to $C$ sending each object and arrow (in $S$) to itself (in $C$) is a functor, the *inclusion functor*. Such functor is automatically faithful. In case the inclusion functor is also full, we say that $S$ is a full subcategory of $C$.

Given $C$ and $D$ categories, a map $\Xi : C \to D$ is a *contravariant functor* if it is a functor from $C^{op}$ to $D$. Given two functors $\Phi, \Psi : C \to D$, a *natural
transformation $\tau : \Phi \to \Psi$ is a function that assigns to each object $c$ of $\mathcal{C}$ an arrow of $\mathcal{D}$, $\tau_c = \tau(c) : \Phi(c) \to \Psi(c)$ in such a way that for every arrow $f : c \to c'$ in $\mathcal{C}$ the following diagram commutes:

\[
\begin{array}{ccc}
  c & \overset{\Phi(c)}{\longrightarrow} & \Psi(c) \\
  \downarrow{\Phi(f)} & & \downarrow{\Psi(f)} \\
  c' & \overset{\Phi(c')}{\longrightarrow} & \Psi(c') \\
\end{array}
\]

Notice that a natural transformation can be seen as a morphism of functors. A natural transformation $\tau$ with every component $\tau_c$ invertible in $\mathcal{D}$ is called a natural isomorphism, in symbols $\tau : \Phi \cong \Psi$.

A functor $\Phi : \mathcal{C} \to \mathcal{D}$ is an equivalence of categories, and the categories $\mathcal{C}$ and $\mathcal{D}$ are equivalent, when there is a functor $\Psi : \mathcal{D} \to \mathcal{C}$ and natural isomorphisms $\Phi\Psi \cong 1_\mathcal{C}$ and $\Psi\Phi \cong 1_\mathcal{D}$.

**Proposition 1.3.1** (Theorem 1, page 91 [89]). For a functor $\Phi : \mathcal{C} \to \mathcal{D}$ the following are equivalent:

1. $\Phi$ is an equivalence of categories.
2. $\Phi$ is full, faithful and essentially surjective.

If a category $\mathcal{C}$ is equivalent to the category $\mathcal{D}^{\text{op}}$, for some category $\mathcal{D}$, then $\mathcal{C}$ and $\mathcal{D}$ are dually equivalent, and there is a duality of categories.

**Remark 1.3.2.** The notion of categorical equivalence, or of categorical duality, is fairly powerful. Indeed it allows to establish a close connection between different categories, allowing to suitably translate theorems and properties from one category to the other, and hence it is of particular interest also in algebraic logic.
Part I

Varieties of residuated lattices with an MV-retraction term
Chapter 2

Varieties generated by disconnected $\delta$-rotations of residuated lattices

The vanguard of the disconnected rotation construction goes back to Wróski’s reflection construction for BCK-algebras of 1983 [116]. Then, in 2000 [78], Jenei introduces the connected rotation construction for t-norms, subsequently extended by him to arbitrary residuated posets in [79]. In the same paper, also the disconnected rotation construction was introduced for arbitrary residuated posets. The reader is also referred to [67]. Both the connected and the disconnected rotation constructions have been shown to have a powerful role in the construction of residuated structures. For example in the structural description of perfect and bipartite MTL-algebras [99], of nilpotent minimum algebras [32], of Nelson algebras [33], and of Glivenko MTL-algebras [42]. Even more recently, we exploited them in [5] to obtain a characterization of a large class of MTL-algebras, that we named SBP$_0$-algebras, and will shall see a generalized version of this work in the following sections and chapters.

2.1 Disconnected $\delta$-rotation

In this section we will present the notion of $\delta$-rotation of a residuated lattice, that will extend Jenei’s disconnected rotation on residuated lattices. In order to do so, we first define a class of operators denoted by $\delta$, that, in analogy with Cignoli and Torrens $dl$-operators [42], we shall call weakly $dl$-operators, or $wdl$-operators for short. It is worth pointing out that, while Cignoli and Torrens
define dl-operators on what they name generalized (i.e. unbounded) MTL-algebras, or GMTL-algebras, in what follows we will extend the definition of wdl-operators to all residuated lattices.

**Definition 2.1.1.** Let $R$ be a residuated lattice. A map $\delta : R \rightarrow R$ is called *wdl-admissible* if it satisfies the following conditions:

\[(\delta_1)\] \(x \rightarrow \delta(x) = 1\)
\[(\delta_2)\] \(\delta(\delta(x)) = \delta(x)\)
\[(\delta_3)\] \(\delta(x \cdot y) = \delta(\delta(x) \cdot \delta(y))\)
\[(\delta_4)\] \(\delta(x \land y) = \delta(x) \land \delta(y)\)
\[(\delta_5)\] \(\delta(x \lor y) = \delta(x) \lor \delta(y)\)

**Notation 2.1.2.** For every $R \in \text{RL}$ let $\delta_L, \delta_D : R \rightarrow R$ respectively denote the following maps: $\delta_L : x \in R \mapsto 1 \in R$, while $\delta_D : x \in R \mapsto x \in R$. It is easy to see that they are wdl-admissible. These maps will be used later in our constructions.

**Remark 2.1.3.** Definition 2.1.1 coincides with the cited one of Cignoli and Torrens in [42], except for the fact that dl-operators further require the following condition:

\[(\delta_6)\] \(\delta(x \rightarrow y) = x \rightarrow \delta(y)\).

Moreover, Definition 2.1.1 is equivalent to the one we give in [5], defined on prelinear semihoops (that are term-equivalent to GMTL-algebras). However, in [5] the following condition is also imposed:

\[(\delta_6)'\] \(\delta(x \rightarrow y) \leq x \rightarrow \delta(y)\).

Now, we prove $(\delta_6)'$ to be redundant. Indeed, given $\delta$ wdl-admissible operator of $R \in \text{RL}$, for any $x, y \in R$ we have that, applying $(\delta_1)$ and $(\delta_2)$:

\[x \cdot \delta(x \rightarrow y) \leq \delta(x) \cdot \delta(x \rightarrow y) \leq \delta(\delta(x) \cdot \delta(x \rightarrow y)) = \delta(x \cdot (x \rightarrow y)) \leq \delta(y)\]
thus $x \cdot \delta(x \rightarrow y) \leq \delta(y)$, whence by residuation, $\delta(x \rightarrow y) \leq x \rightarrow \delta(y)$.

In order to show that wdl-admissible operators are actually weaker than Cignoli and Torrens’s dl-admissible ones, we provide an easy example. Let us consider the algebra $R = (\{a, b, c, 1\}, \cdot, \rightarrow, \land, \lor, 1)$ whose lattice order is the total order $a < b < c < 1$, and $\cdot$ and $\rightarrow$ are defined by the following tables:

<table>
<thead>
<tr>
<th>$\cdot$</th>
<th>a</th>
<th>b</th>
<th>c</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>a</td>
<td>a</td>
<td>a</td>
<td>a</td>
</tr>
<tr>
<td>b</td>
<td>a</td>
<td>a</td>
<td>a</td>
<td>b</td>
</tr>
<tr>
<td>c</td>
<td>a</td>
<td>a</td>
<td>c</td>
<td>c</td>
</tr>
<tr>
<td>1</td>
<td>a</td>
<td>b</td>
<td>c</td>
<td>1</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$\rightarrow$</th>
<th>a</th>
<th>b</th>
<th>c</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>b</td>
<td>c</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>c</td>
<td>b</td>
<td>b</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>a</td>
<td>b</td>
<td>c</td>
<td>1</td>
</tr>
</tbody>
</table>

It is easy to see that $R$ is a (prelinear) residuated lattice. Now, if we take as $\delta$ the map that is the identity on $\{b, c, 1\}$ and such that $\delta(a) = b$, a straightforward computation shows that $\delta$ is a wdl-admissible operator (that moreover constitutes a different example of wdl-admissible operator, with respect of the two already given, $\delta_D$ and $\delta_L$). Moreover, we have that $\delta(b \rightarrow a) < b \rightarrow \delta(a)$. Indeed, $\delta(b \rightarrow a) = \delta(c) = c < 1 = b \rightarrow b = b \rightarrow \delta(a)$. Hence, such operator satisfies $(\delta 6)'$ but not $(\delta 6)$.

**Remark 2.1.4.** It is also worth noticing that any wdl-admissible operator $\delta$ over a RL is a *nucleus* in the sense of [104]. Let us recall that a nucleus on a residuated lattice $L$ is a closure operator $\gamma$ on $L$ such that $\gamma(a) \cdot \gamma(b) \leq \gamma(a \cdot b)$, for all $a, b \in L$. Let us first show that $\delta$ is a closure operator on $R$, i.e. it satisfies:

1. $x \leq \delta(x)$,
2. $x \leq y$ implies $\delta(x) \leq \delta(y)$,
3. $\delta(\delta(x)) \leq \delta(x)$.

Meaning that it is, respectively, expansive, monotone and idempotent. The fact that $\delta$ is expansive is equivalent to $(\delta 1)$, while $(\delta 3)$ is stronger than the
expansion property, and monotonicity is implied by both \((\delta 4)\) or \((\delta 5)\). Indeed, using \((\delta 4)\), if \(x \leq y\) then \(\delta(x) = \delta(x \land y) = \delta(x) \land \delta(y)\), which implies that \(\delta(x) \leq \delta(y)\). Finally, for any closure operator, \((\delta 3)\) is equivalent to satisfying \(\delta(x) \cdot \delta(y) \leq \delta(x \cdot y)\) (see [66, Lemma 3.33 (1)]). In particular, since for a nucleus \((\delta 1), (\delta 2)\) and \((\delta 3)\) hold, every wdl-admissible operator \(\delta\) is a nucleus operator respecting \(\land\) and \(\lor\).

**Definition 2.1.5.** Let \(R = (\cdot, \to, \land, \lor, 1)\) be a RL, let \(\delta: R \to R\) be a wdl-admissible operator and define \(\mathcal{R}_\delta(R) = (\{1\} \times R) \cup (\{0\} \times \delta[R])\) to be the disjoint union of \(R\) and \(\delta[R]\). Further let us define on \(\mathcal{R}_\delta(R)\) the following operations:

\[
\begin{align*}
(i, a) \lor_R (j, b) &= \begin{cases} 
(1, a \lor b) & \text{if } i = j = 1, \\
(0, a \land b) & \text{if } i = j = 0, \\
(1, b) & \text{if } i < j.
\end{cases} \\
(i, a) \land_R (j, b) &= \begin{cases} 
(1, a \land b) & \text{if } i = j = 1, \\
(0, a \lor b) & \text{if } i = j = 0, \\
(0, a) & \text{if } i < j.
\end{cases} \\
(i, a) \cdot_R (j, b) &= \begin{cases} 
(1, a \cdot b) & \text{if } i = j = 1, \\
(0, 1) & \text{if } i = j = 0, \\
(0, b \to a) & \text{if } i < j.
\end{cases} \\
(i, a) \to_R (j, b) &= \begin{cases} 
(1, a \to b) & \text{if } i = j = 1, \\
(1, b \to a) & \text{if } i = j = 0, \\
(0, \delta(a \cdot b)) & \text{if } i > j, \\
(1, 1) & \text{if } i < j.
\end{cases}
\]

We will call \(\mathcal{R}_\delta(R) = (\mathcal{R}_\delta(R), \land_R, \lor_R, \cdot_R, \to_R, (0, 1), (1, 1))\) the **disconnected \(\delta\)-rotation** of \(R\).

**Remark 2.1.6.** We can define the negation as usual, as \(\neg_R(i, a) = (i, a) \to_R (0, 1)\), and thus we obtain:

\[
\neg_R(1, a) = (0, \delta(a)), \quad \neg_R(0, b) = (1, b);
\]
Chapter 2. Disconnected \( \delta \)-rotations of residuated lattices

\[ \neg R \neg R(1, a) = (1, \delta(a)), \quad \neg R \neg R(0, b) = (0, b). \]

We will now prove that disconnected \( \delta \)-rotations of residuated lattices are in BRL, and moreover they are directly indecomposable. In the following sections we shall see more precisely which properties they satisfy, and we will characterize the varieties they generate.

**Theorem 2.1.7.** For every residuated lattice \( R \) and for every wdl-admissible operator \( \delta \), the algebra

\[ \mathfrak{R}_\delta(R) = (\mathfrak{R}_\delta(R), \cdot_R, \rightarrow_R, \wedge_R, \vee_R, (0, 1), (1, 1)) \]

is a directly indecomposable bounded residuated lattice.

**Proof.** We follow the same lines of the proof of [42, Theorem 2.2], in which the authors prove that, if \( \delta \) is dl-admissible, starting with a prelinear residuated lattice the resulting structure is an MTL-algebra. However, inspecting the proof, prelinearity of \( R \) is only involved in the proof that \( \mathfrak{R}_\delta(R) \) also results to be prelinear. By virtue of Remark 2.2.10, we only need to show that our weaker form of \( (\delta 6) \) still ensures that the operations are well-defined. In particular, let us prove that \( \cdot_R \) is well-defined, the other cases being straightforward from the definition of the operations, and also given by a direct inspection on the proof of [42, Theorem 2.2]. This fact follows because \( (\delta 1), (\delta 2) \) and \( (\delta 6)' \) imply that \( \delta(b \rightarrow \delta(a)) \leq b \rightarrow \delta(\delta(a)) = b \rightarrow \delta(a) \leq \delta(b \rightarrow \delta(a)) \), whence \( \delta(b \rightarrow \delta(a)) = b \rightarrow \delta(a) \). Therefore, \( (a, i) \cdot_R (b, j) \) is well-defined in the case \( i < j \), and since the other cases are easily checked, \( \mathfrak{R}_\delta(R) \) is a bounded residuated lattice.

We notice that, via Remark 2.1.6, the only Boolean elements of \( \mathfrak{R}_\delta(R) \) are \((0, 1), (1, 1)\). Indeed, an element \( x \) of a bounded residuated lattice is Boolean iff \( x \lor \neg x = 1 \) (as proved in [84]). Thus, \((1, a)\) is Boolean iff \( (1, a) \lor_R (0, \delta(a)) = (1, 1) \), iff \( a = 1 \), while \((0, b) \lor_R (1, b) = (1, 1) \) iff \( b = 1 \). Thus \( \mathfrak{R}_\delta(R) \) is directly indecomposable (see Proposition 1.2.2.

**Example 2.1.8.** In order to help the intuition, we shall now see some examples of this construction. Let us consider the two wdl-admissible operators \( \delta_L, \delta_D \). Let \( R \) be a RL. We recall that \( \delta_L : x \in R \mapsto 1 \in R \), while \( \delta_D : x \in R \mapsto x \in R \). Thus, for \( \delta = \delta_L \), we have \( \mathfrak{R}_\delta(R) = (\{1\} \times R) \cup (\{0\} \times \{1\}) \):
Chapter 2. Disconnected $\delta$-rotations of residuated lattices

We shall call this construction the lifting of the residuated lattice $R$, that coincides with the ordinal sum $2 \oplus R$. Whereas with $\delta = \delta_D$, we have $R_\delta(R) = (\{1\} \times R) \cup (\{0\} \times R)$:

Notice that this is the disconnected rotation of the residuated lattice $R$ [79].

Let us now see special cases of the previous examples.

**Residuated lattices** The lifting of any residuated lattice is a directly indecomposable Stonean residuated lattice, and every directly indecomposable algebra in their variety $SRL$ is of this form [34].

**Cancellative hoops** The lifting of a cancellative hoop is a directly indecomposable product algebra, and every directly indecomposable product algebra is of this form [41]. Disconnected rotations of cancellative hoops coincide instead with perfect MV-algebras [12].
Chapter 2. Disconnected $\delta$-rotations of residuated lattices

**Gödel hoops** The lifting of a Gödel hoop is a directly indecomposable Gödel algebra, and as recalled in Proposition 1.2.4, Gödel hoops are the 0-free subreducts of Gödel algebras. Thus every directly indecomposable Gödel algebra is of this form. Disconnected rotations of Gödel hoops correspond to directly indecomposable NM-algebras without negation fixpoint.

**GMTL-algebras** Liftings of GMTL-algebras coincide with directly indecomposable SMTL-algebras, while their disconnected rotations are perfect IMTL-algebras as in [99].

**Generalized Heyting algebras** A generalized Heyting algebra is the 0-free reduct of an Heyting algebra. Thus, liftings of generalized Heyting algebras are the Heyting algebras that are directly indecomposable. Disconnected rotations correspond instead to directly indecomposable regular Nelson residuated lattices without negation fixpoint [33].

In the next section we will study and characterize the subvarieties of BRL generated by disconnected $\delta$-rotations of residuated lattices. In [5], we also require prelinearity, that however has no role in the construction, although some of the most prominent examples satisfy prelinearity condition. It is worth noticing that prelinearity implies distributivity of the underlying lattice, that it is also, more surprisingly, not necessary for our construction. We shall mention that what follows also generalizes the work [34] on Stonean residuated lattices, in a sense that we will make more precise later on.

### 2.2 rDL-algebras

In [43], Cignoli and Torrens define a hierarchy of subvarieties of BRL:

$$WL_1 \nsubseteq WL_2 \nsubseteq \ldots \nsubseteq WL_n \nsubseteq \ldots$$

where, for $k > 0$, algebras in $WL_k$ are bounded residuated lattices satisfying the following identity:

$$(wl_k) \ k.x \vee k.(\neg x) = 1$$

43
and $k.x$ is recursively defined by: $0.x = 0$, $(n + 1).x = x \oplus n.x$. These varieties are such that, for every $n > 0$, the following term:

$$\Lambda_n(x) := n.x^n$$

(2.1)

is the unique admissible Boolean retraction term for any subvariety $V$ of $WL_n$, i.e. it defines a homomorphism from any algebra $A \in V$ onto its largest Boolean subalgebra, and, moreover, such term is essentially unique, meaning that if $t$ is a Boolean retraction term for $V$, then the equation $t(x) = \Lambda_n(x)$ holds in $V$. The authors of [43] also prove that it is possible to axiomatize the largest subvariety $V_n$ of each $WL_n$ admitting a Boolean retraction term, via the extra axiom:

$$(DL_n) \ n.x^n = (n.x)^n.$$  

Alternatively, $V_n$ can be axiomatized as the subvariety of $BRL$ also satisfying $\Lambda_n(x) \lor \Lambda_n(\neg x) = 1$.

Algebras in $V_n$ have been proven to satisfy important structural properties, that we will show in what follows. First of all, recall that given any bounded residuated lattice $A$, its radical $\text{Rad}(A)$ is the intersection of its maximal filters, while the coradical of $A$ is defined as $\text{CoRad}(A) = \{x \in A \mid \neg x \in \text{Rad}(A)\}$ (see [99, 42]). Thus it can be proved the following:

**Proposition 2.2.1.** Given $A$ in $V_n$:

(i) $A$ is directly indecomposable iff $A = \text{Rad}(A) \cup \text{CoRad}(A)$;

(ii) $x > y$ for all $x \in \text{Rad}(A)$ and $y \in \text{CoRad}(A)$.

**Proof.** (i) is proved in [43, Lemma 3.5]. For (ii), using (i) and [43, Theorem 3.2, 3.3], noticing that $\text{Rad}(A)$ is upward closed while $\text{CoRad}(A)$ is downward closed, the claim easily follows. \qed

For $n = 2$, equation (wl$_2$) becomes the so-called weak-prelinearity axiom:

$$(wl_2) \ (\neg x \rightarrow \neg \neg x) \lor (\neg \neg x \rightarrow \neg x) = 1.$$
Chapter 2. Disconnected $\delta$-rotations of residuated lattices

that is indeed implied by the prelinearity condition. Algebras in $WL_2$ will be\textit{ weakly prelinear} bounded residuated lattices. Moreover, $(DL_2)$ coincides with equation:

$$(DL) \ 2x^2 = (2x)^2,$$

first introduced by Di Nola and Lettieri to axiomatize the variety of \textit{perfect} MV-algebras [51]. In analogy, directly indecomposable MTL-algebra satisfying $(DL)$ have been named \textit{perfect MTL-algebras} in [98, 99]. In the same papers, the variety generated by perfect MTL-algebras was denoted by $BP_0$.

\textbf{Notation 2.2.2.} We call $DL$ the variety of bounded residuated lattices satisfying $(DL)$, and algebras in $DL$ $DL$-algebras. The subvarieties of weakly prelinear and prelinear $DL$-algebras will be denoted respectively by $wDL$ and $sDL$ (where $s$ stands for semilinear), and their algebras $wDL$-algebras and $sDL$-algebras. Conforming to this notation, we are settling the following definition.

\textbf{Definition 2.2.3.} An $rDL$-algebra is a $wDL$-algebra satisfying the De Morgan law:

$$(DM1) \ \neg(x \land y) = \neg x \lor \neg y$$

plus the following condition:

$$(r) \ \neg(x^2) \to (\neg\neg x \to x) = 1.$$  

The variety of $rDL$-algebras will be called $rDL$.

The variety of $wDL$-algebras satisfying the involutivity condition $\neg\neg x = x$, that we name $wDL$, is a subvariety of $rDL$. Pseudocomplemented $rDL$-algebras correspond to Stonean residuated lattices, as follows from observation (1.20) in [43]. We denote such variety with $SRL$. Moreover, the variety of prelinear $rDL$ algebras is denoted with $srDL$, and the algebras $srDL$-algebras.

\textbf{Remark 2.2.4.} $sDL$-algebras are $BP_0$-algebras in [98, 99], and $srDL$-algebras are the $SBP_0$-algebras defined in [5]. Indeed, $(DM1)$ holds in any MTL-algebra [53, Proposition 1]. Moreover, observe that the other De Morgan law:
Chapter 2. Disconnected $\delta$-rotations of residuated lattices

\begin{equation}
\neg(x \lor y) = \neg x \land \neg y
\end{equation}

holds in every residuated lattice [66, Corollary 3.14, (2)].

**Proposition 2.2.5.** For any wDL-algebra satisfying De Morgan laws the following are equivalent:

1. For every $x \in \text{CoRad}(A)$, $x = \neg \neg x$.
2. $\text{CoRad}(A) = \neg \text{Rad}(A) = \{ \neg x \mid x \in \text{Rad}(A) \}$.
3. $A$ is an rDL-algebra.

**Proof.** Let us first show that (1) implies (2). If for every $x \in \text{CoRad}(A)$, $x = \neg \neg x$, then $\neg x = y \in \text{Rad}(A)$ (see for instance [43, Lemma 3.2(1)]). Then clearly $x = \neg y$, with $y \in \text{Rad}(A)$, and (2) holds. Viceversa, if $x \in \text{CoRad}(A)$, then $x = \neg y$, with $y \in \text{Rad}(A)$. Thus, $\neg \neg x = \neg \neg y = \neg y = x$, and we have proved that (2) implies (1), hence they are equivalent.

Now, we prove that (3) $\iff$ (1). To this end, let us start assuming (3). Let $A$ be an rDL-algebra, and let $x \in \text{CoRad}(A)$. Then, $(2x)^2 = 0$, since $(2x)^2$ is a Boolean retraction term for rDL. Hence, via (DL), $x^2 = 0$ and $\neg(x^2) = 1$. Thus, by (r), $\neg \neg x \rightarrow x = 1$ in $A$, that is to say $\neg \neg x \leq x$. Since $x \leq \neg \neg x$ holds in every BRL (see Proposition 1.2.1 (x)), (1) holds. Conversely, let $A$ be a wDL-algebra satisfying (1). Without loss of generality, let us assume that $A$ is directly indecomposable, whence, by Proposition 2.2.1(1), $A = \text{Rad}(A) \cup \text{CoRad}(A)$ and let $x \in A$. If $x \in \text{CoRad}(A)$, then $\neg \neg x = x$, and (r) clearly holds. On the other hand, if $x \in \text{Rad}(A)$, $x^2 \in \text{Rad}(A)$ as well ($\text{Rad}(A)$ is a filter of $A$) and $\neg(x^2) \in \text{CoRad}(A)$ (since we are assuming (1) which we already proved to be equivalent to (2)). As for $\neg \neg x \rightarrow x$, notice that $x \in \text{Rad}(A)$ implies that $\neg \neg x \rightarrow x \in \text{Rad}(A)$. Thus, $\neg(x^2) \leq \neg \neg x \rightarrow x$ and (r) holds in $A$.

For every rDL-algebra $A$ we define:

\begin{align*}
\mathcal{B}(A) &= \{x \in A \mid x \lor \neg x = 1\}, \\
\mathcal{R}(A) &= \{x \in A \mid x > \neg x\}, \\
\mathcal{C}(A) &= \{x \in A \mid x < \neg x\}.
\end{align*}
Chapter 2. Disconnected δ-rotations of residuated lattices

**Proposition 2.2.6.** For every $rDL$-algebra $A$, $\mathcal{B}(A)$, $\mathcal{R}(A)$, $\mathcal{C}(A)$ are respectively the domain of the largest Boolean subalgebra of $A$, the residuated lattice $\text{Rad}(A)$, and the coradical of $A$.

*Proof.* The fact that $\mathcal{B}(A)$ is the domain of the largest Boolean subalgebra of $A$ was proved in [84] for bounded residuated lattices. In order to prove that $\mathcal{R}(A)$ is the radical of $A$, recall Proposition 2.2.1 and [43, Theorem 3.2, 3.3]. If $x \in \text{Rad}(A)$, then $\neg x \in \text{CoRad}(A)$, thus $x > \neg x$. Viceversa, if $x > \neg x$ then let us consider the directly indecomposable components $A_i$, with $i \in I$, of $A$, then calling $x_i$ the $i$-th component of $x$, we have $x_i > \neg x_i$, which implies, again via [43, Theorem 3.2, 3.3], that $x_i \in \text{Rad}(A_i)$. But $\prod_{i \in I} \text{Rad}(A_i) = \text{Rad}(\prod_{i \in I} A_i)$ [43, Lemma 3.7], thus the claim follows. The proof that $\mathcal{C}(A)$ is the coradical of $A$ is analogous.  

**Notation 2.2.7.** Without danger of confusion, we will henceforth denote by $\mathcal{B}(A)$ the largest Boolean subalgebra of $A$, with $\mathcal{R}(A)$ the residuated lattice of domain $\text{Rad}(A)$, and with $\mathcal{C}(A)$ the coradical of $A$.

**Remark 2.2.8.** (1) In the light of the results shown above, we can now provide a slight explanation of the axiom $(r)$ we used to define $rDL$-algebras. Proposition 2.2.5 shows that a $wDL$-algebra satisfies $(r)$ iff it has an involutive coradical, or, equivalently, iff the coradical is the negation of the radical. Thus, $rDL$-algebras can be regarded as weakly prelinear bounded residuated lattices admitting a Boolean retraction term (and hence whose directly indecomposable elements are *perfect* in the terminology of [99]), satisfying De Morgan laws, and whose coradical is involutive.

(2) Identity $(r)$ may seem a strong condition to impose, but is needed to characterize the varieties generated by disconnected $\delta$-rotations of residuated lattices, as we shall see in what follows. Moreover, in an arbitrary directly indecomposable $wDL$-algebra $A$, it is not possible to recover the coradical starting from the radical. Indeed, while in $rDL$-algebras the elements of the coradical are exactly the negations of elements in the radical, in an arbitrary $wDL$-algebra we may have, for every element $x \in \text{Rad}(A)$, an infinite set of elements \{ $y \in \mathcal{C}(A) : \neg y = x$ \}, thus to reconstruct $\mathcal{C}(A)$ from the radical.

47
we would intuitively need some extra information. Thus, (r) yields a structure where the coradical is determined by the radical, via the construction of \( \delta \)-rotation.

We shall now deepen the result of Theorem 2.1.7 and show that directly indecomposable rDL-algebras are exactly disconnected \( \delta \)-rotations of residuated lattices.

**Theorem 2.2.9.** For every residuated lattice \( R \) and for every wdl-admissible operator \( \delta \), the algebra

\[
\mathcal{R}_\delta(R) = (\mathcal{R}_{\delta}(R), \cdot_R, \rightarrow_R, \wedge_R, \vee_R, (0, 1), (1, 1))
\]

is a directly indecomposable rDL-algebra. Furthermore, given a directly indecomposable rDL-algebra \( A \), \( \sim \) is a wdl-admissible operator on its radical \( \mathcal{R}(A) \), and \( A \) is isomorphic to \( \mathcal{R}_{\sim}(\mathcal{R}(A)) \).

**Proof.** We already proved in Theorem 2.1.7 that \( \mathcal{R}_\delta(R) \) is a directly indecomposable bounded residuated lattice. Now, we need to prove that (DL), (DM1) and (r) are valid in \( \mathcal{R}_\delta(R) \). These equations can be actually checked by a direct computation. We prove for instance the case of (r). Let \( x \in \mathcal{R}_\delta(R) \) and, in particular, \( x = (0, a) \), where \( a = \delta(a') \), for some \( a' \in R \). Thus, \( (0, a)^2 = (0, a) \cdot_R (0, a) = (0, 1) \), whence \( \neg(0, a)^2 = (1, 1) \). Similarly, \( \neg\neg(0, a) \rightarrow_R (0, a) = (0, a) \rightarrow_R (0, a) = (1, 1) \). Therefore, (r) holds. The case \( x = (1, a) \) is similar and omitted. This shows that \( \mathcal{R}_\delta(R) \) is a directly indecomposable rDL-algebra.

In order to conclude the proof let \( A \) be a rDL-algebra. Then, let us firstly observe that \( \sim \) is wdl-admissible. Indeed, (d1) and (d2) follow by the monotonicity and idempotency of \( \neg \), while (d4) and (d5) immediately follow from De Morgan laws, that hold in rDL-algebras, as from Remark 2.2.4. To prove (d3) notice that \( (x \cdot y) \rightarrow z = x \rightarrow (y \rightarrow z) \) and \( x \rightarrow \neg y = \neg \neg x \rightarrow \neg y \) hold in every bounded residuated lattice (Proposition 1.2.1 (v), (xiii)). Thus, \( \neg \neg(\neg \neg a \cdot \neg \neg b) = \neg((\neg \neg a \cdot \neg \neg b) \rightarrow 0) = \neg((\neg \neg a \rightarrow \neg \neg b) = \neg((\neg \neg a \rightarrow \neg b) = \neg(a \rightarrow \neg b) = \neg((a \cdot b) \rightarrow 0) = \neg(\neg a \cdot b) \). Moreover, \( \sim : \mathcal{R}(A) \rightarrow \mathcal{R}(A) \) since \( \mathcal{R}(A) = \text{Rad}(A) \). Thus, \( \neg \neg \) is wdl-admissible.
Finally, following [42, Theorem 3.5 (ii)] and [44], let $\psi_A : R_{\neg \neg}(\mathcal{R}(A)) \rightarrow A$ defined as

$$
\psi_A(i, x) = \begin{cases} 
  x & \text{if } i = 1, \\
  \neg x & \text{if } i = 0.
\end{cases}
$$

That $\psi_A$ is an injective homomorphism and $\psi_A$ is surjective iff $A$ is directly indecomposable follow by direct inspection on the proof provided in [42, Theorem 3.5 (ii)] and [44] and noticing that the claim is implied by the Equation (1) of [44], that is implied by Equation (2) of the same note [44], which is in turn equivalent to Condition (r).

Remark 2.2.10. As it can be seen inspecting the proof of [42, Theorem 2.2], imposing Cignoli and Torrens’ condition ($\delta_6$) is equivalent to ask the algebra $R_{\delta}(R)$ to satisfy Glivenko equation

$$
(\text{GL}) \quad \neg \neg (\neg \neg x \rightarrow x) = 1
$$

which, however, plays no role in our construction. As we showed in Remark 2.1.3, our condition is weaker, and the example in the same remark proves that wdl-admissibility does not force our structures to satisfy (GL).

Some examples of rDL-algebras are the following:

- $P$, the variety of product algebras;
- $G$, the variety of Gödel algebras;
- $\text{DLMV}$, i.e. the subvariety of rDL generated by perfect MV-algebras, axiomatized by prelinearity, divisibility and involutivity;
- $\text{NM}^-$, the subvariety of wDL generated by nilpotent minimum chains without negation fixpoint, and axiomatised by MTL $+ (x^2) \lor (x \rightarrow x^2) = 1$;

The following are some examples of not prelinear structures:

- $\text{SRL}$, the variety of Stonean residuated lattices;
- $\text{SHA}$, the variety of Stonean Heyting algebras;
• $\text{NR}^-$, the variety of regular Nelson lattices without negation fixpoint, that in [33] is referred as $\text{NM} \cap \mathcal{S}$, i.e. splitting regular Nelson lattices.

We end this section showing how every element of an rDL-algebra $\mathbf{A}$ can be expressed by means of an element of the Boolean skeleton, and an element of the radical. Notice that this suggests that every rDL-algebra can be properly decomposed in such two parts, and indeed it will play a fundamental role in the proofs of the categorical equivalences of Section 2.4.

Figure 2.1: Subvarieties of $\text{BRL}$, with the most relevant subvarieties of $\text{wDL}$.
Chapter 2. Disconnected δ-rotations of residuated lattices

Proposition 2.2.11. In every rDL-algebra $A$ the following hold:

(i) For every element $a \in A$, there are $b \in \mathcal{B}(A)$ and $c \in \mathcal{A}(A)$ such that

\[ a = (b \vee \neg a) \wedge (-b \vee c). \]

(ii) If $A$ is an Stonean residuated lattice, for every element $a \in A$, there are $b \in \mathcal{B}(A)$ and $c \in \mathcal{A}(A)$ such that $a = b \wedge c$.

Proof. We prove (i). Without loss of generality, we will prove the claim for directly indecomposable rDL-algebras. By Theorem 2.1.7, any directly indecomposable rDL-algebra $A$ is isomorphic to the algebra with domain $\{1\} \times \mathcal{B}(A) \cup \{(0) \times \delta(\mathcal{A}(A))\}$, where $\delta = \neg\neg$. Hence any element $a$ can be written either as $(1, a)$ or $(0, \delta(a'))$.

We will show that $a = (b_a \vee \neg c_a) \wedge (-b_a \vee c_a)$, where $b_a = \neg((-a^2)^2)$ and $c_a = a \vee \neg a$. First we need to prove that $b_a \in \mathcal{B}(A)$ and $c_a \in \mathcal{A}(A)$. Let us prove that $b_a$ is Boolean, that is to say, $b_a \vee \neg b_a = \neg((-a^2)^2) = -((-a^2)^2)^2 = 1$. If $a = (1, a)$ then $\neg((-a^2)^2) = -((-a^2)^2)^2 = -((1, 0) = (1, 1)$. Otherwise, if $a = (\delta(a'), 0)$ then $\neg((-a^2)^2) = (1, 1)$. Now we prove that $c_a \in \mathcal{A}(A)$, that is, $c_a \geq \neg c_a$. Again, if $a = (a, 1)$ then $(a, 1) \vee (\delta(a), 0) = (a, 1) \vee (\delta(a), 0) = (a, 1)$, $\neg(a, 1) = (a, 1) \vee (\delta(a), 0) = c_a$. Otherwise, if $a = (0, \delta(a'))$ then $\neg((0, \delta(a')) \vee (1, \delta(a')) = (1, \delta(a')) \vee (1, \delta(a')) = (1, 1, \delta(a'))$. Hence, the expression above reduces to:

\[ \neg((-a^2)^2)^2 \vee (1, a) \wedge (-\delta(0, a^2)^2 \vee 0, \delta(a)) \]

which equals to $(0, 1) \vee (1, a) = (1, a) \wedge (1, 1) = (1, a)$. Otherwise, if $a = (0, \delta(a'))$, we obtain:

\[ \neg((1, 1) \vee (1, \delta(a'))) \wedge ((0, 1) \vee (0, \delta(a'))) = (1, 1) \wedge (0, \delta(a')) = (0, \delta(a')). \]

Claims (ii) follows, since it is a special case of (i). \qed
2.3 RL-triplets and rDL-algebras

What follows in this section takes inspiration from our work \[92\] on product algebras, whose ideas we are going to apply to a much wider setting. Proposition 2.2.11 suggests that we can decompose an rDL-algebra in a pair of the kind \((B, R)\), with \(B\) a Boolean algebra, and \(R\) a residuated lattice. However, such a pair is not enough to characterize an rDL-algebra univocally, as one can see from \[92, \text{Theorem 3.3}\] for the case of product algebras: there can be non-isomorphic rDL-algebras that have isomorphic Boolean skeleton and isomorphic radical.

We will hence define residuated lattice-based triplets and we will show how we can associate univocally an rDL-algebra to any triplet and any wdl-admissible operator \(\delta\) and viceversa.

**Definition 2.3.1.** An RL-triplet is a triplet \((B, R, \vee_e)\) where \(B\) is a Boolean algebra, \(R\) is a residuated lattice such that \(B \cap R = \{1\}\) and \(\vee_e : B \times R \to R\) satisfies the following conditions where, for fixed \(b \in B\) and \(c \in R\), \(\nu_b(x) = b \vee_e x\) and \(\lambda_c(x) = x \vee_e c\):

(V1) For every \(b \in B\), and \(c \in R\), \(\nu_b\) is an endomorphism of \(R\) and the map \(\lambda_c\) is a lattice homomorphism from (the lattice reduct of) \(B\) into (the lattice reduct of) \(R\).

(V2) \(\nu_0\) is the identity on \(R\) and \(\nu_1\) is constantly equal to 1. (Note that since \(\nu_b(c) = \lambda_c(b)\), for all \(c \in R\), \(\lambda_c(1) = 1\) and \(\lambda_c(0) = c\).

(V3) For all \(b, b' \in B\) and for all \(c, c' \in R\), \(\nu_b(c) \lor \nu_{b'}(c') = \nu_{b \lor b'}(c \lor c') = \nu_b(\nu_{b'}(c \lor c'))\).

We shall call the map \(\vee_e\) external join.

For the sake of a lighter notation, but without danger of confusion, we will henceforth call triplets the structures \((B, R, \vee_e)\) where \(R\) is a residuated lattice.

**Remark 2.3.2.** Given a pair \((B, R)\), there always exists an external join \(\nu_e : B \times R \to R\). Indeed, let \(U\) be an ultrafilter of \(B\), and consider the map
such that:
\[ b \vee_e c = \begin{cases} 
1 & \text{if } b \in U \\
0 & \text{otherwise.} 
\end{cases} \]

It is easy to see that such map satisfies Conditions (V1)-(V3) of Definition 2.3.1.

It follows directly from Definition 2.3.1 that the following properties hold for any external join.

**Proposition 2.3.3.** Let \((B, R, \vee_e)\) be a triplet. Then for all \(b, b' \in B\) and \(c, c' \in R\), the following conditions hold:

1. \((J1)\) \((b \vee_e c) \vee c' = b \vee_e (c \vee c') = (b \vee_e c) \vee (b \vee_e c')\),
2. \((J2)\) \(b \vee_e (c \wedge c') = (b \vee_e c) \wedge (b \vee_e c')\),
3. \((J3)\) \((b \vee b') \vee_e c = b \vee_e (b' \vee_e c) = (b \vee_e c) \vee (b' \vee_e c)\),
4. \((J4)\) \((b \wedge b') \vee_e c = (b \vee_e c) \wedge (b' \vee_e c)\),
5. \((J5)\) \(1 \vee_e c = b \vee_e 1 = 1\) and \(0 \vee_e c = c\).

Let \((B, R, \vee_e)\) be a triplet. For all \(b \in B\), let
\[ \Theta_b = \{(c, c') \in R \times R \mid \nu_{-b}(c) = \nu_{-b}(c')\}. \]

Let us denote with \(\text{Max} B\) the set of ultrafilters of \(B\).

**Lemma 2.3.4.** \(B\) is a subdirect product of the family \(\{B/p\}_{p \in \text{Max} B}\).

**Proof.** The claim follows by Birkhoff subdirect representation theorem [14] plus [29, Corollary 1.9] (see also [29, Corollary 1.12]).

We display each \(b \in B\) as \(b = (b/p)_{p \in \text{Max} B}\). Notice that, for each \(p \in \text{Max} B\), it holds that \(B/p \cong \{0, 1\}\), whence either \(b/p = 0/p\) or \(b/p = 1/p\).

For each \(p \in \text{Max} B\), let
\[ \Theta_p = \{(c, c') \in R \times R \mid \exists b \in p((c, c') \in \Theta_b)\}. \]

Obviously \(\Theta_p = \{(c, c') \in R \times R \mid \exists b \in p(\nu_{-b}(c) = \nu_{-b}(c'))\}\).
Chapter 2. Disconnected \( \delta \)-rotations of residuated lattices

**Lemma 2.3.5.** \( \Theta_p \) is a congruence of \( R \) for each \( p \in \text{Max} \, B \).

**Proof.** Clearly, \( \Theta_p \) is reflexive and symmetric. Assume now that there are \( b', b'' \in p \) such that \( (c, c') \in \Theta_{b'} \) and \( (c', c'') \in \Theta_{b''} \). Whence, \( \nu_{-b'}(c) = \nu_{-b'}(c') \) and \( \nu_{-b''}(c') = \nu_{-b''}(c'') \). Now, using (V3), \( \nu_{-(b' \lor b'')}(c) = \nu_{-b'}(\nu_{-b''}(c)) = \nu_{-b'}(\nu_{-b''}(c')) = \nu_{-b'}(\nu_{-b''}(c'')) = \nu_{-(b' \lor b'')}(c'') \). Clearly, \( b' \land b'' \in p \), and hence \( (c, c'') \in \Theta_p \), proving that \( \Theta_p \) is an equivalence relation over \( R \). Let now \( * \) be any of the binary operations of \( R \). Assume again that there are \( b', b'' \in p \) such that \( (c, c') \in \Theta_{b'} \) and \( (d, d') \in \Theta_{b''} \). Reasoning as before, \( (c, c'), (d, d') \in \Theta_{b' \lor b''} \), and then \( (c * d, c' * d') \in \Theta_{b' \lor b''} \), proving that \( \Theta_p \) is a congruence of \( R \). \( \square \)

**Lemma 2.3.6.**

\[
\bigcap_{p \in \text{Max} \, B} \Theta_p = \{(c, c) \mid c \in R\}.
\]

**Proof.** Pick \( (c, c') \in \bigcap_{p \in \text{Max} \, B} \Theta_p \). Then, for each \( p \in \text{Max} \, B \) there exists \( b(p) \in p \) such that \( \nu_{-b(p)}(c) = \nu_{-b(p)}(c') \). Notice that, by Lemma 2.3.4,

\[
\bigvee_{p \in \text{Max} \, B} b(p) = 1,
\]

since for each \( d \in p \), \( d/p = 1/p \).

By compactness (either of propositional logic, since the set of all \( -b(p) \) is unsatisfiable in the theory having \( B \) as a model, or topological, since each \( b(p) \) is identifiable with a clopen, and hence an open set of the Stone Space of \( B \), which is compact), there exist a natural number \( k > 0 \) and \( b_1, b_2, \ldots, b_k \in \{b(p)\}_{p \in \text{Max} \, B} \) such that

\[
\bigvee_{i=1}^k b_i = 1.
\]

Now, from (V1) and (V2),

\[
\begin{align*}
c &= \nu_0(c) & = & \nu_{-\bigvee_{i=1}^k b_i}(c) \\
    &= \nu_{\bigwedge_{i=1}^k -b_i}(c) & = & \bigwedge_{i=1}^k \nu_{-b_i}(c) \\
    &= \bigwedge_{i=1}^k \nu_{-b_i}(c') & = & \nu_{\bigwedge_{i=1}^k -b_i}(c') \\
    &= \nu_{-\bigvee_{i=1}^k b_i}(c') & = & \nu_0(c') \\
    &= c'.
\end{align*}
\]  

(2.2)
thus proving that \( \cap_{p \in \text{Max } B} \Theta_p = \{(c, c) \mid c \in R\} \).

For each \( p \in \text{Max } B \), let \( R/p \) be the quotient of \( R \) by the congruence \( \Theta_p \).

**Lemma 2.3.7.** \( R \) is a subdirect product of the family \( \{R/p\}_{p \in \text{Max } B} \).

**Proof.** Immediate, after recalling that \( \text{Max } B \) is the only map \( /\text{uni2294}∶ \) is a subdirect product of the family \( \text{Lemma 2.3.8.} \)

For each \( p \in \text{Max } B \), let \( \vee_p : B/p \times R/p \to R/p \) be defined, for each \( c \in R/p \), by:

\[
0/p \vee_p c = c, \quad 1/p \vee_p c = 1.
\]

**Lemma 2.3.8.** \((B/p, R/p, \vee_p)\) is a triplet for each \( p \in \text{Max } B \). Moreover, \( \vee_p \) is the only map \( \sqcup : B/p \times R/p \to R/p \) such that \((B/p, R/p, \sqcup)\) is a triplet.

**Proof.** Immediate, after recalling that \( B/p \cong \{0, 1\} \).

**Lemma 2.3.9.** For each \((b, c) \in B \times R\),

\[
b \vee_e c = (b/p \vee_p c/p)_{p \in \text{Max } B}.
\]

**Proof.** We shall show that for each \( p \in \text{Max } B \) it holds that

\[
(b \vee_e c)/p = b/p \vee_p c/p.
\]

Recall that \( b/p \in \{0/p, 1/p\} \).

Assume first \( b/p = 0/p \). Then we must show \( (b \vee_e c)/p = c/p \). By definition of \( \Theta_p \), the latter equation holds if and only if there exists \( b' \in p \) such that \(-b' \vee_e (b \vee_e c) = -b' \vee_e c \). Now, take \( b' = -b \). Then \( b'/p = -b/p = -0/p = 1/p \), whence \( b' \in p \). The case is settled by observing that \(-(-b \vee_e (b \vee_e c)) = b \vee_e (b \vee_e c) = (b \vee b) \vee_e c = b \vee_e c = -b \vee_e c \).

Assume now \( b/p = 1/p \). Then we must show \( (b \vee_e c)/p = 1/p \), that is, there exists \( b' \in p \) such that \(-b' \vee_e (b \vee_e c) = -b' \vee_e 1 = 1 \). It suffices to take \( b' = b \), that is in \( p \) by our standing assumption that \( b/p = 1/p \). As a matter of fact, \(-b \vee_e (b \vee_e c) = (-b \vee b) \vee_e c = 1 \vee_e c = 1 \).

As there are no other cases left, we have settled the proof.

55
Chapter 2. Disconnected $\delta$-rotations of residuated lattices

Let $(B, R, \lor_e)$ be a triplet, let $\delta : R \to R$ be a wdl-admissible operator. For every $(b, c), (b', c') \in B \times R$, let us write $(b, c) \sim (b', c')$ iff $b = b', \neg b \lor_e c = \neg b' \lor_e c'$ and $b \lor_e \delta(c) = b' \lor_e \delta(c')$. It is not difficult to show that $\sim$ is an equivalence relation on $B \times R$. As usual, we will denote $B \times R/\sim$ the quotient of $B \times R$ modulo $\sim$.

**Definition 2.3.10.** For every triplet $(B, R, \lor_e)$ and for every wdl-admissible operator $\delta : R \to R$, let us define the algebra

$$B \otimes_\delta^e R = (B \times R/\sim, \circ, \Rightarrow, \cap, \cup, [0, 1], [1, 1])$$

where the operations are defined by the following stipulation: for all $(b, c), (b', c') \in B \times R$:

$[b, c] \circ [b', c'] = [b \land b', \lor_{b \lor_b b'}(c' \to c) \land \lor_{b \lor_b b'}(c \to c') \land \lor_{b \lor_b b'}(c \land c')]$

$[b, c] \Rightarrow [b', c'] = [b \to b', \lor_{b \lor_b b'}(\delta(c') \to \delta(c)) \land \lor_{b \lor_b b'}(\delta(c \land c') \land \lor_{b \lor_b b'}(c \to c'))$

$[b, c] \cap [b', c'] = [b \land b', \lor_{b \lor_b b'}(c \land c') \land \lor_{b \lor_b b'}(c \land c') \land \lor_{b \lor_b b'}(c \land c')]$

$[b, c] \cup [b', c'] = [b \lor b', \lor_{b \lor_b b'}(c \land c') \land \lor_{b \lor_b b'}(c \land c') \land \lor_{b \lor_b b'}(c \land c')]$

For what follows it is worth pointing out how the operation $\neg$ is computed:

$$-[b, c] = [b, c] \Rightarrow [0, 1] = [-b, \delta(c)]. \quad (2.3)$$

**Lemma 2.3.11.** For every triplet $(B, R, \lor_e)$, and every wdl-admissible operator $\delta$, the operations of $B \otimes_\delta^e R$ are well-defined.

**Proof.** Let us check that the operations are well defined with respect of the equivalence relation $\sim$. Let us suppose that $[b', c'] \sim [b'', c'']$. This means by definition that $b' = b''$, $-b' \lor_e c' = -b'' \lor_e c''$ and $b' \lor_e \delta(c') = b'' \lor_e \delta(c'')$.

Via Lemma 2.3.7 and Lemma 2.3.9, we can use the subdirect representation of $R$ and $B$, and we obtain that when $b' = 1$, then $c' = c''$, while if $b' = 0$, $\delta(c') = \delta(c'')$. We shall prove the following, the proofs for the lattice operations being similar and hence omitted:

1. $[b, c] \circ [b', c'] \sim [b, c] \circ [b'', c'']$;
2. $[b, c] \Rightarrow [b', c'] \sim [b, c] \Rightarrow [b'', c'']$;
3. \([b', c'] \Rightarrow [b, c] \sim [b'', c''] \Rightarrow [b, c].\]

Let us prove 1. The condition becomes \(b \cdot b' = b \cdot b''\), which is clearly satisfied, and the other two equations are:

\[
\nu_{(b,b')}(\nu_{b \triangleright b'}(c' \rightarrow c) \land \nu_{-b \triangleright b'}(c \rightarrow c') \land \nu_{-b \triangleright b'}(c \cdot c'))
\]

\[
= \nu_{(b,b')}(\nu_{b \triangleright b'}(c'' \rightarrow c) \land \nu_{-b \triangleright b'}(c \rightarrow c'') \land \nu_{-b \triangleright b'}(c \cdot c'')) \quad (2.4)
\]

\[
\nu_{(b,b')}(\delta(\nu_{b \triangleright b'}(c' \rightarrow c) \land \nu_{-b \triangleright b'}(c \rightarrow c') \land \nu_{-b \triangleright b'}(c \cdot c')))
\]

\[
= \nu_{(b,b')}(\delta(\nu_{b \triangleright b'}(c'' \rightarrow c) \land \nu_{-b \triangleright b'}(c \rightarrow c'') \land \nu_{-b \triangleright b'}(c \cdot c''))) \quad (2.5)
\]

Again, via Lemma 2.3.9, we will consider the subdirect representation. Notice that if \(b' = 1\), then \(c' = c''\), thus the conditions clearly holds. Let us consider the case \(b' = 0\). Notice than hence Eq. (2.4) becomes \(1 = 1\) and then holds for any \(b\). Now, if \(b = 0\) then also Eq. (4.5) becomes \(1 = 1\). While if \(b = 1\) Eq. (4.5) reduces to \(\delta(c \rightarrow c') = \delta(c \rightarrow c'')\), which holds since \(\delta(c \rightarrow c') = c \rightarrow \delta(c')\), \(\delta(c \rightarrow c'') = c \rightarrow \delta(c'')\), and \(\delta(c') = \delta(c'')\) by hypothesis.

Let us now prove 2. The condition becomes \(b \rightarrow b' = b \rightarrow b''\), again satisfied, and the other two:

\[
\nu_{(b,b')}(\nu_{b \triangleright b'}(\delta(c') \rightarrow \delta(c) \land \nu_{-b \triangleright b'}(\delta(c \cdot c') \land \nu_{-b \triangleright b'}(c \rightarrow c'))
\]

\[
= \nu_{(b,b')}(\nu_{b \triangleright b'}(\delta(c'') \rightarrow \delta(c) \land \nu_{-b \triangleright b'}(\delta(c \cdot c'') \land \nu_{-b \triangleright b'}(c \rightarrow c''))) \quad (2.6)
\]

\[
\nu_{(b,b')}(\delta(\nu_{b \triangleright b'}(\delta(c') \rightarrow \delta(c) \land \nu_{-b \triangleright b'}(\delta(c \cdot c') \land \nu_{-b \triangleright b'}(c \rightarrow c')))
\]

\[
= \nu_{(b,b')}(\delta(\nu_{b \triangleright b'}(\delta(c'') \rightarrow \delta(c) \land \nu_{-b \triangleright b'}(\delta(c \cdot c'') \land \nu_{-b \triangleright b'}(c \rightarrow c''))) \quad (2.7)
\]

Reasoning as before, we shall check the case where \(b' = 0\). If \(b = 0\) then \(b \rightarrow b' = 1\), thus Eq. (2.7) is trivially satisfied, while Eq. (2.6) becomes \(\delta(c') \rightarrow \delta(c) = \delta(c'') \rightarrow \delta(c)\), which holds since by hypothesis \(\delta(c') = \delta(c'')\). If \(b = 1\), then \(b \rightarrow b' = 0\), Eq. (2.6) trivially holds and Eq. (2.7) becomes \(\delta\delta(c \cdot c') = \delta\delta(c \cdot c'')\), that holds since \(\delta\delta(c \cdot c') = \delta(c \cdot c') = \delta(\delta(c) \cdot \delta(c')) = \delta(\delta(c) \cdot \delta(c'')) = \delta(c \cdot c') = \delta(c \cdot c'').\)
Chapter 2. Disconnected $\delta$-rotations of residuated lattices

We shall now prove 3, thus $b' \to b = b'' \to b$, easily satisfied, and:

$$\nu_{(\nu \to b)}(\nu_{\nu \to b}(\delta(c) \to \delta(c')) \land \nu_{\nu \to b}(\delta(c' \cdot c)) \land \nu_{\nu \to b}(c' \to c))$$

$$= \nu_{(\nu \to b)}(\nu_{\nu \to b}(\delta(c) \to \delta(c')) \land \nu_{\nu \to b}(\delta(c'' \cdot c)) \land \nu_{\nu \to b}(c'' \to c)) \quad (2.8)$$

Again, we check for $b' = 0$. Notice that hence $b' \to b = 1$, thus Eq. 2.9 is always satisfied. Now, if $b = 1$, Eq. 2.8 reduces to $1 = 1$. If $b = 0$, Eq. 2.8 reduces to $\delta(c) \to \delta(c') = \delta(c) \to \delta(c'')$, that holds since by hypothesis $\delta(c') = \delta(c'')$. Thus, the operations are well defined.

**Remark 2.3.12.** Notice that, if $b \neq b'$ then $[b, c] \neq [b', c']$. Moreover, $[1, c]$ and $[1, c']$ differ iff $c \neq c'$.

Whenever $p \in \text{Max } B$, and thanks to Lemma 2.3.8, we will denote $B/p \otimes_p^\delta R/p$ the structure whose domain and operations are defined as above starting from the triplet $(B/p, R/p, \nu_p)$ as in Lemma 2.3.8.

**Lemma 2.3.13.** For each $p \in \text{Max } B$, $B/p \otimes_p^\delta R/p \cong R_p(\delta(R/p))$ is a directly indecomposable rDL-algebra.

**Proof.** The Boolean algebra $B/p \cong \{0, 1\}$, and a straightforward check shows that $\{0, 1\} \otimes_p^\delta R/p \cong R_p(\delta(R/p))$ in Theorem 2.2.9. In that theorem we prove that the algebra $R_p(\delta(R))$ is a directly indecomposable rDL-algebra for any residuated lattice $R$. □

**Theorem 2.3.14.** $B \otimes_e^\delta R$ is a rDL-algebra. Moreover, $B(\otimes_e^\delta R) \cong B$ and $R(\otimes_e^\delta R) \cong R$.

**Proof.** By Lemma 2.3.7, Lemma 2.3.9 and Lemma 2.3.13, $B \otimes_e^\delta R$ is a subdirect product of the family of rDL-algebras $\{B/p \otimes_p^\delta R/p\}_{p \in \text{Max } B}$. As a matter of fact, the universe of $B \otimes_e^\delta R$ is the set of all tuples $\{(b/p, d/p)_{p \in \text{Max } B}\}$ such

58
that if $b/p = 1$ then $d$ is any element of $R$, while if $b/p = 0$ then $d = \delta(c)$ for some $c \in R$. Moreover, for each binary operation $\ast$ of $B \otimes^\delta_e R$ it holds that

$$(b, c) \ast (b', c') = ((b/p, c/p) \ast (b'/p, c'/p))_{p \in \text{Max } B}.$$  

Therefore, $B \otimes^\delta_e R$ is a subalgebra of $\prod_{p \in \text{Max } B} B/p \otimes^\delta_p R/p$ such that each projection $\pi_p : B \otimes^\delta_e R \to B/p \otimes^\delta_p R/p$ is surjective.

In order to conclude the proof, remember that $\mathcal{A}(B \otimes^\delta_e R)$ has domain $\{x \in B \otimes^\delta_e R \mid x \lor \neg x = 1\}$. Thus, define $\lambda_B : B \to \mathcal{A}(B \otimes^\delta_e R)$ in the following way: for every $b \in B$, $\lambda_B(b) = [b, 1]$. The first thing we have to show is that each $[b, 1]$ is Boolean, i.e., $[b, 1] \lor \neg [b, 1] = [1, 1]$. Indeed, $\neg [b, 1] = [\neg b, \delta(1)] = [\neg b, 1]$ hence $[b, 1]$ is clearly a Boolean element of $B \otimes^\delta_e R$. Moreover, if $[b, c] \in \mathcal{A}(B \otimes^\delta_e R)$, then $[b, c] \sim [b, 1]$. Indeed, for every $[b, c] \in \mathcal{A}(B \otimes^\delta_e R)$,

$$[b, c] \lor \neg [b, c] = [b, c] \cup [\neg b, \delta(c)] = [1, (b \lor_c \delta(c)) \lor \neg (b \lor_c c)].$$

Thus, $[1, (b \lor_c \delta(c)) \lor \neg (b \lor_c c)] = [1, 1]$ iff $b \lor_c \delta(c) \lor \neg (b \lor_c c) = 1$, that is $b \lor_c \delta(c) = 1$ and $\neg (b \lor_c c) = 1$. This implies that $[b, c] \sim [b, 1]$. Thus, $\lambda_B$ is surjective. Notice that $\lambda_B$ is clearly injective since $b \neq b'$ implies $[b, 1] \neq [b', 1]$ (recall Remark 2.3.12) and hence $\lambda_B(b) \neq \lambda_B(b')$. Let now $b, b' \in B$, hence $\lambda_B(b \lor b') = [b \lor b', 1] = [b, 1] \cup [b', 1] = \lambda_B(b) \cup \lambda_B(b')$. In a very similar way it can be shown that $\lambda_B(\neg b) = \neg \lambda_B(b)$ and $\lambda_B(0) = [1, 0]$.

As for the last claim, let $\lambda_R : R \to \mathcal{A}(B \otimes^\delta_e R)$ be the map $\lambda_R(c) = [1, c]$. Obviously $\lambda_R$ maps $R$ in $\mathcal{A}(B \otimes^\delta_e R)$ because $[1, c] > [0, \delta(c)]$. Moreover, it is fairly easy to prove that $\lambda_R$ is injective. As to prove that $\lambda_R$ is surjective, let $[b, c] \in \mathcal{A}(B \otimes^\delta_e R)$, thus $[b, c] > [\neg b, c]$, i.e., $[b, c] \cup [\neg b, \delta(c)] = [b, c]$. Now, $[b, c] \cup [\neg b, \delta(c)] = [1, (b \lor_c \delta(c)) \lor \neg (b \lor_c c)] = [b, c]$ implies that $b = 1$. Hence, by substitution, one has $[b, c] = [1, c]$. Thus $\lambda_R$ is a bijection. To conclude the proof notice that, for all $c, c' \in R$ and for $\ast \in \{\lor, \land, \to, \cdot\}$, $\lambda_R(c \ast c') = [1, c \ast c']$. Hence $\lambda_R$ is an isomorphism. 

**Notation 2.3.15.** In what follows we will adopt the notation used in the proof of Theorem 2.3.14 above: for every $(B, R, \lor_c)$ and for every wdl-admissible operator $\delta$, $\lambda_B$ and $\lambda_R$ denote the isomorphisms from $B$ to $\mathcal{A}(B \otimes^\delta_e R)$ and from $R$ to $\mathcal{A}(B \otimes^\delta_e R)$ respectively.
Remark 2.3.16. The original definition of the operator $\lor_e$ of external join provided in [92, Definition 4.1] makes use, with respect to Definition 2.3.1, of the following additional condition that turns out to be redundant:

(V4) For all $b \in B$ and for all $c, c' \in C$, $(b \lor_e c) \cdot c' = (-b \lor_e c') \land (b \lor_e (c \cdot c'))$.

Indeed, it is not difficult to show that (V4) can be proved from (V1)–(V3).

As a matter of fact, from Lemma 2.3.9, it is sufficient to prove that, for every $p \in \text{Max } B$,

$$(b/p \lor_e c/p) \cdot c'/p = (-b/p \lor_e c'/p) \land (b/p \lor_e (c/p \cdot c'/p)).$$

The latter can be easily proved by an easy computation taking into account that, for every $p$, $b/p$ is either 0 or 1.

Corollary 2.3.17. For every triplet $(B, R, \lor_e)$ and for every wdl-admissible operator $\delta$, if $B \otimes_\delta e R$ is directly indecomposable, then $B \cong \{0, 1\}$.

Proof. Immediately follows from Theorem 2.3.14 and Proposition 2.2.6.

Corollary 2.3.18. Let $(B, R, \lor_e)$ be a triplet, let $\delta$ be a wdl-admissible operator. Then the following hold:

1. $B$ and $R$ embed into $B \otimes_\delta e R$ via the maps $\lambda_B$ and $\lambda_R$,

2. for every $b \in B$ and every $c \in R$, $\lambda_R(b \lor_e c) = \lambda_B(b) \lor \lambda_R(c)$,

Furthermore, with respect to Proposition 2.2.11, the representation of the elements of $B \otimes_\delta e R$ provided in (i) and (ii) is unique. Indeed,

3. For any element $[b, c] \in B \otimes_\delta e R$, $[b, c] = ([b, 1] \cup \neg[1, \delta(c)]) \cap (\neg[b, 1] \cup [1, c])$.

In the particular cases of Stonean residuated lattices and wIDL-algebras we have:

4. If $B \otimes_\delta e R$ is a Stonean residuated lattice, for each of its elements, $[b, c] = [b, 1] \cap [1, c]$,.
(5) If \( B \otimes_\delta R \) is an wIDL-algebra, for each of its elements, \([b, c] = ([b, 1] \cup \neg [1, c]) \cap ([b, 1] \cup [1, c]). \)

**Proof.** Claim (1) is trivial. As to prove (2), notice that we have \( \lambda_R(b \vee c) = [1, b \vee c] = [b \vee 1, ((-b \vee 0) \vee (c \vee 1)) \land ((-b \vee 1) \vee (c \vee 0)) \) = \( [b, 1] \cup [1, c] = \lambda_B(b) \cup \lambda_R(c). \) It is left to prove (3), being (4) and (5) special cases. We have: ([1, 1] \cup \neg [1, \delta(c)]) \cap ([b, 1] \cup [1, c]) = ([b, 1] \cup [0, \delta(c)]) \cap ([\neg b, \delta(1)] \cup [1, c]) = ([b, 1] \cup [0, \delta(c)]) \cap ([\neg b, 1] \cup [1, c]) = [b, b \vee \delta(c)] \cap [1, \neg b \vee c] = [b, b \vee \delta(c) \vee (\neg b \vee c)] = [b, (b \vee \delta(c)) \land (\neg b \vee c)]. \) Now, recall that \([b, c] = [b', c'] \) iff \( b = b', \) \( -b \vee c = -b' \vee c' \) and \( b \vee \delta(c) = b' \vee \delta(c') \), thus let us check these three conditions:

- Trivially \( b = b. \)

- \( \neg b \vee c \) is equal to \( (b \vee \delta(c)) \land (\neg b \vee c) \) since \( \neg b \vee (b \vee \delta(c)) \land (\neg b \vee c) = (\neg b \vee (b \vee \delta(c)) \land (\neg b \vee c)) = ((\neg b \land b) \vee \delta(c)) \land (\neg b \vee c) = (b \vee \delta(c)) \land (\neg b \vee c) \)  

- \( b \vee \delta(c) \) equals \( b \vee \delta((b \vee \delta(c)) \land (\neg b \vee c)) \). Indeed, \( b \vee \delta((b \vee \delta(c)) \land (\neg b \vee c)) = b \vee \delta((b \vee \delta(c)) \land (\neg b \vee c)) = b \vee ((b \vee \delta(c)) \land (\neg b \vee c)) = (b \vee \delta(c)) \land (\neg b \vee c) = b \vee \delta(c). \)

Note that in the last point we have used the fact that \( \delta(b \vee c) = b \vee \delta(c). \)

Let us recall the notation introduced in Remark 2.1.2, where for every \( R \in RL, \delta_D, \delta_L : R \to R \) respectively denote the identity map, and the map constantly equal to 1. The following result specializes Theorem 2.3.14 above.

**Theorem 2.3.19.** Let \( R \) be a residuated lattice and let \((B, R, \vee_e) \) be a triplet. Then the following hold

1. \( B \otimes_\delta R \) is a Stonean residuated lattice;

2. \( B \otimes_\delta R \) is an wIDL-algebra.
Chapter 2. Disconnected $\delta$-rotations of residuated lattices

Proof. In both cases, since all properties we need to show are equationally described, it is sufficient to prove them for the directly indecomposable components of $B \otimes^\delta_e R$. Thanks to Corollary 2.3.17 each directly indecomposable component of $B \otimes^\delta_e R$ has $\{0, 1\}$ as Boolean skeleton.

(1) We have that $[0, c] \cap \neg[0, c] = [0, c] \cap ([0, c] \Rightarrow [0, 1]) = [0, c] \cap [1, 1] = [0, c]$ (recall that in $B \otimes^\delta_e R$, $[0, c] = [0, 1]$ for every $c \in R$), and $[1, c] \cap \neg[1, c] = [1, c] \cap ([1, c] \Rightarrow [0, 1]) = [1, c] \cap [0, 1] = [0, c]$.

(2) In order to prove the claim we have to show that $B \otimes^\delta_D R$ is involutive. Indeed, notice that, by (2.3) and the very definition of $\delta_D$, $\neg\neg[b, c] = [\neg\neg b, c] = [b, c]$. Hence the claim is settled.

2.4 Categorical equivalences

In this section we are going to establish categorical equivalences between subcategories of rDL-algebras and properly defined categories whose objects are RL-triplets.

Definition 2.4.1. The category $T_{RL}$ has residuated lattice-based triplets as objects while morphisms are good morphism pairs, i.e., if $(B, R, \lor_e)$ and $(B', R', \lor'_e)$ are two triplets, a pair $(f, g)$ is a good morphism pair if $f : B \to B'$ is a Boolean homomorphism, $g : R \to R'$ is a RL-homomorphism, and for every $(b, c) \in B \times R$, $g(b \lor_e c) = f(b) \lor'_e g(c)$.

Let $R$ be a subvariety of $RL$ and let $T_R$ the full subcategory of $T_{RL}$ whose objects are triplets $(B, R, \lor_e)$ where $R \in R$. Let $SRL_R$ and $wIDL_R$ be the full subcategory respectively of $SRL$ and $wIDL$ consisting of algebras $A$ such that $R(A) \in R$.

Definition 2.4.2. Let us define $\Xi^\delta_L : T_R \to SRL_R$ and $\Xi^\delta_D : T_R \to wIDL_R$ in the following manner: for every triplet $(B, R, \lor_e)$, for every good morphism $(f, g) : (B, R, \lor_e) \to (B', R', \lor'_e)$, and for every $X \in \{L, D\}$,

$$\Xi^L_R((B, R, \lor_e)) = B \otimes^\delta_L R$$
and for every \([b, c] \in B \otimes^\delta_e R\),

\[
\Xi_R^\delta_e(f, g)([b, c]) = [f(b), g(c)].
\]

Before showing that \(\Xi_R^\delta_L\) and \(\Xi_R^\delta_D\) are functors that establish a categorical equivalence, we need a preparatory result.

**Lemma 2.4.3.** For every residuated lattice homomorphism \(g : R \to R'\), \(g(\delta_L(c)) = \delta_L(g(c))\) and \(g(\delta_D(c)) = \delta_D(g(c))\).

**Proof.** The claim follows by the very definition of \(\delta_L\) and \(\delta_D\). \(\square\)

**Theorem 2.4.4.** The maps \(\Xi_R^\delta_L\) and \(\Xi_R^\delta_D\) are functors.

**Proof.** By Theorem 2.3.19, for every triplet \((B, R, \vee_e)\), \(B \otimes^\delta_e R\) is a Stonean residuated lattice and \(B \otimes^\delta_D R\) is an wDL-algebra. Thus, we only need to show that if \((f, g)\) is a good morphism, \(\Xi_R^\delta_L(f, g)\) is a morphism of SRL and \(\Xi_R^\delta_D(f, g)\) is a morphism of wDL. We will show the claim for the case of SRL. The case of wDL is analogous and left to the reader.

Let \((B, R, \vee_e)\) and \((B', R', \vee'_e)\) be triplets and let \((f, g)\) a good morphism between them. Let \([b, c], [b', c'] \in B \otimes^\delta_e R\). Then, \(\Xi_R^\delta_L(f, g)([b, c] \otimes [b', c']) = \Xi_R^\delta_L(f, g)([b, c]) \otimes \Xi_R^\delta_L(f, g)([b', c'])\).

As for \(\Xi_R^\delta_L(f, g)([b, c] \Rightarrow [b', c']) = \Xi_R^\delta_L(f, g)([b, c]) \Rightarrow \Xi_R^\delta_L(f, g)([b', c'])\).

\[
\Xi_R^\delta_L(f, g)(b \Rightarrow b', (\vee_{b \vee b'}(c' \Rightarrow c) \land (\vee_{b \vee b'}(c' \Rightarrow c'))) \land (\vee_{b \vee b'}(c \Rightarrow c')))
= (f(b \Rightarrow b'), (\vee_{f(b \vee b')}(c' \Rightarrow c) \land (\vee_{f(b \vee b')}(c' \Rightarrow c'))) \land (\vee_{f(b \vee b')}(c \Rightarrow c')))
= (f(b \Rightarrow b'), (\vee_{f(b \vee b')}(c' \Rightarrow c) \land (\vee_{f(b \vee b')}(c' \Rightarrow c'))) \land (\vee_{f(b \vee b')}(c \Rightarrow c')))
\]

\[
\Xi_R^\delta_D(f, g)(b \Rightarrow b', (\delta_L(c') \Rightarrow \delta_L(c)) \land (\delta_L(c) \Rightarrow \delta_D(c')) \land (\delta_D(c') \Rightarrow \delta_D(c)) \land (\delta_D(c) \Rightarrow \delta_D(c')))
= (f(b \Rightarrow b'), (\delta_L(c') \Rightarrow \delta_L(c)) \land (\delta_L(c) \Rightarrow \delta_D(c')) \land (\delta_D(c') \Rightarrow \delta_D(c)) \land (\delta_D(c) \Rightarrow \delta_D(c')))
\]

63
By Lemma 2.4.3 plus the fact that \( g \) is a homomorphism of residuated lattices, 
\[ g(\delta_L(c')) = \delta_L(g(c')) = \delta_L(g(c) \cdot g(c')) \].
Hence, \( \Xi_R(f,g)([b,c] \Rightarrow [b',c']) = \Xi_R(f,g)(b,c) \Rightarrow [b',c'] \).

The proofs for the lattice operations are similar and left to the reader. \( \Xi_R(f,g) \) clearly preserves the constants, hence \( \Xi_R(f,g) \) is a homomorphism. In order to prove that \( \Xi_R(f,g) \) is a functor, we need to show that it preserves the identity map, which is trivial, and compositions. To prove the latter, if \( (f,g) \) is a good morphism pair from \( (B,R,\lor_e) \) to \( (B',R',\lor'_e) \), and \( (f',g') \) is a good morphism from \( (B',R',\lor'_e) \) to \( (B'',R'',\lor''_e) \), it results:
\[ \Xi_R((f' \circ g)(f,g))(b,c) = \Xi_R(f,g)(f'(f(b)),g'(g(c))) = (\Xi_R(f,g))(f')) \circ (\Xi_R(f,g))(g') \].
Hence the claim is settled.

**Definition 2.4.5.** Let us define \( \Phi^\delta_L : \text{SRL}_R \rightarrow \text{JR}_R \) and \( \Phi^\delta_D : \text{wldL}_R \rightarrow \text{JR}_R \) in the following manner:

- For every \( A \in \text{SRL}_R \), \( \Phi^\delta_L(A) = (\mathcal{B}(A), \mathcal{A}(A), \lor) \) and for any morphism \( k : A \rightarrow A' \), \( \Phi^\delta_L(k) = (k_1, k_2) \).

- For every \( A \in \text{wldL}_R \), \( \Phi^\delta_R(A) = (\mathcal{B}(A), \mathcal{A}(A), \lor) \) and for any morphism \( k : A \rightarrow A' \), \( \Phi^\delta_R(k) = (k_1, k_2) \).

**Theorem 2.4.6.** The maps \( \Phi^\delta_L \) and \( \Phi^\delta_D \) are functors.

**Proof.** We prove the claim for \( \Phi^\delta_L : \text{SRL}_R \rightarrow \text{JR}_R \), the other case being similar. Given any \( A \in \text{SRL}_R \), it is easy to see that \( \Phi^\delta_L(A) = (\mathcal{B}(A), \mathcal{A}(A), \lor) \) is in \( \text{JR}_R \), since clearly the natural join satisfies the requested properties of Definition 2.3.1, and \( \mathcal{A}(A) \in \text{JR}_R \) by definition. Moreover, for any morphism \( k : A \rightarrow A' \), \( \Phi^\delta_L(k) = (k_1, k_2) \) is a good morphism pair. Indeed, since \( \mathcal{B}(A) \) is a Boolean subalgebra of \( A \), \( \mathcal{A}(A) \) is a GMTL-subalgebra of the 0-free reduct of
A, and \( k \) is a homomorphism, it follows that \( k_{1_{\mathcal{R}(A)}} \) and \( k_{1_{\mathcal{R}(A)}} \) are respectively a Boolean homomorphism from \( \mathcal{R}(A) \) to \( \mathcal{R}(A') \) and a RL-homomorphism from \( \mathcal{R}(A) \) to \( \mathcal{R}(A') \). Also, being \( k \) a homomorphism, trivially \( k_{1_{\mathcal{R}(A)}}(b \lor c) = k(b \lor c) \lor' k(c) = k_{1_{\mathcal{R}(A)}}(b) \lor' k_{1_{\mathcal{R}(A)}}(c) \).

It is left to prove that \( \Phi^\delta_L \) preserves the identity map, which is trivial, and composition of morphisms:

\[
\Phi^\delta_L (k \circ j)(b, c) = ((k \circ j)_{1_{\mathcal{R}(A)}}, (k \circ j)_{1_{\mathcal{R}(A)}})(b, c) = ((k \circ j)(b), (k \circ j)(c)) = (k(j(b)), k(j(c))) = (k_{1_{\mathcal{R}(A)}}, k_{1_{\mathcal{R}(A)}})(j(b), j(c)) = (k_{1_{\mathcal{R}(A)}}, k_{1_{\mathcal{R}(A)}})((j_{1_{\mathcal{R}(A)}}, j_{1_{\mathcal{R}(A)}})(b, c)) = \Phi^\delta_L (k) \circ \Phi^\delta_L (j)(b, c).
\]

\[
\square
\]

**Theorem 2.4.7.** Given any subvariety \( R \) of RL, the following holds:

- The pair \( (\Xi^\delta_L, \Phi^\delta_L)_R \) establishes a categorical equivalence between \( SRL_R \) and \( \mathcal{I}_R \).
- The pair \( (\Xi^\delta_D, \Phi^\delta_D)_R \) establishes a categorical equivalence between \( \mathcal{I}DL_R \) and \( \mathcal{I}_R \).

**Proof.** Let us prove that \( \Xi^\delta_L \) and \( \Xi^\delta_D \) are full, faithful and essentially surjective. Let us start by proving the claim for \( \Xi^\delta_D \).

Let \((B, R, \vee_e)\) and \((B', R', \vee'_e)\) be triplets and let \( \tau : B \otimes^e D_R \rightarrow B' \otimes^e D_{R'} \) be an sDL-morphism. Let \( f : B \rightarrow B' \) be defined as follows: for every \( b \in B \), \( f(b) = b' \), where by Theorem 2.3.14, \( b' \) is the unique element of \( B' \) such that \( \tau([b, 1]) = [b', 1] \). Notice that, since \( [b, 1] \in \mathcal{R}(B \otimes^e D_R) \) and \( \tau \) is a homomorphism, \( \tau([b, 1]) \in \mathcal{R}(B' \otimes^e D_{R'}) \). Analogously, let \( g : R \rightarrow R' \) be defined as follows: for every \( c \in R \), \( g(c) = c' \), where \( c' \) is the unique element of \( R' \) such that \( \tau([1, c]) = [1, c'] \). Notice that \( b' \) and \( c' \) are uniquely determined because of Remark 2.3.12. Thus, \( f \) and \( g \) are well-defined. The maps \( f \) and \( g \) are homomorphisms since by Theorem 2.3.14 they are, respectively, the restriction of \( \tau \) to \( \mathcal{R}(B \otimes^e D_R) \cong B \) and \( \mathcal{R}(B \otimes^e D_R) \cong R \). Moreover, \( g(b \vee_e c) = f(b) \lor' g(c) \). Indeed, \( g(b \vee_e c) = c' \) such that \( \tau([1, b \vee_e c]) = [1, c'] \). \( f(b) = b' \) such that \( \tau([b, 1]) = [b', 1] \) and \( g(c) = c'' \) such that \( \tau([1, c]) = [1, c''] \). Now, by
Corollary 2.3.18 (2), \( \lambda_R(b \vee c) = \lambda_B(b) \cup \lambda_R(c) \), i.e. \([1, b \vee c] = [b, 1] \cup [1, c].\) Since \( \tau \) is an homomorphism, we get \( \tau([1, b \vee c]) = \tau([b, 1]) \cup \tau([1, c]), \) thus \([1, c'] = [b', 1] \cup [1, c''] = [1, b' \vee e c''], \) which implies \( g(b \vee e c) = c' = b' \vee e c'' = f(b) \vee e g(c). \) Hence, \((f, g)\) is a morphism in \( \mathcal{T}_R, \) whence \( \Xi^D_R \) is full.

As to prove that \( \Xi^D_R \) is faithful, let \((f, g), (f', g') : (B, R, \vee e) \to (B', R', \vee'_e)\) and assume that, for every \([b, c] \in B \otimes^D e R\)

\[
\Xi^D_R((f, g))([b, c]) = \Xi^D_R((f', g'))([b, c]),
\]

that is, \([f(b), g(c)] = [f'(b), g'(c)].\) By Remark 2.3.12, this means that \( f(b) = f'(b) \) for every \( b.\) Hence, in particular, if \( b = 1, [1, g(c)] = [1, g'(c)].\) Thus, again by Remark 2.3.12, \( g(c) = g'(c), \) whence \((f, g) = (f', g')\) and \( \Xi^D_R \) is faithful.

Finally, let \( A \in \mathbf{wlDL}_R.\) We need to prove that

\[
A \cong \Xi^D_R \Phi^D_R(A) = \mathcal{B}(A) \otimes^D e \mathcal{B}(A).
\]

For any \( a \in A, a = (b \vee \neg c) \land (\neg b \vee c), \) where \( b \in \mathcal{B}(A), c \in \mathcal{B}(A).\) Let \( \alpha : A \to \mathcal{B}(A) \otimes^D e \mathcal{B}(A)\) be such that \( \alpha(a) = [b, c].\) By Corollary 2.3.18 (1), (5) it follows that \( \alpha \) is well defined and injective. Surjectivity is trivial. It is enough to prove that \( \alpha \) is an order preserving monoid homomorphism. Let \( A \) be represented as a subdirect product of \( \prod_{p \in \text{Spec}(A)} A/p.\) Hence, every component \( a_p \) of \( a \in A \) is in the form \((b \vee \neg c) \land (\neg b \vee c)/p = (b/p \vee \neg c/p) \land (\neg b/p \vee c/p)\) where now \( b/p \in \{0, 1\}.\) For the sake of a lighter notation, let us write \( b_p\) instead of \( b/p\) and \( c_p\) for \( c/p.\) Let \( a, a' \in A \) with \( a \leq a'.\) Then, \( a_p \leq a'_p\) for every \( p \in \text{Spec}(A)\) whence

\[
(b_p \vee \neg c_p) \land (\neg b_p \vee c_p) \leq (b'_p \vee \neg c'_p) \land (\neg b'_p \vee c'_p), \quad (2.10)
\]

Let us show that \( \alpha(a) \leq \alpha(a')\) that is, \([b, c] \cap [b', c'] = [b, c],\) that is to say, \([b \land b', \nu_{b \land c'}(c \land c') \land \nu_{b \land c'}(c) \land \nu_{\neg b \lor b'}(c') \land \nu_{\neg b \lor b'}(c \times c')] = [b, c]\) and hence, componentwise,

\[
b_p \land b'_p = b_p \land \nu_{b \land b'_p}(c_p \land c'_p) \land \nu_{\neg b \lor b'_p}(c_p) \land \nu_{\neg b \lor b'_p}(c'_p) = c_p \quad (2.11)
\]

Chapter 2. Disconnected \( \delta \)-rotations of residuated lattices
• If \( b_p = b'_p = 0 \), Equation (2.11) reduces to \( 0 = 0 \) and \( c_p \lor c'_p = c_p \). The latter holds by Equation (2.10) since it reduces to \( \neg c_p \leq \neg c'_p \).

• If \( b_p = b'_p = 1 \), Equation (2.11) reduces to \( 1 = 1 \) and \( c_p \land c'_p = c_p \). Again the latter equation holds by Equation (2.10) since it reduces to \( c_p \leq c'_p \).

• If \( b_p = 0 \) and \( b'_p = 1 \), Equation (2.11) reduces to \( 0 = 0 \) and \( c_p = c_p \).

• If \( b_p = 1 \) and \( b'_p = 0 \), Equation (2.10) is false since it reduces to \( c_p \leq \neg c'_p \) which is impossible since \( c_p \) and \( c'_p \) belong to \( \mathcal{R}(A/p) \).

Hence \( \alpha \) is order preserving. Let us now show that \( \alpha \) is a homomorphism of monoids. First of all for all \( \alpha(1) = [1, 1] \) by definition. Moreover, if \( a, a' \in A \), let us prove that \( \alpha(a \cdot a') = [b, c] \circ [b', c'] \).

**Claim 2.4.8.** \( a \cdot a' = (\beta \lor \neg c) \land (\neg \beta \lor c) \lor (\neg \beta \lor c') \lor (c \land c') \) for \( \beta = b \land b' \) and \( \gamma = ((b \lor \neg b') \lor (c' \land c')) \lor ((\neg b \lor \neg b') \lor (c' \land c')) \).

**Proof.** (of Claim 2.4.8). First of all \( a \cdot a' = (b \lor \neg c) \land (\neg \beta \lor c) \lor (\neg \beta \lor c') \lor (c \land c') \). Further, \( a \cdot a' \in A \) and hence there are \( \beta \in \mathcal{R}(A) \) and \( \gamma \in \mathcal{R}(A) \) such that \( a \cdot a' = (\beta \lor \neg c) \land (\neg \beta \lor c) \). Let us prove that, indeed, \( \beta = b \land b' \) and \( \gamma = ((b \lor \neg b') \lor (c' \land c')) \lor ((\neg b \lor \neg b') \lor (c' \land c')) \). Clearly \( \beta \in \mathcal{R}(A) \) and, since \( \mathcal{R}(A) \) is a filter of \( A \) and hence \( x \lor c \in \mathcal{R}(A) \) for every \( x \in \mathcal{R}(A) \) and \( c \in \mathcal{R}(A) \). Thus, \( \gamma \in \mathcal{R}(A) \). Let us prove the claim componentwise, that is

\[
(b_p \lor -c_p) \land (-b_p \lor c_p) \land (b'_p \lor -c'_p) \land (-b'_p \lor c'_p) = (\beta_p \lor -\gamma_p) \land (\neg \beta_p \lor \gamma_p), \tag{2.12}
\]

where, conventionally, \( \beta_p = b_p \land b'_p \) and \( \gamma_p \) is defined analogously.

Again we enter a case distinction.

• If \( b_p = b'_p = 0 \), \( \beta_p = 0 \) and \( \gamma_p = 1 \), whence \( (\beta_p \lor -\gamma_p) \land (\neg \beta_p \lor \gamma_p) = 0 \) and the left-hand expression of Equation (2.12) reduces to \( -c_p \land -c'_p = 0 \).

• If \( b_p = b'_p = 1 \), \( \beta_p = 1 \) and \( \gamma_p = c_p \land c'_p \), \( (\beta_p \lor -\gamma_p) \land (\neg \beta_p \lor \gamma_p) = c_p \land c'_p \) and \( (b_p \lor -c_p) \land (-b_p \lor c_p) \land (b'_p \lor -c'_p) \land (-b'_p \lor c'_p) = c_p \land c'_p \) as desired.

67
Claim 2.4.9. If $b_p = 0$ and $b'_p = 1$, then $\beta_p = 0$, $\gamma_p = c'_p \rightarrow c_p$, $(\beta_p \lor \lnot \gamma_p) \land (\lnot \beta_p \lor \gamma_p) = \lnot (c_p \rightarrow c'_p)$ and the left-hand expression of Equation (2.12) is $\lnot c_p \cdot c'_p$ which equals $\lnot (c_p \rightarrow c'_p)$.

If $b_p = 0$ and $b'_p = 1$, then $\beta_p = 1$, $\gamma_p = c_p$, $(\beta_p \lor \lnot \gamma_p) \land (\lnot \beta_p \lor \gamma_p) = \lnot (c_p \rightarrow c'_p)$ and the left-hand expression of Equation (2.12) reduces to $c_p \cdot \lnot c'_p$ which equals $\lnot (c_p \rightarrow c'_p)$.

And Claim 2.4.8 is completely proved.

Thus, $\alpha(a \cdot a') = [\beta, \gamma] = [b \land b', ((b \lor \lnot b') \lor c'(c \rightarrow c')) \land ((\lnot b \lor \lnot b') \lor e(c \rightarrow c'))] = [b, c] \lor [b', c']$. Hence, $\alpha$ is a monoid homomorphism.

We now have to prove that $\alpha$ preserves the implication. That is to say, if $x, x' \in X$, let us prove that $\alpha(x \rightarrow x') = [b, c] \Rightarrow [b', c']$.

Claim 2.4.9. $x \rightarrow x' = (\beta \lor \lnot \gamma) \land (\lnot \beta \lor \gamma)$ for $\beta = b \rightarrow b'$ and $\gamma = \nu_{b \lor b'}(\delta(c') \rightarrow \delta(c)) \lor \nu_{b \lor b'}(\delta(c \cdot c') \lor \lnot b' \lor b'(c \rightarrow c'))$.  

Proof. (of Claim 2.4.9). Clearly $x \rightarrow x' = ((b \lor \lnot c) \land (\lnot b \lor c)) \Rightarrow ((b' \lor \lnot c) \land (\lnot b' \lor c'))$. Let us prove the claim on the components, that is to say for each $p \in \text{Spec} X$:

$((b_p \lor \lnot c_p) \land (\lnot b_p \lor c_p)) \Rightarrow ((b'_p \lor \lnot c'_p) \land (\lnot b'_p \lor c'_p)) = (\beta_p \lor \lnot \gamma_p) \land (\lnot \beta_p \lor \gamma_p)$ (2.13)

We shall denote with (A) the left-hand side of Equation (2.13), and with (B) its right-hand side.

Again we enter a case distinction.

- If $b_p = b'_p = 0$, $\beta_p = 1$, and $\gamma_p = c'_p \rightarrow c_p$, whence (B) $= c'_p \rightarrow c_p$ and (A) reduces to $\lnot c_p \rightarrow \lnot c'_p$. They coincide, as can be seen directly via the definition of $\rightarrow$ in the directly indecomposable components.

- If $b_p = b'_p = 1$, $\beta_p = 1$ and $\gamma_p = c_p \rightarrow c'_p$, thus both (A) and (B) reduce to $c_p \rightarrow c'_p$ as desired.

- If $b_p = 0$ and $b'_p = 1$, then $\beta_p = 1$, $\gamma_p = 1$, then (B) equals 1 and (A) reduces to $\lnot c_p \rightarrow c'_p = 1$. 

68
Chapter 2.Disconnected $\delta$-rotations of residuated lattices

- If $b_p = 1$ and $b_p' = 0$, then $\beta_p = 0$, $\gamma_p = c_p \cdot c_p'$. Hence, (A) reduces to $c_p \rightarrow \neg c_p'$ and (B) reduces to $\neg (c_p \cdot c_p')$. The fact that they coincide can be proved in the components using the definition of the operations.

Thus Claim 4.4.9 is proved.\hfill $\Box$

Hence, $\alpha(x \rightarrow x') = [b, c] \rightarrow [b', c']$, thus $\alpha$ is an isomorphism, and the functor is essentially surjective.

In order to prove that $(\Xi^\delta_* \Phi^\delta_*)$ establishes a categorical equivalence between $SRL_R$ and $T_R$, notice that the previous proof showing that $\Xi^\delta_D$ is full and faithful perfectly adapts to the case of $\Xi^\delta_L$. On the other hand proving that $\Xi^\delta_L$ is essentially surjective strongly depends on the representation of the elements of an Stonean residuated lattice (recall Proposition 2.2.11), thus, the previous proof cannot be immediately adapted to this case. However, the elements of a Stonean residuated lattice $A$ can be expressed as $b \land c$ where $b \in \mathcal{B}(A)$ and $c \in \mathcal{R}(A)$. This situation is completely analogous to the case of product algebras investigated in [92]. Thus, proving our claim can be derived by direct inspection on the proof of [92, Theorem 6.1] and following the same lines of the case above.\hfill $\Box$

The following corollary collects results concerning categorical equivalences between relevant subcategories of $rDL$. We recall that we denote with $CH$ the variety of *cancellative hoops*, with $GH$ the variety of *Gödel hoops*, and we refer to unbounded Heyting algebras as *generalized Heyting algebras*, and to their variety as $GHA$.

**Corollary 2.4.10.** For every subvariety $R$ of $RL$, the algebraic categories $SRL_R$ and $wIDL_R$ are equivalent. In particular, the following categories are equivalent:

(i) $SRL$ and $wIDL$,

(ii) $SMTL$ and $sIDL$,

(iii) $P$ and $DLMV$,

(iv) $G$ and $NM^-$,
\(v\) \(SHA\) and \(NR^-\).

**Proof.** For every SRL \(A\), \(\mathcal{R}(A)\) is a residuated lattice, hence \(SRL_{RL} = SRL\). Analogously, \(wIDL_{RL} = wIDL\). Now, we need to show that \(SRL_{GMTL} = SMTL\), \(wIDL_{GMTL} = sIDL\), \(SRL_{CH} = P\), \(wIDL_{CH} = DLMV\), \(SRL_{GH} = G\) and \(wIDL_{GH} = NM^-\), \(SRL_{GHA} = H\) and \(wIDL_{GHA} = NR^-\). Let us prove that \(SRL_{GH} = G\), the other cases being similar. Let \(A\) be a Gödel algebra, then clearly \(A \in SRL_{GH}\). Conversely, if \(A\) is a SRL such that \(\mathcal{R}(A) \in GH\), then every directly indecomposable component of \(A\) is of the form \(2@H\), where \(H\) is a Gödel hoop, hence \(A\) is a Gödel algebra. Then the claim directly follows from Theorem 2.4.7. \(\square\)

### 2.5 A general construction to build rDL-algebras

The two operators \(\delta_L\) and \(\delta_D\) which we use in the previous section to provide a categorical equivalence between (subvarieties of) Stonean residuated lattices and (subvarieties of) \(wIDL\)-algebras are not the unique examples of \(wdl\)-admissible operators, recall for instance the example seen in Remark 2.1.3. Moreover, take \(A\) to be a rDL-algebra not belonging to \(SRL \cup wIDL\). The map \(\neg: \mathcal{R}(A) \to \mathcal{R}(A)\), where \(\neg\) is the operation of \(A\), is \(wdl\)-admissible (by Theorem 2.1.7) but it does not coincide neither with \(\delta_D\), since \(\neg\) is not involutive, nor with \(\delta_L\), since \(\neg\) is not a negation of Stonean residuated lattices.

Thus, in order to cope with the whole class of rDL-algebras and also to provide a more general construction than the one encoded by triplets, we now present the category of quadruples.

In order to keep the same level of generality adopted so far, let us fix any subvariety \(R\) of \(RL\), and define \(\Omega_R\) to be the following category:

\[\begin{align*}
\text{(Q1)} & \quad \text{The objects are quadruples } (B, R, \vee_e, \delta) \text{ where } R \in R, (B, R, \vee_e) \in \mathcal{T}_R \text{ and } \delta: R \to R \text{ wdl-admissible.} \\
\text{(Q2)} & \quad \text{The morphisms are pairs } (f, g): (B_1, R_1, \vee_{e_1}, \delta_1) \to (B_2, R_2, \vee_{e_2}, \delta_2), \text{ such that } (f, g) \text{ is a good morphism from } (B_1, R_1, \vee_{e_1}) \text{ to } (B_2, R_2, \vee_{e_2}) \text{ and for all } x \in R_1, g(\delta_1(x)) = \delta_2(g(x)).
\end{align*}\]
Chapter 2.Disconnected δ-rotations of residuated lattices

Let \( rDL_R \) be the full subcategory of \( rDL \) consisting of algebras \( A \) such that \( \mathcal{A}(A) \in R \).

**Remark 2.5.1.** Notice that the condition (Q2) that fixes the morphisms in the category of quadruples requires that \( g \) is a homomorphism with respect to wdl-admissible operators (i.e., \( g(\delta_1(x)) = \delta_2(g(x)) \)). This requirement should not surprise since it makes explicit the same property which we proved to be satisfied by \( \delta_L \) and \( \delta_D \) in Lemma 2.4.3.

**Definition 2.5.2.** Let us define \( \Phi_R : rDL_R \rightarrow Q_R \) in the following manner:

- \( \Phi_R(A) = (\mathcal{A}(A), \mathcal{A}(A), \vee, \neg \neg) \), for every \( A \in rDL_R \).
- \( \Phi_R(k) = (k|_{\mathcal{A}(A)}, k|_{\mathcal{A}(A)}) \), for any homomorphism \( k : A \rightarrow A' \).

**Theorem 2.5.3.** The map \( \Phi_R \) is a functor.

**Proof.** Clearly, \( (\mathcal{A}(A), \mathcal{A}(A), \vee) \in \mathcal{I}_R \), and from Theorem 2.1.7 we have that the double negation \( \neg \neg \) is a wdl-admissible map from \( \mathcal{A}(A) \) to \( \mathcal{A}(A) \). For any morphism \( k : A \rightarrow A' \), we already proved that \( \Phi_R(k) \) is a good morphism from \( (\mathcal{A}(A), \mathcal{A}(A), \vee) \) to \( (\mathcal{A}(A'), \mathcal{A}(A'), \vee') \). Now, \( \Phi_R(k) \) is a map from \( (\mathcal{A}(A), \mathcal{A}(A), \vee, \neg \neg) \) to \( (\mathcal{A}(A'), \mathcal{A}(A'), \vee', \neg' \neg') \), hence it is left to show that \( k(\neg \neg(x)) = \neg' \neg'(k(x)) \), for all \( x \in \mathcal{A}(A) \), which follows from the fact that \( k \) is a homomorphism. The proof of the fact that \( \Phi_R \) preserves the identity map and composition is the same as in the proof of Theorem 2.4.6. \( \square \)

**Definition 2.5.4.** Let us define \( \Xi_R : Q_R \rightarrow rDL_R \) in the following manner: for every \( (B, R, \vee_e, \delta) \in Q_R \), and for every good morphism \( (f, g) : (B, R, \vee_e, \delta) \rightarrow (B', R', \vee'_e, \delta') \),

- \( \Xi_R((B, R, \vee_e, \delta)) = B \otimes^d_e R \).
- \( \Xi_R(f, g)((b, c)) = [f(b), g(c)] \), for every \( [b, c] \in B \otimes^d_e R \).

**Theorem 2.5.5.** The map \( \Xi_R \) is a functor.
that, since \( \tau \), recalling how negation is computed on a pair from Equation 2.3, we have

\[
if \tau \leftarrow prove that for any rDL-homomorphism \( \tau \).
\]

**Proof.** We need to prove that \( \Xi_R \) is full, faithful and essentially surjective. Faithfulness can be proved in the same way as in Theorem 2.4.7, since we used general results, not depending on a particular choice of the wdl-admissible operators. In order to show the fullness of \( \Xi_R \), following the same lines of the proof of Theorem 2.4.7, and defining the pair \((f, g)\) in the same way, it is left to prove that for any rDL-homomorphism \( \tau : B_1 \oplus R_1 \rightarrow B_2 \oplus R_2 \), if \( \tau[1, c] = [1, c'] \) then \( \tau[1, \delta_1(c)] = [1, \delta_2(c')] \). This follows from the fact that, since \( \tau \) is a rDL-homomorphism, \( \tau(-1-1[1, c]) = -2-2\tau([1, c]) \). But, recalling how negation is computed on a pair from Equation 2.3, we have \( \tau(-1-1[1, c]) = \tau([1, \delta_1(\delta_1(c))]) = \tau([1, \delta_1(c)]) \), and \( -2-2\tau([1, c]) = -2-2[1, c'] = [1, \delta_2\delta_2(c')] = [1, \delta_2(c')] \).

Now, let us prove that \( \Xi_R \) is essentially surjective. Notice that the proof will again be similar to the one for wDL. Now, let \( A \in rDL \). We need to prove that \( A \cong \Xi_R \Phi_R(A) = \mathcal{B}(A) \otimes \delta_e \mathcal{R}(A) \), where \( \delta = -\cdot \), by the definition of \( \Phi_R \). For any \( a \in A \), \( a = (b \lor -\cdot c) \land (-b \lor c) = (b \lor -\delta(c)) \land (-b \lor c) \), where \( b \in \mathcal{B}(A), c \in \mathcal{R}(A) \). Let \( \alpha : A \rightarrow \mathcal{B}(A) \otimes \delta_e \mathcal{R}(A) \) be such that \( \alpha(a) = [b, c] \).

By Corollary 2.3.18 (1),(3) it follows that \( \alpha \) is well defined and injective, while surjectivity is trivial. Again, it is enough to prove that \( \alpha \) is an order preserving monoid homomorphism. Let \( A \) be represented as a subdirect product of \( \prod_{p \in \text{Spec}(A)} A/p \). Hence, every component \( a_p \) of \( a \in A \) is in the form \( (b \lor -\delta(c)) \land (-b \lor c)/p = (b/p \lor -\delta(c/p)) \land (-b/p \lor c/p) \) where now \( b/p \in \{0,1\} \).

For the sake of a lighter notation, let us write \( b_p \) instead of \( b/p \) and \( c_p \) for \( c/p \), and recall that \( \delta = -\cdot \). Let \( a, a' \in A \) with \( a \leq a' \). Then, \( a_p \leq a'_p \) for every
Let us show that $\alpha(a) \leq \alpha(a')$ that is, $[b, c] \cap [b', c'] = [b, c]$, that is to say, $[b \land b', \lor_{b \lor b'}(c \lor c')] \land \lor_{b \lor b'}(c \lor c') \land \lor_{b \lor b'}(c' \lor c') = [b, c]$ and hence, componentwise,

$$b_p \land b'_p = b_p \quad \text{(2.15)}$$

- If $b_p = b'_p = 0$, Equation (2.15) reduces to $0 = 0$, $1 = 1$ and $\delta(c_p \lor c'_p) = \delta(c_p)$. The latter holds by Equation (2.14) since it reduces to $-\delta(c_p) \leq -\delta(c'_p)$.

- If $b_p = b'_p = 1$, Equation (2.15) reduces to $1 = 1$ and $c_p \land c'_p = c_p$. Again the latter equation holds by Equation (2.14) since it reduces to $c_p \leq c'_p$.

- If $b_p = 0$ and $b'_p = 1$, Equation (2.15) reduces to $0 = 0$, $1 = 1$ and $\delta(c_p) = \delta(c'_p)$.

- If $b_p = 1$ and $b'_p = 0$, Equation (2.14) is false since it reduces to $c_p \leq -\delta(c'_p)$ which is impossible since $c_p$ and $\delta(c'_p)$ belong to $\mathcal{R}(A/p)$.

Hence $\alpha$ is order preserving. In order to conclude the proof, let us show that $\alpha$ is a homomorphism of monoids. First of all $\alpha(1) = [1, 1]$ by definition. Moreover, if $a, a' \in A$, let us prove that $\alpha(a \cdot a') = [b, c] \cap [b', c']$.

**Claim 2.5.7.** $a \cdot a' = (\beta \lor \neg \delta(\gamma)) \land \neg (\beta \lor \neg \delta(\gamma))$ for $\beta = b \land b'$ and $\gamma = ((b \lor b') \lor (c' \lor c')) \land ((b \lor b') \lor (c' \lor c'))$.

**Proof.** (of Claim 2.5.7.) First of all $a \cdot a' = (b \lor \neg \delta(c)) \land (b \lor \neg \delta(c')) \land (b \lor \neg \delta(c')) \land (b \lor \neg \delta(c'))$. Further, $a \cdot a' \in A$ and hence there are $\beta \in \mathcal{R}(A)$ and $\gamma \in \mathcal{R}(A)$ such that $a \cdot a' = (\beta \lor \neg \delta(\gamma)) \land (\beta \lor \neg \delta(\gamma))$. Let us prove that, indeed, $\beta = b \land b'$ and $\gamma = ((b \lor b') \lor (c' \lor c')) \land ((b \lor b') \lor (c' \lor c'))$. Clearly $\beta \in \mathcal{R}(A)$ and, since $\mathcal{R}(A)$ is a RL, it is closed under $\to$, and $\cdot$. Moreover,
\[ \mathcal{R}(A) \] is a filter of \( A \) and hence \( x \lor c \in \mathcal{R}(A) \) for every \( x \in \mathcal{R}(A) \) and \( c \in \mathcal{R}(A) \). Thus, \( \gamma \in \mathcal{R}(A) \). Let us prove the claim componentwise, that is

\[
(b_p \lor -\delta(c_p)) \land (-b_p \lor c_p) \land (-b'_p \lor c'_p) = (\beta_p \lor -\delta(\gamma_p)) \land (-\beta_p \lor \gamma_p), \tag{2.16}
\]

where, conventionally, \( \beta_p = b_p \land b'_p \) and \( \gamma_p \) is defined analogously.

Again we enter a case distinction.

- If \( b_p = b'_p = 0 \), \( \beta_p = 0 \), and \( \gamma_p = 1 \), whence \((\beta_p \lor -\delta(\gamma_p)) \land (-\beta_p \lor \gamma_p) = 0 \) and the left-hand expression of Equation (2.16) reduces to \(-\delta(c_p) \land -\delta(c'_p) = 0 \).

- If \( b_p = b'_p = 1 \), \( \beta_p = 1 \) and \( \gamma_p = c_p \cdot c'_p \), \((\beta_p \lor -\delta(\gamma_p)) \land (-\beta_p \lor \gamma_p) = c_p \cdot c'_p \) and \((b_p \lor -\delta(c_p)) \land (-b_p \lor c_p) \land (b'_p \lor -\delta(c'_p)) \land (-b'_p \lor c'_p) = c_p \cdot c'_p \) as desired.

- If \( b_p = 0 \) and \( b'_p = 1 \), then \( \beta_p = 0 \), \( \gamma_p = c'_p \rightarrow c_p \), \((\beta_p \lor -\delta(\gamma_p)) \land (-\beta_p \lor \gamma_p) = -\delta(c'_p \rightarrow c_p) \) and the left-hand expression of Equation (2.16) is \(-\delta(c_p) \cdot c'_p \). We need to show that \(-\delta(c_p) \cdot c'_p = -\delta(c'_p \rightarrow c_p) \). In every rDL-algebra, \( \neg\neg x = \neg x \) holds and hence, since \( \delta = \neg \), it is left to show that \(-c_p \cdot c'_p = -(c'_p \rightarrow c_p) \). The latter equation hence follows by the very definition of \( \cdot \) and \( \rightarrow \) in every directly indecomposable rDL-algebra.

- If \( b_p = 1 \) and \( b'_p = 0 \), then \( \beta_p = 0 \), \( \gamma_p = c_p \rightarrow c'_p \), \((\beta_p \lor -\delta(\gamma_p)) \land (-\beta_p \lor \gamma_p) = -\delta(c_p \rightarrow c'_p) \) while the left-hand expression of Equation (2.16) reduces to \(-\delta(c'_p \cdot c_p) \) which, by the same argument presented in the point above, is equal to \(-\delta(c_p \rightarrow c'_p) \).

And Claim 4.4.8 is completely proved. \( \square \)

We have that \( \alpha(a \cdot a') = [\beta, \gamma] = [b \land b', ((b \lor \neg b') \lor (c' \rightarrow c)) \land (\neg b \lor \neg b' \lor c') \land (\neg b \lor \neg b' \lor c')] = [b, c] \sqcap [b', c'] \), thus \( \alpha \) is a monoid homomorphism.

We now have to prove that \( \alpha \) preserves the implication. That is to say, if \( x, x' \in X \), let us prove that \( \alpha(x \rightarrow x') = [b, c] \Rightarrow [b', c'] \).

**Claim 2.5.8.** \( x \rightarrow x' = (\beta \lor \neg \delta(\gamma)) \land (\neg \beta \lor \gamma) \) if \( \beta = b \rightarrow b' \) and \( \gamma = \nu_{b \lor b'}(\delta(c') \rightarrow \delta(c)) \land \nu_{\neg b \lor b'}(c \rightarrow c') \land \nu_{\neg b \lor \neg b'}(c \rightarrow c'). \)
Proof. (of Claim 2.5.8). Clearly \( x \rightarrow x' = ((b \lor -c) \land (-b \lor c)) \rightarrow ((b' \lor -c') \land (-b' \lor c')) \). Let us prove the claim on the components, that is to say for each \( p \in \text{Spec} X \):

\[
((b_p \lor -c_p) \land (-b_p \lor c_p)) \rightarrow ((b'_p \lor -c'_p) \land (-b'_p \lor c'_p)) = (\beta_p \lor -\gamma_p) \land (-\beta_p \lor \gamma_p) \tag{2.17}
\]

We shall denote with (A) the left-hand side of Equation (2.17), and with (B) its right-hand side.

Again we enter a case distinction.

- If \( b_p = b'_p = 0 \), \( \beta_p = 1 \), and \( \gamma_p = c_p \rightarrow c'_p \), whence (B) equals \( c_p \rightarrow c'_p \) as desired.

- If \( b_p = b'_p = 1 \), \( \beta_p = 1 \) and \( \gamma_p = c_p \rightarrow c'_p \), thus both (A) and (B) reduce to \( c_p \rightarrow c'_p \) as desired.

- If \( b_p = 0 \) and \( b'_p = 1 \), then \( \beta_p = 1 \), \( \gamma_p = 1 \), then (B) equals 1 and (A) reduces to \( -c_p \rightarrow -c'_p = 1 \).

- If \( b_p = 1 \) and \( b'_p = 0 \), then \( \beta_p = 0 \), \( \gamma_p = \delta(c_p \cdot c'_p) \). Hence, (A) reduces to \( c_p \rightarrow -c'_p \) and (B) reduces to \( -\delta(c_p \cdot c'_p) = -(c_p \cdot c'_p) \). Again, the fact that they coincide can be proved in the components using the definition of the operations. Indeed \( -(1, c_p) \cdot (1, -c'_p) = (0, \delta(c_p \cdot c'_p)) \), and \( (1, c_p) \rightarrow -(1, c'_p) = (0, \delta(c_p \cdot \delta(c_p))) \). Now, by the definition of wdl-admissible operator, we get that \( \delta(c_p \cdot \delta(c_p)) = \delta(\delta(c_p \cdot \delta(c_p))) = \delta(\delta(c_p \cdot \delta(c_p))) = \delta(c_p \cdot c_p) \), thus (A) = (B).

Thus Claim 4.4.9 is completely proved.\[\Box\]

Hence, \( \alpha(x \rightarrow x') = [b, c] \rightarrow [b', c'] \), thus \( \alpha \) is an isomorphism. Hence, \( \Xi_R \) is essentially surjective and the claim is settled.\[\Box\]

Finally, let \( \mathcal{R} \) be any subclass of \( \text{RL} \) and let \( \mathbf{I}(\mathcal{R}) \) be the collection of the isomorphic copies, in \( \text{RL} \), of the elements in \( \mathcal{R} \). Let \( \Delta_{\mathcal{R}} \) be an indexed family, on \( \mathbf{I}(\mathcal{R}) \), of wdl-admissible operators such that, for every \( R \in \mathbf{I}(\mathcal{R}) \), \( \delta_R : R \rightarrow R \).
Chapter 2. Disconnected $\delta$-rotations of residuated lattices

We will say that $\Delta_R$ is $\mathcal{R}$-compatible if for every homomorphism $g : R_1 \to R_2$, with $R_1, R_2 \in \mathbf{I}(\mathcal{R})$, $g \circ \delta_{R_1} = \delta_{R_2} \circ g$.

Let hence $\mathcal{Q}(\Delta_R)$ be the subcategory of $\mathcal{Q}_{\mathcal{RL}}$ whose objects are quadruples of the form $(B, R, \vee e, \delta_R)$ where $R \in \mathbf{I}(\mathcal{R})$ and $\delta_R \in \Delta_R$ and morphisms are pairs $$(f, g) : (B_1, R_1, \vee_1 e, \delta_{R_1}) \to (B_2, R_2, \vee_2 e, \delta_{R_2})$$ which are morphism pairs of $\mathcal{Q}_{\mathcal{RL}}$.

**Proposition 2.5.9.** Let $\mathcal{R}$ be any subclass of $\mathcal{RL}$ and let $\Delta_R = \{ \delta_R \mid R \in \mathbf{I}(\mathcal{R}) \}$ and $\Delta'_R = \{ \delta'_R \mid R \in \mathbf{I}(\mathcal{R}) \}$ be two indexed families of wdl-admissible operators on $\mathbf{I}(\mathcal{R})$. The map $\Psi : \mathcal{Q}(\Delta_R) \to \mathcal{Q}(\Delta'_R)$ that on objects is defined as:

$$\Psi(B, R, \vee e, \delta_R) = (B, R, \vee e, \delta'_R)$$

and it is the identity on morphisms is an equivalence of the categories $\mathcal{Q}(\Delta_R)$ and $\mathcal{Q}(\Delta'_R)$ if $\Delta_R$ and $\Delta'_R$ are $\mathcal{R}$-compatible.

**Proof.** We need to prove that $\Psi$ is a functor and it establishes a categorical equivalence between $\mathcal{Q}(\Delta_R)$ and $\mathcal{Q}(\Delta'_R)$ if $\Delta_R$ and $\Delta'_R$ are $\mathcal{R}$-compatible.

Let us hence suppose that $\Delta_R$ and $\Delta'_R$ are $\mathcal{R}$-compatible, and let us first prove that $\Psi$ is a functor. Clearly, for every $(B, R, \vee e, \delta_R) \in \mathcal{Q}(\Delta_R)$, $\Psi((B, R, \vee e, \delta_R)) \in \mathcal{Q}(\Delta'_R)$. Moreover, because of the $\mathcal{R}$-compatibility of the sets of wdl-operators, $(f, g)$ is a morphism of $\mathcal{Q}(\Delta_R)$ iff it is a morphism of $\mathcal{Q}(\Delta'_R)$. Finally, $\Phi$ preserves the identical morphism and composition of morphisms, whence it is a functor. In order to prove that $\Phi$ is an equivalence of the categories $\mathcal{Q}(\Delta_R)$ and $\mathcal{Q}(\Delta'_R)$, let us show that it is full and faithful and that each object of $\mathcal{Q}(\Delta'_R)$ is isomorphic to the image of an object in $\mathcal{Q}(\Delta_R)$ via $\Psi$. It is straightforward to see that $\Psi$ is faithful since it is the identity on morphisms, while its fullness is due to the $\mathcal{R}$-compatibility of $\Delta_R$ and $\Delta'_R$. Lastly, each $(B, R, \vee e, \delta'_R) \in \mathcal{Q}(\Delta'_R)$ is clearly isomorphic to $\Psi(B, R, \vee e, \delta_R) = (B, R, \vee e, \delta'_R)$. We have hence proved that $\mathcal{Q}(\Delta_R)$ and $\mathcal{Q}(\Delta'_R)$ are categorically equivalent. □
Chapter 2. Disconnected $\delta$-rotations of residuated lattices

Remark 2.5.10. Notice that without the $R$-compatibility of $\Delta_R$ and $\Delta'_R$, the map $\Psi$ defined in Proposition 2.5.9 may not be a functor, since a morphism $(f, g)$ of $\mathcal{Q}(\Delta_R)$ is not assured to be a morphism of $\mathcal{Q}(\Delta'_R)$ since it could fail condition (Q2) in the definition of morphism pairs between quadruples.

Proposition 2.5.11. Let $R$ be any subclass of $RL$. Let us consider $\Delta^L_R = \{\delta^R_L | R \in I(R)\}$ and $\Delta^D_R = \{\delta^R_D | R \in I(R)\}$ where $\delta^R_L$ and $\delta^R_D$ denote respectively the maps $\delta_L$ and $\delta_D$ over $R$. Then $\mathcal{Q}(\Delta^L_R)$ and $\mathcal{Q}(\Delta^D_R)$ are equivalent categories.

Proof. Immediate, by Lemma 2.4.3 and Proposition 2.5.9. \hfill \Box

In particular,

- $\mathcal{Q}(\Delta^L_{CH})$ and $\mathcal{Q}(\Delta^D_{CH})$ are equivalent categories.

- $\mathcal{Q}(\Delta^L_{GH})$ and $\mathcal{Q}(\Delta^D_{GH})$ are equivalent categories.

Notice that Proposition 2.5.11 and Theorem 2.4.7 provide us with an alternative proof of Corollary 2.4.10.

2.6 Chen and Grätzer’s triple construction

In previous sections we have seen a generalization of the triplet-construction as we first introduced in [92], that results in characterizing a large class of residuated lattices that have a retraction term to their Boolean skeleton. However, in the context of (non residuated) lattice theory, a similarly inspired triple-construction can be found in relation to Stone algebras, namely pseudocomplemented and bounded distributive lattices where the pseudo-complement $^*$ is such that $a^* \lor a^{**} = 1$ holds. Indeed, in a paper of 1969 [38], Chen and Grätzer study the connection of Stone algebras with respect to a category of triples whose objects are formed by a Boolean algebra, a distributive lattice and a map that associates to each Boolean element a filter of the distributive lattice. More precisely, we call $(B, D, \phi)$ a CG-triple if $B$ is a Boolean algebra, $D$ is a distributive lattice, and the $\phi$-operator is a lattice homomorphism from $B$ to the lattice of filters of $D$, $\text{Fil}(D)$. The authors prove that there is a one-one correspondance between Stone algebras and CG-triplet, extending such
correspondence also to Stone algebras homomorphisms and suitably defined pair of morphisms on CG-triplets. These results are generalized in [81] and [107], respectively to p-algebras and semigroups.

More recently, in [34] the authors apply these ideas to the case of Stonean residuated lattices. In the same paper, they also point out that for the case of product algebras, the two formulations of triples are equivalent, and they show how to define CG-triples starting from triplets defined in [92] and viceversa. The authors also notice that the same reasoning extends to the intersection between Stonean residuated lattices and srDL-algebras. Following [34], we shall now show that, for any Boolean algebra \( B \) between Stonean residuated lattices and srDL-algebras. Following [34], we shall now show that, for any Boolean algebra \( B \) and in case the residuated lattice \( D \) is distributive, there is a bijection between external-join operators from \( B \times D \) to \( D \) and \( \phi \)-operators from \( B \) to \( \mathcal{F}il(D) \).

**Definition 2.6.1.** Let \( B \) be a Boolean algebra, \( D \) a distributive residuated lattice, and \( \vee_e : B \times D \rightarrow D \) be an external join operator as in Definition 2.3. Let \( \phi_{\vee_e} : B \rightarrow \mathcal{F}il(D) \) be the map defined as follows:

\[
\phi_{\vee_e}(b) = \{ d \in D : d = -b \vee_e d \},
\]

**Lemma 2.6.2.** The map \( \phi_{\vee_e} \) is a lattice homomorphism from \( B \) to \( \mathcal{F}il(D) \).

*Proof.* First, we need to show that \( \phi_{\vee_e} \) is well-defined, i.e. that \( \phi_{\vee_e}(b) \) is a filter of \( D \) for any \( b \in B \). In all that follows we will use the properties of \( \vee_e \) listed in Proposition 2.3.3. If \( d \in \phi_{\vee_e}(b), d \leq d' \), then \( d' = d \vee d' = (-b \vee_e d) \vee d' = -b \vee_e (d \vee d') = -b \vee_e d' \), thus \( d' \in \phi_{\vee_e}(b) \). If \( d, d' \in \phi_{\vee_e}(b) \), then \( d \cdot d' = (-b \vee_e d) \cdot (-b \vee_e d') = -b \vee_e (d \cdot d') \), thus \( d \cdot d' \in \phi_{\vee_e}(b) \). Also, clearly \( 1 \in \phi_{\vee_e}(b) \), thus \( \phi_{\vee_e}(b) \) is a filter of \( D \).

Now we shall prove that \( \phi_{\vee_e}(b \land b') = \phi_{\vee_e}(b) \cap \phi_{\vee_e}(b') \), and \( \phi_{\vee_e}(b \lor b') = \phi_{\vee_e}(b) \lor \phi_{\vee_e}(b') \), where with \( \lor \) we mean the join of the lattice of filters, that is to say, the filter generated by the union. Let us first prove that \( \phi_{\vee_e}(b \land b') = \phi_{\vee_e}(b) \cap \phi_{\vee_e}(b') \). First, we have that:

\[
\phi_{\vee_e}(b \land b') = \{ d \in D : d = -(b \land b') \vee_e d \} = \{ d \in D : d = -(b \lor -b') \vee_e d \},
\]

and

\[
\phi_{\vee_e}(b) \cap \phi_{\vee_e}(b') = \{ d \in D : d = -b \vee_e d \} \cap \{ d' \in D : d' = -b' \vee_e d' \}.
\]

78
Chapter 2. Disconnected $\delta$-rotations of residuated lattices

Let $d \in \phi_{\vee e}(b) \cap \phi_{\vee e}(b')$. Then $d = -b \vee_e d = -b \vee_e (-b' \vee_e d) = (-b \vee -b') \vee_e d$, hence $d \in \phi_{\vee e}(b \wedge b')$. Viceversa, let $d \in \phi_{\vee e}(b \wedge b')$. Then $(-b \vee -b') \vee_e d = d$, but $-b \vee_e d, -b' \vee_e d \leq (-b \vee -b') \vee_e d = d$, thus, since clearly $d \leq -b \vee_e d, -b' \vee_e d$, we have that $d = -b \vee_e d = -b' \vee_e d$. Hence $d \in \phi_{\vee e}(b) \cap \phi_{\vee e}(b')$.

Let us now prove that $\phi_{\vee e}(b \vee b') = \phi_{\vee e}(b) \vee \phi_{\vee e}(b')$. We have that:

$$\phi_{\vee e}(b \vee b') = \{d \in D : d = -(b \vee b') \vee_e d\} = \{d \in D : d = (-b \wedge -b') \vee_e d\},$$

and

$$\phi_{\vee e}(b) \vee \phi_{\vee e}(b') = \{d \in D : d = -b \vee_e d\} \cup \{d' \in D : d' = -b' \vee_e d'\} > .$$

Let us first assume that $d \in \phi_{\vee e}(b \vee b')$. Then $d = (-b \wedge -b') \vee_e d = (-b \vee_e d) \wedge (-b' \vee_e d)$. Since $(-b \vee_e d) \in \{d \in D : d = -b \vee_e d\}$ and $(-b' \vee_e d) \in \{d' \in D : d' = -b' \vee_e d'\}$, we have that $d$ belongs to the filter generated by the union of $\phi_{\vee e}(b)$ and $\phi_{\vee e}(b')$, that is to say, $d \in \phi_{\vee e}(b) \vee \phi_{\vee e}(b')$. Let now $d \in \phi_{\vee e}(b) \vee \phi_{\vee e}(b')$. Then $d_1 \cdot d_2 \leq d$, with $d_1 = -b \vee_e d_1$ and $d_2 = -b' \vee_e d_2$.

Then $-b \wedge -b' = -b \cdot -b' \leq (-b \vee_e d_1) \cdot (-b' \vee_e d_2) \leq d$. But from $-b \wedge -b' \leq d$ we get $(-b \wedge -b') \vee_e d \leq d$, and since it also hold that $d \leq (-b \wedge -b') \vee_e d$, then $d \in \phi_{\vee e}(b \vee b')$.

Hence the proof is settled.

Viceversa, given a lattice homomorphism $\phi : B \to \Fil(D)$, we can define a corresponding external join $\vee_{\phi} : B \times D \to D$. In order to do so, we need some preparatory results. Given a residuated lattice $A$, a lattice filter $G$ of $A$ is a central filter if there exists another filter $G'$ such that for every other filter $F$ it holds:

$$F = (F \cap G) \vee (F \cap G') = (F \vee G) \cap (F \vee G') \quad (2.18)$$

where $\vee$ denotes the filter generated by the union. Central filters are the central elements of the lattice $\Fil(A)$, i.e., the neutral and complemented elements (see [91, Theorem 4.13]). Hence it is possible to adapt the results of Grätzer on lattice ideals in [73, Page 152], to prove the following lemma:

**Lemma 2.6.3** ([34], Lemma 4.1). Let $D$ be a residuated lattice, and let $G$ be a central filter of $\Fil(D)$. For each $x \in D$ there is a unique $z \in D$ such that $[x] \cap G = [z]$. 

79
Lemma 2.6.4. Let D be a distributive residuated lattice, then \( \phi(b) \) is central in \( \text{Fil}_\ell(D) \) for each \( b \in B \).

Proof. We will prove that \( \phi(b) \) satisfies Equation 2.18 with \( G' = \phi(-b) \), that is to say, for any \( F \in \text{Fil}_\ell(D) \):

\[
F = (F \cap \phi(b)) \lor (F \cap \phi(-b)) \quad (2.19)
\]

\[
F = (F \lor \phi(b)) \cap (F \lor \phi(-b)) \quad (2.20)
\]

As for Equation 2.19, observe that \( (F \cap \phi(b)) \lor (F \cap \phi(-b)) \subseteq F \), thus we prove the other inclusion. Let \( x \in F \). Since \( \phi(b) \lor \phi(-b) = \phi(b \lor -b) = D \), we have that there exist \( d \in \phi(b), d' \in \phi(-b) \) such that \( d \land d' \leq x \). Thus \( (d \land d') \lor x = x \), with \( (d \land d') \lor x = (x \lor d) \land (x \lor d') \). Notice also that \( x \lor d \in F \cap \phi(b) \), and \( x \lor d' \in F \cap \phi(-b) \), thus \( x \in (F \cap \phi(b)) \lor (F \cap \phi(-b)) \) and Equation 2.19 is satisfied.

In order to prove Equation 2.20, observe that \( F \subseteq (F \lor \phi(b)) \cap (F \lor \phi(-b)) \), thus we prove the other inclusion. Let hence \( x \in (F \lor \phi(b)) \cap (F \lor \phi(-b)) \). This means that there exist \( f, f' \in F, d \in \phi(b), d' \in \phi(-b) \) such that \( f \land d, f' \land d' \leq x \). Hence, \( (f \land d) \lor (f' \land d') \leq x \). Moreover, \( (f \land d) \lor (f' \land d') = (f \lor (f' \land d')) \land (d \lor (f' \land d')) = (f \lor f') \land (f \lor d') \land (d \lor f') \land (d \lor d') \). Notice that \( d \lor d' \in \phi(b) \cap \phi(-b) = \phi(b \lor -b) = \{1\} \), thus \( d \lor d' = 1 \). Hence we have \( (f \lor f') \land (f \lor d') \land (d \lor f') \land (d \lor d') = (f \lor f') \land (f \lor d') \land (d \lor f') \land 1 \geq (f \lor f') \land f \lor f' \in F \). Thus, being \( x \) greater than an element of \( F \), \( x \in F \), and the proof is settled. \( \square \)

Hence it follows from Lemma 2.6.3 that for each \( b \in B \) we can define a function \( \rho_b : D \rightarrow \phi(b) \) in the following way:

\[
z = \rho_b(d) \text{ iff } [d] \cap \phi(b) = [z].
\]

Definition 2.6.5. Let \( \lor_\phi : B \times D \rightarrow D \) be defined by the following:

\[
b \lor_\phi d = \rho_{-b}(d).
\]

Lemma 2.6.6. The map \( \lor_\phi \) is an external join operator.
Lemma 2.6.7. Let $D$ be a Boolean algebra, $B$ a residuated lattice. Then:

1. Given any external join $\lor_e : B \times D \to D$, $\lor_e = \lor_{\phi_e}$.

2. Given any $\phi$ lattice homomorphism from $B$ to $\text{Fill}(D)$, then $\phi = \phi_{\lor_e}$.

Proof. Let us first prove 1. We shall show that for any $b \in B$, $d \in D$, $b \lor_e d = e$ iff $b \lor_{\phi_e} d = e$. Let us suppose that $b \lor_e d = e$. Now, $b \lor_{\phi_e} d = \rho_{\phi_e}(d)$, and $\rho_{\phi_e}(d) = [d] \cap \phi_{\lor_e}(-b)$. Thus we shall prove that $e \in [d]$, $e \in \phi_{\lor_e}(-b)$, and $x \in [d] \cap \phi_{\lor_e}(-b)$ then $e \leq x$. First, $e \in [d]$ since $d \leq e$. Then, $e = b \lor_e d = b \lor_{\phi_e} (b \lor_e d) = b \lor_{\phi_e} e$, thus $e \in \phi_{\lor_e}(-b)$. Finally, if $x \in [d] \cap \phi_{\lor_e}(-b)$, then $d \leq x$, thus $e = b \lor_e d \leq b \lor_e x = x$. Hence, $b \lor_{\phi_e} d = e$. We now prove the other direction, i.e., let us suppose that $b \lor_{\phi_e} d = e$ and prove that $b \lor_e d = e$. From $b \lor_{\phi_e} d = e$ we obtain that $\rho_{\phi_e}(d) = e$, thus $[e] = [d] \cap \phi_{\lor_e}(-b)$. Hence, $d \leq e$, and $e \in \phi_{\lor_e}(-b)$, thus $e = b \lor_e e$. Now, $d \leq b \lor_e d$, and $b \lor_e d \in \phi_{\lor_e}(-b)$, indeed $b \lor_e (b \lor_e d) = b \lor_e d$. Hence, $b \lor_e d \in [e]$, thus $e \leq b \lor_e d$. Now, $b \lor_e d = (b \lor_e d \lor e) = b \lor_e (d \lor e) = b \lor_e e = e$. Hence, we proved 1.

Let us now prove 2. We shall show that for any $b \in B$, $\phi(b) = \phi_{\lor_e}(b)$. Let us first prove that $\phi \leq \phi_{\lor_e}$. Let $x \in \phi(b)$, then $[x] \cap \phi = [x]$, thus $x = \rho_{\phi}(x)$. Hence $x = \rho_{\phi}(x) \cap \phi(b)$, and thus $x \in \phi_{\lor_e}$, as desired. Let us now prove that $\phi_{\lor_e} \leq \phi$. Let $x \in \phi_{\lor_e}$. Then $x = \rho_{\phi}(x)$, and thus $x \in \phi_{\lor_e}$. This implies that $[x] = [x] \cap \phi(b)$, hence $x \in \phi(b)$ and the proof is settled. 

\[ \square \]
Chapter 2. Disconnected $\delta$-rotations of residuated lattices

Proposition 2.6.8. Let $B$ be a Boolean algebra, $D$ a distributive residuated lattice. Then external join operators from $B \times D$ to $D$ are in bijection with $\phi$-operators from $B$ to $\text{Fil}(D)$.

Proof. Follows directly from Lemma 2.6.7. \hfill \Box

Thus, our triplets defined with the external join operator are equivalent to triples defined with the $\phi$ operator whenever the residuated lattice is distributive.
Chapter 3

(Weak) Boolean products and dual triple construction

In this chapter we will focus our investigation on the space of prime filters. We will first provide a further description of rDL-algebras in terms of (weak) Boolean products. The latter are important tools in the algebraic analysis of logics, since every algebra in a variety $V$ which is representable as a weak Boolean product has a sheaf-representation as well. Indeed, as it is shown in [30], weak Boolean products are the global sections of (not necessarily Hausdorff) sheaves of algebras over Boolean spaces. Although a representation of rDL-algebras in terms of weak Boolean products can be obtained by general theorems (see for instance [42, Theorem 1.6]), it is worth pointing out that the result we are going to exhibit is proven in a direct and explicit way. As an immediate consequence of our proof, we will describe a nontrivial class of rDL-algebras which can be represented as Boolean products of directly indecomposable structures.

Then, in the following sections, we will show how the decomposition of rDL-algebras in terms of their Boolean skeleton and their radical can be performed in the poset of prime lattice filters. In particular, we will obtain an order isomorphism for the poset of prime lattice filters of rDL-algebras with respect to a structure constructed starting from Boolean ultrafilters, prime lattice filters of the radical, together with the information enclosed by the external join operator and the wdl-admissible operator. Interestingly, we shall
see that the information needed for such a construction is given by how prime
filters of the radical result to be fix points of the external join with respect
to Boolean elements belonging to a maximal ideal. This investigation can be
seen as a preliminary work for a dualized triplet construction. An analogous
direction has been taken in Pogel’s PhD thesis [101] and by Priestley in [103],
where they study dualized constructions for Chen and Grätzer triples related
to Stone algebras, that we have presented in Section 2.6.

3.1 rDL-algebras and (weak) Boolean products

Definition 3.1.1. A weak Boolean product of an indexed family \((A_s)_{s \in S}\),
\(S \neq \emptyset\), of algebras in a variety \(V\) is a subdirect product \(A \leq \prod_{s \in S} A_s\), where \(S\) can be endowed with a Boolean space topology such that:

1. the set \([x = y] = \{ s \in S \mid x_s = y_s \}\) is open for \(x, y \in A\).

2. If \(x, y \in A\) and \(N\) is a clopen subset of \(S\), then \(x_{1_N} \cup y_{1_{S\setminus N}} \in A\).

If \([x = y]\) is clopen, \(A \leq \prod_{s \in S} A_s\) is a Boolean product.

Theorem 3.1.2. Every rDL-algebra \(A\) is a weak Boolean product of the in-
dexed family \(B(A)/\text{p} \otimes^\delta R(A)/\text{p}\), for some wdl-admissible operator \(\delta\), and for \(p \in \text{Max} A\).

Proof. Let \(A\) be a rDL-algebra and let \((B(A), R(A), \vee, \delta) = \Phi_{RL}(A)\). By
Theorems 2.3.14 and 2.5.6, \(A\) is a subalgebra of \(\prod_{p \in \text{Max} A} B(A)/p \otimes^\delta R(A)/p\). For the sake of a lighter notation, let us write \(S\) for \(\text{Max} A\).
Hence, it is left to show that, according with Definition 3.1.1, the following
claims hold:

Claim 3.1.3. For every \(x, y \in A\), \([x = y]\) is open in \(S\).

Proof. (of Claim 3.1.3). Let \(x, y \in A\) and let \(b_1, b_2 \in B(A)\) and \(c_1, c_2 \in R(A)\)
such that \(x = (\neg b_1 \vee c_1) \land (b_1 \vee \neg c_1)\) and \(y = (\neg b_2 \vee c_2) \land (b_2 \vee \neg c_2)\). Let
Let us define
\[ O = \left( N_{b_1} \cap N_{b_2} \cap \bigcup_{b \in B_1} N_b \right) \cup \left( N_{-b_1} \cap N_{-b_2} \cap \bigcup_{-b' \in B_2} N_{-b'} \right). \]

Notice that \( O \) is open since each \( N_b \) is a clopen in \( S \) and arbitrary union of open is open and finite intersection of open sets is open. Now, we prove that \([x = y] \subseteq O\). Let \( p \in [x = y] \). Hence \( x/p = y/p \), i.e. \( b_1/p = b_2/p \), \( -b_1/p \lor c_1/p = -b_1/p \lor c_2/p \), and \( b_1/p \lor \neg c_1/p = b_1/p \lor \neg c_2/p \). Since \( p \) is a maximal filter of \( \mathcal{B}(A) \), then either \( b_1/p = b_2/p = 1 \), or \( b_1/p = b_2/p = 0 \). In the latter case, \( -b_1, -b_2 \in p \) (since \( p \) is maximal) and \( \neg c_1/p = \neg c_2/p \), that is, there is a \( -b \in p \) such that \( b \lor \neg c_1 = b \lor \neg c_2 \). Therefore, \( -b \in B_2 \). Thus, \( p \in (N_{-b_1} \cap N_{-b_2} \cap \bigcup_{b' \in B_2} N_{-b'}) \) and hence \( p \in O \). On the other hand, if \( b_1/p = b_2/p = 1 \), then \( b_1, b_2 \in p \) and \( c_1/p = c_2/p \), i.e., there is a \( b \in p \) such that \( -b \lor c_1 = -b \lor c_2 \) and hence \( b \in B_1 \). Thus \( p \in (N_{b_1} \cap N_{b_2} \cap \bigcup_{b \in B_1} N_b) \subseteq O \).

Finally, \([x = y] = O\) and the claim is settled.

Claim 3.1.4. For every \( x, y \in A \) and \( C \) clopen in \( S \), \( x \upharpoonright_C \cup y \upharpoonright_{S \setminus C} \in A \).

Proof. (of Claim 3.1.4). Let \( x, y \in A \) and let \( C \) be a clopen in \( S \). Then there is a \( b \in \mathcal{B}(A) \) such that \( C = N_b \). We prove that \( x \upharpoonright_{N_b} \cup y \upharpoonright_{S \setminus N_b} = (b \land x) \lor (\neg b \land y) \).

Indeed, for every \( p \in N_b \), \( b/p = 1 \) and hence \( (b/p \land x/p) \lor (\neg b/p \land y/p) = x/p = x \upharpoonright_{N_b}(p) \). On the other hand, if \( p \in S \setminus N_b \), \( -b/p = 1 \) and hence \( b/p \land x/p \lor (\neg b/p \land y/p) = y/p = y \upharpoonright_{S \setminus N_b}(p) \). Hence \( x \upharpoonright_{N_b} \cup y \upharpoonright_{S \setminus N_b} = (b \land x) \lor (\neg b \land y) \) pointwise and \( (b \land x) \lor (\neg b \land y) \in A \).

Now, Claims 3.1.3 and 3.1.4 lead to the expected conclusion.
Notice that, in the proof of Theorem 3.1.2 above, if the Boolean skeleton of $A$ is complete, then Claim 3.1.3 actually shows that $[x = y]$ is clopen. Indeed, assuming $B(A)$ to be complete,

$$\bigcup_{b \in B_1} N_b = N_{b^*} \quad \text{and} \quad \bigcup_{-b' \in B_2} N_{-b'} = N_{b^{**}}$$

where $b^* = \bigvee_{b \in B_1} b$ and $b^{**} = \bigvee_{-b' \in B_2} -b'$. Hence, $O = [x = y]$ is clopen. Thus, the following holds.

**Theorem 3.1.5.** Let $A$ a rDL-algebra whose Boolean skeleton is complete. Then $A$ is a Boolean product of the indexed family $B(A)/\mathfrak{p} \otimes_{\delta} R(A)/\mathfrak{p}$, for some wdl-admissible operator $\delta$, and for $\mathfrak{p} \in \text{Max } B(A)$.

Note that this does not characterize rDL-algebras with complete Boolean skeleton. Indeed, in order to have a Boolean product, we only need that $b^* = \bigvee_{b \in B_1} b$ and $b^{**} = \bigvee_{-b' \in B_2} -b'$ are elements of $B(A)$. This also holds in the case that, for instance, a rDL-algebra $A$ is equal to the direct product of the family $B(A)/\mathfrak{p} \otimes_{\delta} R(A)/\mathfrak{p}$, for $\mathfrak{p} \in \text{Max } B(A)$, where $B(A)$ need not be complete.

### 3.2 Filter pairs

We are now going to study the poset of prime lattice filters (that we shall call in what follows $\ell$-filters) of rDL-algebras, following the ideas given by the decomposition theorems as seen in Chapter 2. What follows can be read as a preliminary investigation towards a dualized version of our triplet construction.

Let $A$ be an rDL-algebra, and let $A_\ell$ be the set of its prime $\ell$-filters, ordered by inclusion.

**Definition 3.2.1.** Let us define $D_A$ as the set of pairs $(u, \eta)$ where:

- **(D1)** $u$ is a prime filter of $B(A)$, the Boolean skeleton of $A$,

- **(D2)** $\eta$ is a prime $\ell$-filter of $R(A)$, the radical of $A$, or $B(A)$ itself,
(D3) Given \( b \in \mathcal{B}(A) \) and \( c \in \mathcal{B}(A) \), if \( b \lor c \in \eta \) then either \( b \in u \) or \( c \in \eta \).

Moreover, let \((u_1, \eta_1) \subseteq (u_2, \eta_2)\) iff \( u_1 \subseteq u_2 \) and \( \eta_1 \subseteq \eta_2 \).

**Remark 3.2.2.** (1) Notice that for any \( u \) prime filter of \( \mathcal{B}(A) \), the pair \((u, \mathcal{B}(A))\) is always in \( \mathcal{D}_A \).

(2) If \( p \) is a prime \( \ell \)-filter of \( \mathcal{B}(A) \), then \( \delta[p] \) is a prime \( \ell \)-filter of \( \delta[\mathcal{B}(A)] \). Indeed, recall that \( \delta[p] = \{ \neg \neg x : x \in p \} \), it is hence easy to see that \( \delta[p] \) is a prime \( \ell \)-filter. Let us suppose that \( x \lor y \in \delta[p] \), for \( x, y \in \delta[\mathcal{B}(A)] \), we shall prove that \( x \in \delta[p] \) or \( y \in \delta[p] \). Let \( x \lor y = \neg \neg c \), with \( c \in p \). But since \( x, y \in \delta[\mathcal{B}(A)] \), then \( \neg \neg x = \neg \neg x', y = \neg \neg y' \), with \( x', y' \in \mathcal{B}(A) \), thus \( x = \neg \neg x' = \neg \neg (\neg \neg x') = \neg \neg x \) and similarly \( y = \neg \neg y \). Thus \( \neg \neg (x \lor y) = \neg \neg x \lor \neg \neg y = \neg \neg c \in p \), thus being \( p \) prime we have two cases: \( \neg \neg x = x \in p \) and hence \( x \in \delta[p] \), or \( \neg \neg y = y \in p \) and hence \( y \in \delta[p] \). The converse also holds if \( \delta[p] \neq \{1\} \). Indeed, if \( q \) is a non trivial prime \( \ell \)-filter of \( \delta[\mathcal{B}(A)] \), it is easy to see that \( \delta^{-1}[q] \) is a prime \( \ell \)-filter of the radical, because \( \delta \) is a lattice homomorphism, thus if \( x, y \in \delta^{-1}[q] \), then also \( x \land y \in \delta^{-1}[q] \), since \( \delta(x \land y) = \delta(x) \land \delta(y) \in q \). If \( x \in \delta^{-1}[q] \), \( x \leq y \), then \( \delta(x) \leq \delta(y) \in q \), hence \( y \in \delta^{-1}[q] \). Finally, if \( x \lor y \in \delta^{-1}[q] \), then \( \delta(x \lor y) = \delta(x) \lor \delta(y) \in q \), that is prime, hence one between \( x \) and \( y \) is in \( \delta^{-1}[q] \).

**Definition 3.2.3.** Let now \( \mathcal{D}_A^\partial \) be a fresh copy of the subset of pairs \( \{(u, \delta[\eta]) : (u, \eta) \in \mathcal{D}_A, \eta \notin \{\mathcal{B}(A), \{1\}\}\} \) ordered with the reversed ordering. Finally, let \( \mathcal{D}_A^\ast = \mathcal{D}_A \cup \mathcal{D}_A^\partial \), where \( \cup \) denotes the disjoint union, thus we define a new order \( \subseteq \) by \((u_1, \eta_1) \subseteq (u_2, \eta_2)\) iff

1. \((u_1, \eta_1), (u_2, \eta_2) \in \mathcal{D}_A \) and \((u_1, \eta_1) \subseteq (u_2, \eta_2)\);
2. \((u_1, \eta_1), (u_2, \eta_2) \in \mathcal{D}_A^\partial \) and \((u_2, \eta_2) \subseteq (u_1, \eta_1)\);
3. \((u_1, \eta_1) \in \mathcal{D}_A^\ast, (u_2, \eta_2) \in \mathcal{D}_A^\partial \) and \( u_1 = u_2 \).

**Remark 3.2.4.** The construction given in Definition 3.2.3 is an analogue to the reflection construction given in [63] in order to prove a dualized version of the categorical equivalence present in the same paper between bounded
Sugihara monoids and Gödel algebras enriched with a Boolean constant. In particular, the equivalence is proven between the Urquhart dual as for relevant algebras, specialized to bounded Sugihara monoids, and the Esakia dual the authors develop for Gödel algebras with a Boolean constant.

Let us call \( \mathcal{D}(A)_* \) the poset of prime \( \ell \)-filters of \( \mathcal{D}(A) \) and \( \mathcal{D}(A)_* \) the collection of ultrafilters of \( \mathcal{D}(A) \). Moreover, for any \( b \in \mathcal{D}(A) \), let us consider the maps \( \nu_b : \mathcal{D}(A) \to \mathcal{D}(A) \) such that \( \nu_b(a) = b \lor a \). Notice that for every \( b \in \mathcal{D}(A) \), \( \nu_b \) is a lattice homomorphism. We can define \( \nu_b : \mathcal{D}(A)_* \to \mathcal{D}(A)_* \) in the following way:

\[
\nu_b(p) = \{ x \in \mathcal{D}(A) : \nu_b(x) \in p \}.
\]

Notice that \( \nu_b \) is well defined, i.e. for every \( p \in \mathcal{D}(A)_* \), we can show that \( \nu_b(p) \in \mathcal{D}(A)_* \), indeed: if \( x \in \nu_b(p) \), \( x \leq y \) it means that \( \nu_b(x) \in p \), but then \( \nu_b(x) = b \lor x \leq b \lor y = \nu_b(y) \in p \), thus \( y \in \nu_b(p) \); if \( x, y \in \nu_b(p) \), then \( x \land y \in \nu_b(p) \), since \( \nu_b(x \land y) = b \lor (x \land y) = (b \lor x) \land (b \lor y) = \nu_b(x) \land \nu_b(y) \in p \); finally, if \( x \lor y \in p \), we have that \( \nu_b(x \lor y) = b \lor (x \lor y) = (b \lor x) \lor (b \lor y) = \nu_b(x) \lor \nu_b(y) \in p \), with \( p \) prime, thus \( \nu_b(x) \in p \) or \( \nu_b(y) \in p \), hence \( x \in \nu_b(p) \) or \( y \in \nu_b(p) \).

**Lemma 3.2.5.** Let \( A \) be an rDL-algebra, for every prime \( \ell \)-filter \( p \) of \( \text{Rad}(A) \) and \( u \) prime filter of \( \mathcal{D}(A) \), \( (u, p) \in \mathcal{D}_A \) iff \( (u, \delta[p]) \in \mathcal{D}_A^\delta \) with \( \delta[p] \notin \{ \mathcal{D}(A), \{1\} \} \).

If \( (u, p) \in \mathcal{D}_A \) then \( \delta[p] \notin \{ \mathcal{D}(A), \{1\} \} \).

**Proof.** It is evident from the definition of \( \mathcal{D}_A^\delta \) that \( (u, p) \in \mathcal{D}_A \) iff \( (u, \delta[p]) \in \mathcal{D}_A^\delta \) with \( \delta[p] \notin \{ \mathcal{D}(A), \{1\} \} \). Let now \( A \) be an rDL-algebra, \( p \) prime \( \ell \)-filter of \( \text{Rad}(A) \) and \( u \) prime filter of \( \mathcal{D}(A) \). Let us first prove that \( (u, p) \in \mathcal{D}_A \) implies (D4). If \( (u, p) \in \mathcal{D}_A \) then \( p \) is such that if \( \bar{b} \in \mathcal{D}(A) \) and \( \bar{c} \in \mathcal{D}(A) \), \( b \lor c \in p \) implies either \( b \in u \) or \( c \in p \). We want to prove that given \( b \in \mathcal{D}(A) \), \( b \lor c \in p \), \( c \in \delta[\mathcal{D}(A)] \setminus \delta(p) \) and \( c' \in \delta[\mathcal{D}(A)] \), if \( \neg c \leq b \lor c' \) then either \( b \in u \) or \( c' \leq b \lor c \). Let us then suppose that \( b \notin u \). If \( c' \notin \delta[\mathcal{D}(A)] \setminus \delta(p) \), i.e. \( c' \in \delta(p) \), then since \( c' \leq b \lor c \), this means that \( b \lor c \in \delta[p] \subseteq p \), being \( \delta[p] \) a filter. From
Chapter 3. Boolean products and dual triple construction

condition (D3) we get that then \( c \in p \). But since \( c \in \delta[\mathcal{A}] \), \( \neg
\neg c = c \in \delta[p] \) which is absurd since by hypothesis \( c \in \mathcal{A} \setminus \delta[p] \). Thus either \( b \in u \) or \( c' \in \delta[\mathcal{A}] \setminus \delta[p] \).

Now let us prove that (D4) implies \((u,p) \in D_A\). Hence, we want to prove (D3) for \((u,p)\). Hence, let us suppose that \( b \in \mathcal{A}, \ c \in \mathcal{A} \), and \( b \lor c \in p \).

We shall prove that then either \( b \in u \) or \( c \in p \). Let us suppose that \( c \notin p \), thus, \( \neg c \notin \delta[p] \) and hence \( \neg c \in \delta[\mathcal{A}] \setminus \delta[p] \). Since \( c \leq \neg c \), we have \( b \lor c \leq b \lor \neg c \in p \). Moreover, \( b \lor \neg c = \neg (b \lor c) \) thus it is in \( \delta[p] \). We can apply (D4) since we have \( b \lor \neg c \leq b \lor \neg c \) where \( b \lor \neg c \in \delta[\mathcal{A}] \), \( \neg c \in \delta[\mathcal{A}] \setminus \delta[p] \). We get that \( b \in u \) or \( b \lor \neg c \in \delta[\mathcal{A}] \setminus \delta[p] \). But \( b \lor \neg c \in \delta[p] \), hence \( b \in u \) and the claim is settled. \( \Box \)

3.3 Prime filters fixed by Boolean ultrafilters

Let \( A \) be an rDL-algebra and let \( a \in A_+ \). For each \( u \in \mathcal{A} \), the fact that \( u \lor \neg u = 1 \) implies that \( u \in a \) or \( \neg u \in a \). No proper \( \ell \)-filter contains a Boolean element and its negation, so this entails that each prime \( \ell \)-filter \( a \) of \( A \) contains an ultrafilter \( u_a \) of \( \mathcal{A} \), and that this ultrafilter is uniquely determined by the prime filter (in fact, it must be the intersection of the prime filter with \( \mathcal{A} \)).

**Lemma 3.3.1.** Let \( A \) be an rDL-algebra and let \( a \) be a prime \( \ell \)-filter of \( A \). Let \( u_a \) be as above. Then \( a \cap \mathcal{A} \) is a not necessarily proper prime \( \ell \)-filter of the radical fixed by each map \( \nu_b : \mathcal{A}_+ \to \mathcal{A}_+ \) for \( b \notin u \).

**Proof.** It is easy to see that \( a \cap \mathcal{A} \) is a prime \( \ell \)-filter of \( \mathcal{A} \), since both \( a \) and \( \mathcal{A} \) are upward closed, closed under meets and moreover, given any \( x, x' \in \mathcal{A} \) such that \( x \lor x' \in a \), then either \( x \in a \cap \mathcal{A} \) or \( x' \in a \cap \mathcal{A} \).

For the rest, let \( u \notin u_a \) and set \( \eta = a \cap \mathcal{A} \). We claim that \( \nu_u(\eta) = \eta \).

Recall that \( \nu_u(\eta) = \{ x \in \mathcal{A} : \nu_u(x) \in \eta \} = \{ x \in \mathcal{A} : u \lor x \in \eta \} \). If \( x \in \eta \), then \( x \leq u \lor x \) implies that \( x \lor u \in \eta \), giving that \( x \in \nu_u(\eta) \) and hence \( \eta \subseteq \nu_u(\eta) \).

For the reverse inclusion, suppose that \( x \in \nu_u(\eta) \). Then \( u \lor x \in \eta \), so as \( \eta \subseteq a \) we have \( u \lor x \in a \). But \( a \) is a prime \( \ell \)-filter of \( A \), so this yields \( u \in a \) or \( x \in a \).
Chapter 3. Boolean products and dual triple construction

The former is impossible since \( u \notin u_a \), so \( x \in a \). Since \( x \) was chosen from \( \mathcal{R}(A) \), this implies that \( x \in a \cap \mathcal{R}(A) \) and that gives the result. \( \square \)

**Lemma 3.3.2.** Let \( A \) be an rDL-algebra, let \( a \) be a prime \( \ell \)-filter of \( A \), and let \( \eta = a \cap \mathcal{R}(A) \). Suppose \( \eta \) is a proper \( \ell \)-filter of the radical, and that \( u \) is an ultrafilter of the Boolean skeleton of \( A \) such that \( \eta \) is fixed by each of the maps \( \nabla_u \) for \( u \notin u_a \). Then \( u \subseteq a \), and hence \( u = u_a \).

**Proof.** Let \( u \in u \). Then \( -u \notin u \) since ultrafilters are proper, so \( \eta \) is fixed by \( \nabla_{-u} \). Were \( u \notin a \), then \( -u \in a \) by primality. Then for each \( x \in \mathcal{R}(A) \), we have that \( -u \leq x \leq -u \lor x \). Since both \( \mathcal{R}(A) \) and \( a \) are upward-closed, this gives \( -u \lor x \in a \cap \mathcal{R}(A) = \eta \). The hypothesis then gives that \( x \in \eta \), which implies that \( \mathcal{R}(A) \subseteq \eta \). This contradicts the assumption that \( \eta \) is proper, so it follows that \( u \in a \). Hence \( u \subseteq a \). \( \square \)

**Definition 3.3.3.** Say that an ultrafilter \( u \) of the Boolean skeleton fixes a prime \( \ell \)-filter \( \eta \) of the radical if there exists a prime \( \ell \)-filter \( a \) of \( A \) such that \( u \subseteq a \) and \( \eta = a \cap \mathcal{R}(A) \).

We will now show that, in case \( A \) is distributive, every prime \( \ell \)-filter of the radical is fixed by some ultrafilter of the Boolean skeleton. Let us recall that an ideal of a lattice \( L \) is a downwards closed subset \( I \) of \( L \), such that if \( x, y \in I \), then \( x \lor y \in I \), and if the lattice is bounded, \( 0 \in I \).

**Lemma 3.3.4.** Let us suppose \( A \) is a distributive lattice, and \( \eta \in \mathcal{R}(A)_* \). Then there exists \( u \in \mathcal{R}(A)_* \) such that \( u \) fixes \( \eta \).

**Proof.** We need to prove that there exists a prime \( \ell \)-filter \( p \) of \( A \) such that \( u \subseteq p \) and \( \eta = p \cap \mathcal{R}(A) \). Let \( i = \mathcal{R}(A) \setminus \eta \). Then it is easy to see that \( i \) is an ideal of the radical. If we consider its downset \( \downarrow i \) in \( A \), then \( \downarrow i \) is an ideal of \( A \). Indeed, it is clearly downward closed and it contains 0. Moreover, if \( a, b \in \downarrow i \) then there exists \( c, d \in i \) such that \( a \leq c, b \leq d \). Hence \( a \lor b \leq c \lor d \), and indeed being \( \eta \) a prime filter, the fact that \( c \lor d \in \eta \) would imply that either \( c \) or \( d \) are in \( \eta \), while they are both in \( i \). Thus \( a \lor b \in \downarrow i \), and we proved that \( \downarrow i \) is an ideal. Moreover, by definition, \( \eta \cap \downarrow i = \emptyset \). Hence, via the Prime Ideal Theorem for distributive lattices, there exists \( p \in A_* \) such that \( p \cap \downarrow i = \emptyset \)

90
and \( \eta \subseteq p \). Hence, let \( u \) be the unique Boolean ultrafilter contained in \( p \). Then \( u \) fixes \( \eta \), indeed \( p \cap R(A) = \eta \), since \( p \cap i = \emptyset \). □

Notice that \( \eta \) being fixed by \( u \) coincides with \( \eta \) being a fixed point of each of the maps \( \nu_u \), \( u \notin u \).

**Lemma 3.3.5.** Let \( A \) be an rDL-algebra and let \( \eta \) be a prime \( \ell \)-filter of the radical. Then \( (u, \eta) \in D_A \) if and only if \( u \) fixes \( \eta \).

**Proof.** Suppose first that \( (u, \eta) \in D_A \). We must show that \( u \) fixes \( \eta \). Let \( u \notin u \). It suffices to show that \( \nu_u(\eta) \subseteq \eta \) since the reverse inclusion always holds, so let \( x \in \nu_u(\eta) \). Then \( \nu_u(x) \in \eta \), i.e. \( u \lor x \in \eta \). The fact that \( (u, \eta) \in D_A \) gives that \( u \notin u \) or \( x \in \eta \). But \( u \notin u \) by assumption, so \( x \in \eta \). It follows that \( \nu_u(\eta) = \eta \) for every \( u \notin u \), and thus that \( u \) fixes \( \eta \) by Lemma 3.3.2.

For the converse, suppose that \( u \) fixes \( \eta \). That \( u \) and \( \eta \) are nonempty prime filters of the Boolean skeleton and the radical, respectively, holds by hypothesis. Let \( u \in \mathcal{B}(A) \) and \( x \in \mathcal{R}(A) \) with \( u \lor x \in \eta \), and suppose that \( u \notin u \). Then the assumption that \( u \) fixes \( \eta \) implies that \( \eta = \nu_u(\eta) \) by Lemma 3.3.1. We have \( u \lor x \in \eta \) implies that \( x \in \nu_u(\eta) \) by definition, so it follows that \( x \in \eta \). This entails that \( u \in u \) or \( x \in \eta \), giving the condition (D3) and hence the result. □

In the proof of Lemma 3.3.4 we also proved that, in case of distributivity, each prime filter of the radical is the intersection of the radical with some prime filter of \( A \). We will now give an alternative proof of this fact, showing in particular how it is actually possible to construct such prime filter.

**Proposition 3.3.6.** Each prime \( \ell \)-filter of the radical that is fixed by some ultrafilter \( u \), is the intersection of the radical with some prime \( \ell \)-filter of \( A \).

**Proof.** Let \( \eta \in \mathcal{R}(A)_u \). Then let \( u \in \mathcal{B}(A)_u \), such that \( u \) fixes \( \eta \). Which means, by Lemma 3.3.5, that \( (u, \eta) \in D_A \). Let now \( p = < u \cup \eta > \). Since \( u \) and \( \eta \) are closed under meets, we have that:

\[
p = \{ a \in A \mid b \land c \leq a, \text{ with } b \in u, c \in \eta \}\]
Let us prove that $p$ is a prime filter of $A$. Let us suppose that $a_1 \lor a_2 \in p$, with $a_1 = (b_1 \lor -c_1) \land (-b_1 \lor c_1)$ and $a_2 = (b_2 \lor -c_2) \land (-b_2 \lor c_2)$. We shall prove that $a_1 \in p$ or $a_2 \in p$, and also that $p$ is proper. Notice that $a_1 \lor a_2 \in p$ means $b \land c \leq a_1 \lor a_2$, for some $b \in u$, $c \in \eta$. We can write:

$$a_1 \lor a_2 = ((b_1 \lor b_2) \lor (-c_1 \lor -c_2)) \land ((b_1 \lor -b_2) \lor (c_1 \lor c_2)) \land ((-b_1 \lor b_2) \lor (c_1 \lor c_2))$$

(see Definition 2.3.10). Thus, we get that, respectively:

$$b \land c \leq ((b_1 \lor b_2) \lor (-c_1 \lor -c_2)),$$

$$b \land c \leq ((b_1 \lor -b_2) \lor c_2),$$

$$b \land c \leq ((-b_1 \lor b_2) \lor c_2),$$

$$b \land c \leq ((-b_1 \lor -b_2) \lor (c_1 \lor c_2))$$

First of all, let us consider the first condition: $b \land c \leq ((b_1 \lor b_2) \lor (-c_1 \lor -c_2))$. This is possible iff $b \leq b_1 \lor b_2$. Indeed, being $A$ isomorphic to $B \otimes^\delta_e R$ (see Theorem 2.5.6), calling $\lambda_B$ the isomorphism from $B$ to $\mathcal{R}(B \otimes^\delta_e R)$, $\lambda_R$ the isomorphism from $R$ to $\mathcal{R}(B \otimes^\delta_e R)$, we get:

$$(\lambda_B(b) \cap \lambda_R(c)) = [b, 1] \cap [1, c] = [b, -b \lor c],$$

$$\lambda_B(b_1 \lor b_2) \cup \lambda_R(c_1 \land c_2) = [b_1 \lor b_2, 1] \cup [0, c_1 \land c_2] = [b_1 \lor b_2, (b_1 \lor b_2) \lor (c_1 \lor c_2)].$$

Now, it is:

$$[b, -b \lor c] \cap [b_1 \lor b_2, (b_1 \lor b_2) \lor (c_1 \lor c_2)] = [b \land (b_1 \lor b_2), \bar{c}],$$

where $\bar{c} \in \mathcal{R}(A)$ is calculated via the operations in Definition 2.3.10. Hence, via the isomorphism, $b \land c \leq ((b_1 \lor b_2) \lor (-c_1 \lor -c_2))$ corresponds to $[b \land (b_1 \lor b_2), \bar{c}] = [b, -b \lor c]$, and this is possible only if $b \leq b_1 \lor b_2$.

Thus it can’t be that both $b_1, b_2 \notin u$. Moreover, notice that $((b_1 \lor -b_2) \lor c_2)$, $((-b_1 \lor b_2) \lor c_2)$ and $((-b_1 \lor -b_2) \lor (c_1 \lor c_2))$ are elements of the radical, but if $a \in \mathcal{R}(A)$ with $b \land c \leq a$, via residuation, we get $c \leq -b \lor a$, hence $-b \lor a \in \eta$, and using (D3) we obtain that either $-b \in u$ or $a \in \eta$, but since $b \in u$, $-b \notin u$, and then $a \in \eta$. Thus, applying this reasoning to the three terms above, plus condition (D3), we get that the following facts hold:

$$(a)\quad c_2 \in \eta \text{ or } b_1 \lor -b_2 \in u$$

$$(b)\quad c_1 \in \eta \text{ or } -b_1 \lor b_2 \in u$$

$$(c)\quad c_1 \lor c_2 \in \eta \text{ or } -b_1 \lor -b_2 \in u$$
Let us consider \((a)\), then we distinguish the two cases \(c_2 \in \eta\) or \(b_1 \vee \neg b_2 \in u\). Let us first suppose that \(c_2 \in \eta\), and check the different cases.

- If \(b_2 \in \eta\), then \(a_2 = (b_2 \vee \neg c_2) \land (\neg b_2 \vee c_2)\) is in \(p\), since \(b_2 \land c_2 \leq a_2\).

- If \(\neg b_2, b_1 \in \eta\), then from \((b)\) we get that \(c_1 \in \eta\), thus \(a_1 \in p\) being \(b_1 \land c_1 \leq a_1\).

Notice that these are the only two cases, because for the observation made above, it can’t be that both \(b_1, b_2 \notin u\). Now, we have to check the other case from \((a)\), thus \(b_1 \vee \neg b_2 \in u\). There are three cases:

- If \(b_1, \neg b_2 \in u\), from \((b)\) we get that \(c_1 \in \eta\) and then \(a_1 \in p\).

- If \(b_1, b_2 \in u\), from \((c)\) we obtain that \(c_1 \vee c_2 \in \eta\), which is a prime filter of the radical, hence \(c_1 \in \eta\) or \(c_2 \in \eta\). In the first case \(a_1 \in p\), in the second case \(a_2 \in p\).

- If \(\neg b_2, b_1 \in u\), again from \((b)\) it follows that \(c_1 \in \eta\), and hence \(a_1 \in p\).

Hence, \(a_1 \in p\) or \(a_2 \in p\). Moreover, \(p\) is clearly proper if \(u\) is proper, because \(b \land c > 0\), for any \(b \in u\) and \(c \in \eta\), thus \(0 \notin p\).

Now let us show that \(p \cap \mathcal{B}(A) = \eta\). Clearly \(\eta \subseteq p \cap \mathcal{B}(A)\). Moreover, we have that \(a \in p \cap \mathcal{B}(A)\) iff \(b \land c \leq a\), for some \(b \in u, c \in \eta\). Via residuation, we get \(c \leq \neg b \lor a\), hence \(\neg b \lor a \in \eta\), and using \((D3)\) we obtain that either \(\neg b \in u\) or \(a \in \eta\), but since \(b \in u\), \(\neg b \notin u\), thus \(a \in \eta\). Hence \(p \cap \mathcal{B}(A) = \eta\).

Notice that also \(p \cap \mathcal{B}(A) = u\). Indeed, clearly, \(p \cap \mathcal{B}(A)\) is a prime filter of the Boolean skeleton of \(A\), and it includes \(u\). But, being \(u\) a maximal filter, \(p \cap \mathcal{B}(A) = u\) iff \(p \cap \mathcal{B}(A)\) is proper, and this holds since we have already observed that \(p\) is proper whenever \(u\) is.

Lemma 3.3.5 yields that if \(u\) is an ultrafilter of the Boolean skeleton, then \((u, \eta) \in \mathcal{D}_A\) if and only if \(u\) fixes \(\eta\). Let \(\eta\) be a prime \(\ell\)-filter of the radical that is fixed by some ultrafilter, then set \(f_\eta = \cap\{u : u \text{ fixes } \eta\}\). Then \(f_\eta\) is a filter of the Boolean skeleton since it is an intersection of filters, but is an ultrafilter if and only if there is a unique ultrafilter of the Boolean skeleton fixing \(\eta\).
Lemma 3.3.7. If \( f_\emptyset \neq \emptyset \), let \( u \notin f_\emptyset \). Then \( \nabla_u(\eta) = \eta \).

Proof. Note that \( u \notin f_\emptyset \) implies that there exists an ultrafilter \( u \) of the Boolean skeleton fixing \( \eta \) and with \( u \notin u \) (since \( f_\emptyset \) is just an intersection of such ultrafilters). Because \( u \) fixes \( \eta \), this entails that \( \nabla_u(\eta) = \eta \) by the above characterization of fixing a prime filter. This gives the result.

Lemma 3.3.8. Suppose that \( u \) is an ultrafilter of the Boolean skeleton. Then \( u \) fixes \( \eta \) if and only if \( f_\emptyset \neq \emptyset \) and \( f_\emptyset \subseteq u \).

Proof. If \( u \) fixes \( \eta \), then clearly \( f_\emptyset \neq \emptyset \) and \( f_\emptyset \subseteq u \), since \( f_\emptyset \) is the intersection of such filters. Suppose now that \( f_\emptyset \neq \emptyset \) and \( f_\emptyset \subseteq u \). Let \( u \notin u \). Then \( f_\emptyset \subseteq u \) implies that \( u \notin f_\emptyset \), so by Lemma 3.3.7 we have \( \nabla_u(\eta) = \eta \). Since this holds for arbitrary \( u \notin u \), this means that \( u \) fixes \( \eta \).

Note that the above says that the ultrafilters fixing \( \eta \) are exactly the ultrafilters that extend the filter \( f_\emptyset \), when \( f_\emptyset \) is non empty.

3.4 An order isomorphism for the poset of prime \( \ell \)-filters

We will now see how to represent the poset of prime \( \ell \)-filters by means of the filter pair-construction \( D^*_A \), and in order to do so we first need some preparatory results. In what follows \( A \) will be an rDL-algebra.

Definition 3.4.1. Given \( p \in A_+ \), let \( p' = \{ x \in A : \neg x \notin p \} \).

Remark 3.4.2. As it is easy to see, if \( p \) is a prime \( \ell \)-filter of \( A \), then \( p' \) is a prime \( \ell \)-filter of \( A \). Indeed: if \( x \in p' \), \( x \leq y \), then since \( \neg y \leq \neg x \), we have that \( \neg y \notin p \), or it would be \( \neg x \notin p \), a contradiction; if \( x, y \in p' \), it means that \( \neg x, \neg y \notin p \), thus \( \neg (x \land y) = \neg x \lor \neg y \notin p \) (recall that De Morgan laws hold in rDL-algebras), indeed if otherwise \( \neg x \lor \neg y \in p \), by primality of \( p \) we get that \( \neg x \in p \) or \( \neg y \in p \), a contradiction; finally, if \( x \lor y \in p' \), it means that \( \neg (x \lor y) = \neg x \land \neg y \notin p \), thus it cannot be that both \( \neg x \) and \( \neg y \) are in \( p \), because \( p \) is closed by meet, and this implies that \( x \in p' \) or \( y \in p' \).
Notice also that in Urquhart duality theory for relevant algebras \( p' \) represents the negation of \( p \) (see [112]). Indeed, it corresponds to Routley star operator for relevant logic introduced to treat negation [105].

**Lemma 3.4.3.** Let \( p \in A_\ast \). Then either \( p \subseteq p' \) or \( p' \subseteq p \).

*Proof.* Suppose that \( p \not\subseteq p' \). Then there exists \( x \in p \) with \( x \not\in p' \). Since \( x \not\in p' \), we have by definition that \( \neg x \in p \). It follows that \( x \land \neg x \in p \). If \( a \in p' \), then the identity \( x \land \neg x \leq a \lor \neg a \), that holds in rDL-algebras as it can be easily proven in the directly indecomposable elements, gives that \( a \lor \neg a \in p \), and by primality \( a \in p \) or \( \neg a \in p \). If \( \neg a \in p \), then \( a \not\in p' \), which is a contradiction. It follows that \( a \in p \), so that \( p' \subseteq p \). \( \square \)

**Lemma 3.4.4.** For every \( p \in A_\ast \), \( p \) and \( p' \) contain the same Boolean ultrafilter.

*Proof.* By way of contradiction, suppose that \( b \in p \cap B(A) \) and \( b \not\in p' \). Since \( 1 = b \lor \neg b \in p' \), the primality of \( p' \) gives \( \neg b \in p' \). But then \( \neg \neg b \not\in p \) by definition, so by involutivity \( b \not\in p \), a contradiction. \( \square \)

We recall that the coradical of \( A \) is denoted with with \( C(A) \).

**Lemma 3.4.5.** \((u, \eta) \in D_A \) if and only if \((u, \neg((C(A) \setminus \eta'))) \in D_A^\delta \).

*Proof.* We will prove that \( \neg((C(A) \setminus \eta')) = \neg \neg \eta \), the claim will follow by the definition of \( D_A^\delta \), recalling that \( \delta \) coincides with the double negation of the algebra \( A \). Let us first suppose that \( x \in \neg(C(A) \setminus \eta') \). Then \( x = \neg y \), with \( y \in C(A) \setminus \eta' \). Then \( y = \neg \neg y \), \( \neg y \in \eta \). Since \( x = \neg \neg y \), thus \( x \in \neg \eta \).

Viceversa, let \( x \in \neg \neg \eta \). Then \( x = \neg \neg y \), with \( y \in \eta \). Since then \( y \in \mathcal{R}(A) \), we have that \( \neg y \in C(A) \). Thus \( \neg y \) is such that \( \neg \neg y \in \eta \), being \( y \leq \neg \neg y \in \eta \), and hence \( \neg y \not\in \eta' \). Thus \( \neg y \in C(A) \setminus \eta' \), and the claim follows. \( \square \)

**Theorem 3.4.6.** Let \( A \) be a rDL-algebra. Then \( A_\ast \) is order isomorphic to \( D_A^\ast \).

*Proof.* Let \( \alpha : A_\ast \rightarrow D_A^\ast \) be defined as follows:

\[
\alpha(p) = \begin{cases} 
(p \cap \mathcal{R}(A), p \cap \mathcal{R}(A)), & \text{if } p \in p', \\
(p \cap \mathcal{R}(A), -(C(A) \setminus p)) & \text{otherwise}.
\end{cases}
\]
We will show that $\alpha$ defines an order isomorphism. Let us first show that it is a well-defined map.

**Claim 3.4.7.** $\alpha$ is a well-defined map from $A_\ast$ to $D_A^\ast$.

*Proof.* We only need to show that, given $p \in A_\ast$, $\alpha(p) \in D_A^\ast$. By Lemma 3.4.3, either $p \subseteq p'$ or $p' \subseteq p$. Let us first suppose that $p \subseteq p'$. It is easy to see that $p \cap B(A)$ is a prime filter of $B(A)$, since both $p$ and $B(A)$ are upward closed, closed under meets and moreover, given any $x, x' \in B(A)$ such that $x \lor x' \in p$, then either $x \in p \cap B(A)$ or $x' \in p \cap B(A)$. With exactly the same reasoning, we can prove that $p \cap B(A)$ is a prime filter of $B(A)$. It is also easy to prove that $\alpha(p)$ respects last condition of Definition 3.2.1. Indeed, let us consider $b \in B(A)$ and $c \in B(A)$ such that $b \lor c \in p \cap B(A)$. But $p$ is prime, hence either $b \in p$ which implies $b \in p \cap B(A)$, or $c \in p$, and hence $c \in p \cap B(A)$. Hence, the proof is settled for the case $p \subseteq p'$. Now, if $p' \subseteq p$, then $(p' \cap B(A), p' \cap B(A)) \in D_A$, as it has been proven in the previous case. The fact that $(p \cap B(A), -(C(A) \setminus p)) \in D_A^\ast$ follows from Lemma 3.4.5. 

We now need to prove that $\alpha$ is injective and surjective.

**Claim 3.4.8.** $\alpha$ is a bijection.

*Proof.* Let us first prove that $\alpha$ is surjective. Recall that $(u, \eta) \in D_A$ iff $u$ fixes $\eta$ by Lemma 3.3.5. Thus, by definition, there exists a prime filter $p$ of $A$ such that $u \subseteq p$ and $p = a \cap B(A)$. Being $u$ an ultrafilter, clearly $p \cap B(A) = u$, thus $\alpha(p) = (u, \eta)$. Let now $(u, \eta) \in D_A^\ast$. By definition, this means that $\eta = \delta[q]$ with $(u, q) \in D_A$. Let $p \in A_\ast$ be such that $\alpha(p) = (u, q)$. Then we shall show that $\alpha(p') = (u, \eta)$. We need to prove that $-(C(A) \setminus p') = \eta = \delta[q]$. Let $x = y - y$, with $y \in C(A) \setminus p'$, which means that $y \in C(A), y \in \neg y \in p$. Hence in particular $\neg y \in p \cap B(A) = q$. Moreover, $\neg y = \neg y = y = x$, thus $x \in \delta[q]$. Viceversa, let $x \in \delta[q]$. Then $x = \neg y, y \in q \subseteq p$. Being $p$ a filter and $y \subseteq \neg y$, we have that also $x = \neg y \in p$. Hence $\neg y \in p'$, with $\neg y \in C(A)$. Thus, $x = \neg y \in -(C(A) \setminus p')$. We have proved that $\alpha$ is surjective.
Let us now prove the injectivity of $\alpha$. Let $p_1 \neq p_2$, and let us show that then $\alpha(p_1) \neq \alpha(p_2)$. Let us write respectively $u_1 = p_1 \cap \mathcal{B}(A)$, $\eta_1 = p_1 \cap \mathcal{R}(A)$, $\delta_1 = \neg(\mathcal{E}(A) \setminus p_1)$, and $u_2 = p_2 \cap \mathcal{B}(A)$, $\eta_2 = p_2 \cap \mathcal{R}(A)$, $\delta_2 = \neg(\mathcal{E}(A) \setminus p_2)$.

Without loss of generality, let us suppose that there exist an $a \in p_1 \setminus p_2$, and via the representation of the elements of rDL-algebras, $a = (b \lor c) \land (b \lor \neg c)$, with $b \in \mathcal{B}(A), c \in \mathcal{R}(A)$. Thus, $a \leq (b \lor c), (b \lor \neg c)$, and then both $(b \lor c), (b \lor \neg c) \in p_1$. Now, if $b \lor c \notin p_2$ then $\eta_1 \neq \eta_2$ and the claim is settled, otherwise, if $b \lor c \in p_2$ it must be $b \lor \neg c \notin p_2$, otherwise $a \in p_2$. Hence, if $b \lor \neg c \notin p_2$, we have that $b \notin p_2$, so $b \notin p_2$, and $c \notin p_2$. Now, recall that $(b \lor c), (b \lor \neg c) \in p_1$, and $p_1$ is prime, thus we distinguish two cases. Either $b, c \notin p_1$, or $b, c \in p_1$. In the first case, we have that $\delta_1 \neq \delta_2$, in the second case $u_1 \neq u_2$. 

\[\square\]

Claim 3.4.9. $p_1 \subseteq p_2$ iff $\alpha(p_1) \leq \alpha(p_2)$, for any $p_1, p_2 \in A_\ast$.

Proof. The fact that $p_1 \subseteq p_2$ implies $\alpha(p_1) \leq \alpha(p_2)$ follows easily from the definition. Let us now suppose that $\alpha(p_1) \leq \alpha(p_2)$. Let us write again $u_1 = p_1 \cap \mathcal{B}(A)$, $\eta_1 = p_1 \cap \mathcal{R}(A)$, $\delta_1 = \neg(\mathcal{E}(A) \setminus p_1)$, and $u_2 = p_2 \cap \mathcal{B}(A)$, $\eta_2 = p_2 \cap \mathcal{R}(A)$, $\delta_2 = \neg(\mathcal{E}(A) \setminus p_2)$. We distinguish three cases:

1. $p_1 \subseteq p_1', p_2 \subseteq p_2'$.

Then $p_1 \cap \mathcal{R}(A) \subseteq p_2 \cap \mathcal{R}(A)$. Let us prove that if $a \in p_1$ then $a \in p_2$. Via the representation of elements of rDL-algebras, $a = (b \lor c) \land (b \lor \neg c)$. Thus, $(b \lor c), (b \lor \neg c) \in p_1$, which is prime. Thus, by $(b \lor c) \in p_1$, we get that or $b \in u_1 \subseteq u_2$, or $c \in \eta_1 \subseteq \eta_2$, in both cases $b \lor c \in p_2$. By $(b \lor c) \in p_1$, we have $b \in u_1 \subseteq u_2$, or $c \in p_1 \cap \mathcal{E}(A) = \emptyset$, hence $b \in u_1 \subseteq u_2$ and $b \lor c \in p_2$.

2. $p_1 \subseteq p_1', p_2 \subseteq p_2$. By the definition of the order on $D_A^n$, this contradicts the fact that $\alpha(p_1) \leq \alpha(p_2)$.

3. $p_1' \subseteq p_1, p_2' \subseteq p_2$.

Then $\neg(\mathcal{E}(A) \setminus p_2) \subseteq \neg(\mathcal{E}(A) \setminus p_1)$. Notice that this implies that $p_1 \cap \mathcal{E}(A) \subseteq p_2 \cap \mathcal{E}(A)$. Indeed, $\neg(\mathcal{E}(A) \setminus p_2) \subseteq \neg(\mathcal{E}(A) \setminus p_1)$ implies $\neg(\mathcal{E}(A) \setminus p_2) \subseteq \neg(\mathcal{E}(A) \setminus p_1)$ and since $\neg(\mathcal{E}(A) \setminus p_1, p_2) = (\mathcal{E}(A) \setminus p_1, p_2)$. 

97
we have that $\mathcal{C}(A) \setminus p_2 \subseteq \mathcal{C}(A) \setminus p_1$ which implies $p_1 \cap \mathcal{C}(A) \subseteq p_2 \cap \mathcal{C}(A)$. Let us now consider again $a \in p_1$, $a = (-b \lor c) \land (b \lor -c)$. Thus, $(-b \lor c), (b \lor -c) \in p_1$, prime. Being any $c \in \mathcal{R}(A)$ both in $p_1$ and $p_2$, clearly $(-b \lor c)$ is in $p_2$. By $b \lor -c \in p_1$, we have $b \in u_1 \subseteq u_2$, or $-c \in p_1 \cap \mathcal{C}(A) \subseteq p_2 \cap \mathcal{C}(A)$, in both cases $b \lor -c \in p_2$, hence $a \in p_2$ and the claim is settled.

Hence, we proved that $\alpha$ is an order isomorphism.

Example 3.4.10. In order to help the intuition, we will now give an example of the construction presented in these last sections. Let us consider Chang MV-algebra $C$, with domain $C = \{0, c, \ldots, nc, \ldots, 1-nc, \ldots, 1-c, 1\}$, that is isomorphic to the disconnected rotation of the cancellative hoop given by the negative cone of the integers $\mathbb{Z}^- = (\mathbb{Z}^+, +, \ominus, \min, \max, 0)$, where $\ominus$ is the difference truncated to 0. Let $C^+ = \{1, 1-c, \ldots, 1-nc, \ldots\}$ and $C^- = \{0, c, \ldots, nc, \ldots\}$. $C$ is a perfect MV-algebra, and is generic for the variety DLMV (see [11]).

Let us now consider the DLMV-algebra $C^2 = C \times C$, as in Figure 3.1. Notice that the Boolean skeleton of $C^2$ is given by the Boolean algebra of 4 elements $\mathcal{B}(C^2) = \{(0,0), (0,1), (1,0), (1,1)\}$, while the radical $\mathcal{R}(C^2)$ is isomorphic to $\mathbb{Z}^- \times \mathbb{Z}^-$, and is the upper square of Figure 3.1.

We shall now construct $\mathcal{D}_{C^2}$ as in Definition 3.2.1. Let us call $u_1$ the Boolean ultrafilter generated by $(1,0)$, $u_2$ the Boolean ultrafilter generated by $(0,1)$, $C^+_1$ the prime $\ell$-filter of the radical given by the segment $\{(1,y) : y \in C^+\}$, and $C^+_2$ the one given by the segment $\{(x,1) : x \in C^+\}$. Looking at Figure 3.2, it is easy to see that the pairs in $\mathcal{D}_{C^2}$ will be of the kind $(u_2, [r_n])$, $(u_1, [r_m])$, plus the pairs $(u_1, \mathcal{R}(C^2))$, $(u_1, C^+_1)$ and $(u_2, \mathcal{R}(C^2))$, $(u_2, C^+_2)$. In particular, via the isomorphism $\alpha$ of Theorem 3.4.6, we have the following correspondences:

- $(u_2, [r_n])$ will correspond to the prime $\ell$-filter of $C^2$ generated by the element $(0,1-nc)$;
- $(u_1, [r_m])$ to the prime $\ell$-filter generated by $(1-nc, 0)$.
Chapter 3. Boolean products and dual triple construction

- \((u_1, \mathcal{P}(C^2))\) to the prime \(\ell\)-filter given by the upper and left squares of Figure 3.1; analogously, \((u_2, \mathcal{P}(C^2))\) corresponds to upper-right squares;

- \((u_1, C^+_1)\) to the segment \(\{(1, y) : y \in C\}\), and \((u_2, C^+_2)\) to \(\{(x, 1) : x \in C\}\).

Now, we want to construct \(D_{C^2}^\alpha\). In \(C^2\) the double negation, and hence the \(\delta\) of the construction, is the identity map. Thus, intuitively, we just need to rotate upwards \(D_{C^2}\) to obtain \(D_{C^2}^\partial\), and then \(D_{C^2}^\alpha\) is as in Figure 3.2. Again via the isomorphism we have the following correspondences:

- \((u_1, \delta([r_m]))\) will correspond to the prime \(\ell\)-filter of \(C^2\) generated by the element \((mc, 0)\);

- \((u_2, \delta([r_n]))\) to the prime \(\ell\)-filter generated by \((0, nc)\);

- \((u_1, \delta[C^+_1])\) to the prime \(\ell\)-filter given by \(C^2 \smallsetminus \{(0, y) : y \in C\}\), and \((u_2, \delta[C^+_2])\) to \(C^2 \smallsetminus \{(x, 0) : x \in C\}\).

It is hence easy to realize that \(D_{C^2}^\alpha\) is order isomorphic to the poset of prime \(\ell\)-filters of \(C^2\).

Notice that the only implicative prime filters are the ones given by, respectively, upper-left and upper-right squares of Figure 3.1, and the two segments
Chapter 3. Boolean products and dual triple construction

\[ D_{C^2} \text{ and } D_m \]

\[
\begin{array}{cccc}
(u_1, \mathcal{R}(C^2)) & (u_2, \mathcal{R}(C^2)) & (u_1, \delta[C_1^+]) & (u_2, \delta[C_2^+]) \\
(u_1, [r_m]) & (u_2, [r_n]) & (u_1, \delta([r_m])) & (u_2, \delta([r_n])) \\
(u_1, C_1^+) & (u_2, C_2^+) & (u_1, [r_m]) & (u_2, [r_n]) \\
\end{array}
\]

\[
\begin{array}{cccc}
(u_1, \mathcal{R}(C^2)) & (u_2, \mathcal{R}(C^2)) & (u_1, \delta[C_1^+]) & (u_2, \delta[C_2^+]) \\
(u_1, [r_m]) & (u_2, [r_n]) & (u_1, \delta([r_m])) & (u_2, \delta([r_n])) \\
(u_1, C_1^+) & (u_2, C_2^+) & (u_1, [r_m]) & (u_2, [r_n]) \\
\end{array}
\]

Figure 3.2: $D_{C^2}$ and $D_m$

\{(1, y) : y \in C\} and \{(x, 1) : x \in C\}. They correspond respectively to the pairs $(u_1, \mathcal{R}(C^2))$, $(u_2, \mathcal{R}(C^2))$, $(u_1, C_1^+)$ and $(u_2, C_2^+)$. Notice also that, as from Lemma 3.3.5, pairs $(u, \eta) \in D_{C^2}$ are such that $u$ fixes $\eta$. In particular, this means that for every $b \not\in u$, $\eta$ is a fix point of $\nu_b$. Indeed, for example, let us consider a pair of the kind $(u_2, [r_n])$. We have that $\nu_{(1,0)}([r_n]) = [r_n]$, since the elements of the filter generated by $r_n$ are the only elements of $\mathcal{R}(C^2)$ whose join with $(1,0)$ is in $[r_n]$. 

100
Chapter 4

Varieties generated by generalized $\delta$-rotations of residuated lattices

We are now going to see how the insights provided in Chapter 2 can be extended to a more general setting. In particular, while disconnected $\delta$-rotations of residuated structures generate varieties of algebras having a Boolean retraction term, we will see how a more general definition of rotation results in generating structures with a retraction in an MV-algebra. The particular choice of the class of MV-algebras used to extend the triplet-framework allows us to reason in a similar way with respect to the Boolean case, that will hence result to be a special case of this new approach. However, the Boolean-based construction was worth being presented separately. Indeed, from one side it lies in a more established setting which allows a deeper understanding of both the generated varieties and the space of prime filters. From the other side, it is important for future approaches to understand what allows the extended construction to work similarly.

4.1 Generalized $\delta$-rotation

We will first recall in detail the structure of finite MV-chains, in order to help the reader's intuition on the generalized notion of rotation we will give right after. We refer to [31], [39]. The $n$-element MV-chain $L_n$ is defined as the
algebra of domain:

\[ L_n = \left\{ 0, \frac{1}{(n-1)}, \ldots, \frac{(n-2)}{(n-1)}, 1 \right\}, \]

with operations defined by: \( x \cdot y = \max(0, x + y - 1) \), \( x \rightarrow y = 1 - x + y \), \( x \land y = \min(x, y) \), \( x \lor y = \max(x, y) \). Moreover, we can define Lukasiewicz sum \( x \oplus y = \min(1, x + y) \) and negation \( \neg x = 1 - x \). The following facts are straightforward.

**Proposition 4.1.1.** In \( L_n \) the following properties hold:

1. If \( x < 1 \) then \( x^n = 0 \);
2. If \( x \oplus y < 1 \), then \( x \cdot y = 0 \);
3. If \( x \oplus y = x \oplus z < 1 \) then \( y = z \);
4. If \( x \cdot y = x \cdot z > 0 \) then \( y = z \);
5. \( x \oplus y = x \) iff \( x = 1 \) or \( y = 0 \).

Actually, facts from Proposition 4.1.1 (2) – (5) hold in every MV-chain [39, Lemma 1.6.1].

**Definition 4.1.2.** We shall denote the subvariety of \( \text{MV} \) generated by \( L_n \), of \( \text{MV}_n \)-algebras, with \( \text{MV}_n \). The variety of \( \text{MV}_n \)-algebras can be axiomatized as the subvariety of MV-algebras such that the following equations hold:

\[ x^n = x^{n-1}, \quad (p x^{p-1})^n = n x^p, \]

for every integer \( p = 2, \ldots, n-2 \) that does not divide \( n - 1 \).

**Theorem 4.1.3** ([39]). A finite MV-chain \( L_m \) belongs to \( \text{MV}_n \) iff \( m - 1 \) is a divisor of \( n - 1 \).

Thus every \( \text{MV}_n \)-algebra is a subdirect product of algebras \( \{ L_{m_i} \}_{i \in I} \) where \( m_i - 1 \) divides \( n - 1 \) for every \( i \in I \). We are now ready to settle our generalized notion of rotation.
Chapter 4. Generalized $\delta$-rotations of residuated lattices

Definition 4.1.4. For a fixed $n \in \mathbb{N}$, given a residuated lattice $R$, and given $\delta$ a wdl-admissible operator as defined in 2.1.1, we define the generalized $\delta$-rotation $\mathfrak{R}_L^n(\delta)(R) = (\mathfrak{R}_L^n(\delta), \cdot, \to, \land, \lor, 0, 1)$, in the following way.

$$\mathfrak{R}_L^n(\delta)(R) = (\{0\} \times \delta[R]) \cup \{\{s\} \times \{1\}\}_{s \in L_n \setminus \{0,1\}} \cup \{\{1\} \times R\}$$

Let us now define the operations:

\[
(i,a) \lor_R (j,b) = \begin{cases} 
(1,a \lor b) & \text{if } i = j = 1, \\
(0,a \land b) & \text{if } i = j = 0, \\
(j,b) & \text{if } i < j.
\end{cases}
\]

\[
(i,a) \land_R (j,b) = \begin{cases} 
(1,a \land b) & \text{if } i = j = 1, \\
(0,a \lor b) & \text{if } i = j = 0, \\
(i,a) & \text{if } i < j.
\end{cases}
\]

\[
(i,a) \cdot_R (j,b) = \begin{cases} 
(1,a \cdot b) & \text{if } i = j = 1, \\
(i \cdot j, 1) & \text{if } i,j \in L_n \setminus \{1\}, \\
(j,b) & \text{if } i = 1,j \in L_n \setminus \{1,0\}, \\
(j,a \to b) & \text{if } i = 1,j = 0.
\end{cases}
\]

\[
(i,a) \rightarrow_R (j,b) = \begin{cases} 
(1,a \rightarrow b) & \text{if } i = j = 1, \\
(1,b \rightarrow a) & \text{if } i = j = 0, \\
(i \rightarrow j, 1) & \text{if } i \in L_n \setminus \{0\} \text{ and } j \in L_n \setminus \{1,0\}, \\
& \text{or } i \in L_n \setminus \{1,0\} \text{ and } j = 0, \\
(0,\delta(a \cdot b)) & \text{if } i = 1,j = 0, \\
(1,1) & \text{if } i < j.
\end{cases}
\]

Notation 4.1.5. For the sake of a lighter notation, we shall also write $\mathfrak{R}_n^\delta$ instead of $\mathfrak{R}_L^n(\delta)$.

Remark 4.1.6. If the chain $L_n$ is isomorphic to the Boolean algebra $2$, i.e. for $n = 2$, $\mathfrak{R}_2^\delta(R)$ is the disconnected $\delta$-rotation of Definition 2.1.5.
Remark 4.1.7. It is immediate from the definition of the operations that \((L_n \times \{1\})\) is the universe of a subalgebra of \(\mathcal{R}_n^\delta(R)\) isomorphic to \(L_n\), and \((\{1\} \times R)\) is the universe of a residuated lattice isomorphic to \(R\) (also an implicative filter in \(\mathcal{R}_n^\delta(R)\)). Notice that if \(\delta(x) = 1\) for each \(x \in R\), then \(\mathcal{R}_n^\delta(R)\) is isomorphic to the ordinal sum \(L_n \oplus R\).

With an abuse of notation we shall denote by \(\mathcal{R}_n^\delta(R)\) the subalgebra of \(\mathcal{R}_n^\delta(R)\) whose universe is \((\{0\} \times \delta[R]) \cup (\{1\} \times R)\). Following the ideas of [42] and [5], one can easily obtain the following result:

**Lemma 4.1.8.** The subalgebra \(\mathcal{R}_n^\delta(R)\) of \(\mathcal{R}_n^\delta(R)\) is a bounded commutative residuated lattice.

We are hence able to prove the following:

**Theorem 4.1.9.** \(\mathcal{R}_n^\delta(R)\) is a bounded commutative residuated lattice.

**Proof.** It is easy to see that \((\mathcal{R}_n^\delta(R), \cdot_R, (1, 1))\) is a commutative monoid and that \((\mathcal{R}_n^\delta(R), \wedge_R, \vee_R, (0, 1), (1, 1))\) is a bounded lattice. We shall now prove residuation by cases. That is:

\[ u \cdot_R v \leq_R w \text{ if and only if } u \leq_R v \rightarrow_R w. \]

- If \(u, v, w \in (\{0\} \times \delta[R]) \cup (\{1\} \times R)\), residuation follows from Lemma 4.1.8.

- If \(u, v, w \in (L_n \times \{1\}) \cup (\{1\} \times R)\), then it is immediate as the operations coincide with those of the ordinal sum \(L_n \oplus R\).

- The cases where only one of \(u, v\) belongs to \(\{0\} \times \delta[R]\) and the other two elements to \((L_n \setminus \{0\}) \times \{1\}\) are immediate since both inequalities hold true. The same is true if both \(u, v\) belong to \(\{0\} \times \delta[R]\), as \(u \cdot_R v = (0, 1)\) and \(v \rightarrow_R w \in \{1\} \times R\).

- For the cases \(w \in \{0\} \times \delta[R]\), \(u, v \in ((L_n \setminus \{0\}) \times \{1\})\), it is easy to check that neither inequality is verified.
- If \( u, w \in \{0\} \times \delta[R] \), \( v \in (L_n \setminus \{0,1\}) \times \{1\} \), then \( u \cdot_R v = (0,1) \leq w \) and \( v \rightarrow_R w = (-v,1) \geq u \), so both inequalities always hold.

- If \( v, w \in \{0\} \times \delta[R] \), \( u \in (L_n \setminus \{0,1\}) \times \{1\} \), then \( u \cdot_R v = (0,1) \leq (w,1) \) and \( u \leq_R v \rightarrow_R w = (-v,1) \in \{1\} \times R \), so both inequalities always hold.

\[ \square \]

**Remark 4.1.10.** It is easy to see from the definition of the operations that \( R \) is distributive iff \( R_n^\delta(R) \) is distributive.

Defining the negation as usual, we obtain what follows:

\[
\begin{align*}
\neg_R(1, d) &= (1, d) \rightarrow_R (0,1) = (0, \delta(d)) \\
\neg_R(m, 1) &= (m, 1) \rightarrow_R (0,1) = (\neg m, 1) \\
\neg_R(0, z) &= (0, z) \rightarrow_R (0,1) = (1, z)
\end{align*}
\]

thus the following results are straightforward.

**Corollary 4.1.11.** The only Boolean elements of \( R_n^\delta(R) \) are the bottom and top elements. Therefore \( R_n^\delta(R) \) is a directly indecomposable bounded residuated lattice.

**Corollary 4.1.12.** De Morgan laws hold for \( R_n^\delta(R) \), i.e. \( \neg(x \land y) = \neg x \lor \neg y \).

**Corollary 4.1.13.** The double negation \( \neg\neg \) is a lattice homomorphism in \( R_n^\delta(R) \). Moreover, it is the identity in \( (\{0\} \times \delta[R]) \cup (L_n \times \{1\}) \), and in \( (\{1\} \times R) \) it satisfies \( \neg\neg(1, d) = (1, \delta(d)) \).

**Corollary 4.1.14.** \( (\{1\} \times R) \) is the only maximal filter of \( R_n^\delta(R) \). Moreover, \( (\{1\} \times R) \) is the radical of \( R_n^\delta(R) \) and \( (\{0\} \times \delta[R]) \) is the coradical.

**Proof.** If \( F \) is an implicative filter in \( R_n^\delta(R) \), it induces a filter \( \bar{F} \) in \( L_n \) (given by the first coordinate of \( F \cap (L_n \times \{1\}) \)), and as \( L_n \) is simple we have that \( \bar{F} = \{1\} \) (if \( F \subseteq (\{1\} \times R) \)) or \( \bar{F} = L_n \) (if \( F = R_n^\delta(R) \)). Therefore \( (\{1\} \times R) \) is the only maximal filter of \( R_n^\delta(R) \), and it is hence the radical. The fact
that \((\{0\} \times \delta[R])\) is the coradical follows from the fact that for any element \((0, z) \in (\{0\} \times \delta[R])\), and only for such elements, the negation \(\neg_R(0, z) = (1, z)\) belongs to the radical.

\[\Box\]

**Example 4.1.15.** In order to help the intuition, we will see some examples of this construction. Recall that the two wdl-admissible operators \(\delta_L, \delta_D\) are such that \(\delta_L : x \in R \mapsto 1 \in R\), while \(\delta_D : x \in R \mapsto x \in R\) for every \(R \in RL\). Thus, for \(\delta = \delta_L\), we have:

\[\mathcal{R}_n^\delta(R) = (\{1\} \times R) \cup \{\{s\} \times \{1\}\}_{s \in L_n \setminus \{0,1\}} \cup (\{0\} \times \{1\})\]

![Diagram](image)

We shall call this construction the \(n\)-lifting of the residuated lattice \(R\). Notice that it coincides with the ordinal sum \(L_n \oplus R\).
While with $\delta = \delta_D$, we have:

$$\mathcal{R}_n^\delta = (\{1\} \times R) \cup \{\{s\} \times \{1\}\}_{s \in \mathcal{L}_n \setminus \{0,1\}} \cup \{\{0\} \times R\}$$

We will refer to this construction as the \textit{disconnected n-rotation} of the residuated lattice $\mathbf{R}$.

Special cases of the previous examples are the following.

\textbf{Residuated lattices} For $n=2$, we obtain directly indecomposable rDL-algebras.

\textbf{Basic hoops} Directly indecomposable $\mathbf{BL}_n$-algebras (i.e. $n$-contractive BL-algebras, see [31]) are of the kind $\mathcal{R}_n^\delta(H)$ where $H$ is a basic hoop. We denote the variety they generate as $\mathbf{BL}_n$.

\textbf{Generalized Heyting algebras} All directly indecomposable regular Nelson lattice with negation fixpoint [33] are of the form $\mathcal{R}_n^\delta(H)$, with $H$ a generalized Heyting algebra.

\textbf{Gödel hoops} For $n = 3$, and $H$ a Gödel hoop, $\mathcal{R}_3^\delta(H)$ is a directly indecomposable $\mathbf{NM}^+$-algebra, i.e. a NM-algebra with negation fixpoint, and all directly indecomposable $\mathbf{NM}^+$-algebra are of this form [33].
4.2 Varieties generated by generalized $\delta$-rotations

Our aim is to understand and characterize the varieties generated by generalized $\delta$-rotations of residuated lattices, in a similar fashion to the one presented for the case of disconnected $\delta$-rotations. As one could expect, these varieties will result in having a retraction to an MV-subalgebra. With this idea in mind, given any $n \in \mathbb{N}$, $n \geq 2$, let us consider the following unary term functions:

\[ \nabla_n(x) = \neg(-x^n)^2, \quad \Delta_n(x) = (-\neg x)^2, \quad (4.1) \]

\[ \gamma_n(x) = \Delta_n(x) \land (x \lor \nabla_n(x \lor \neg x)), \quad (4.2) \]

\[ \beta_n(x) = (x \lor \neg x) \lor \neg \nabla_n(x \lor \neg x). \quad (4.3) \]

In order to have an intuition of how these terms behave, let us consider $R$ a residuated lattice and $\delta$ a wdl-admissible operator. Thus, if we evaluate the term functions 4.1, 4.2 and 4.3 in $\mathcal{M}_n(R)$ we obtain what follows:

\[ \nabla_n(x) = \begin{cases} 1 & \text{if } x \in \{1\} \times R \\ 0 & \text{otherwise} \end{cases} \]

\[ \Delta_n(x) = \begin{cases} 0 & \text{if } x \in \{0\} \times \delta[R] \\ 1 & \text{otherwise} \end{cases} \]

\[ \gamma_n(x) = \begin{cases} 1 & \text{if } 1 \in \{1\} \times R \\ x & \text{if } x \in \mathbb{L}_n \times \{1\} \\ 0 & \text{if } x \in \{0\} \times \delta[R] \end{cases} \]

\[ \beta_n(x) = \begin{cases} x & \text{if } x \in \{1\} \times R \\ 1 & \text{if } x \in \mathbb{L}_n \times \{1\} \\ \neg x & \text{if } x \in \{0\} \times \delta[R] \end{cases} \]

In particular, in the examples we have presented, we have:
Chapter 4. Generalized \( \delta \)-rotations of residuated lattices

- For Stonean residuated lattices, \( n = 2 \),
  \[
  \n(x) = \Delta_n(x) = \gamma_n(x) = \neg x, \quad \text{and} \quad 
(x) = \neg x \to x.
  \]

- For directly indecomposable BL\(_n\)-algebras, we have that
  \[
  \gamma_n(x) = \neg x \quad \text{and} \quad \beta_n(x) = \neg x \to x.
  \]

- For regular Nelson lattices, \( n = 3 \), and we have that \( \Delta_n, \n, \) and \( \beta_n \)
coincide with the term functions defined in [33, §5.2], with \( \gamma_n = \phi \).

Remark 4.2.1. Observe that in all examples presented \( \Delta_n(x) \) and \( \n(x) \) are
Boolean elements, and \( \gamma_n \) is a retraction into an MV\(_n\)-algebra.

We are now in position to introduce the class of algebras that will result to
be generated by generalized \( \delta \)-rotations of residuated lattices.

Definition 4.2.2. For a fixed \( n \in \mathbb{N} \), let MVR\(_n\) be the class of bounded residu-
ated lattices such that:

1. (M1) The term functions \( \Delta_n, \n \) result in Boolean elements, that is:
   \[
   \n(x) \lor \neg \n(x) = 1, \quad \Delta_n(x) \lor \neg \Delta_n(x) = 1;
   \]

2. (M2) The term function \( \gamma_n \) is a retraction into an MV\(_n\)-algebra;

3. (M3) De Morgan laws hold, i.e.:
   \[
   \neg(x \land y) = \neg x \lor \neg y;
   \]

4. (M4) \( \neg \Delta_n(x) \leq \neg x \to x \).

Proposition 4.2.3. MVR\(_n\) is a subvariety of BRL.
Chapter 4. Generalized $\delta$-rotations of residuated lattices

Proof. It suffices to notice that MVR$_n$ is equationally definable. In particular the fact that the term function $\gamma_n$ is a retraction into an MV$_n$-algebra means that it is a homomorphism with respect to the operations of the residue lattice (which can be written equationally), $\gamma_n(\gamma_n(x)) = \gamma_n(x)$, and the elements of the form $\gamma_n(x)$ give an MV$_n$-algebra, that again can be stated via identities of MV$_n$-equational theory.

Remark 4.2.4. Observe that, by the comments above, subvarieties of Stonean residuated lattices are subvarieties of MVR$_2$, subvarieties of regular Nelson lattices and in particular NM-algebras are subvarieties of MVR$_3$, and the variety of BL$_n$-algebras is a subvariety of MVR$_n$.

Definition 4.2.5. Recalling Example 4.1.15, for fixed $n \geq 2$, we will call SMVR$_n$ the variety generated by $n$-liftings of residuated lattices and IMVR$_n$ the variety generated by disconnected $n$-rotations of residuated lattices.

Remark 4.2.6. Both SMVR$_n$ and IMVR$_n$ are subvarieties of MVR$_n$. Moreover, SMVR$_2$ = SRL and IMVR$_2$ = wDL. In particular, to help the intuition, (M4) generalizes condition (r) in Definition 2.2.3.

We can prove the following technical lemmas that will be useful later on.

Lemma 4.2.7. Let $A \in$ MVR$_n$. Then if $b \in \mathcal{B}(A)$, $b = \gamma_n(b)$. Therefore as $\gamma_n(A)$ is a subalgebra of $A$, we have that $\mathcal{B}(\gamma_n(A)) = \mathcal{B}(A)$.

Proof. If $b$ is Boolean, then $b \lor \neg b = 1$, so $\gamma_n(b) = \Delta_n(b)$. But as $\neg b$ is also Boolean,

$$\Delta_n(b) = (-\neg b)^{n-1} = \neg \neg b = b.$$ 

Lemma 4.2.8. Let $A \in$ MVR$_n$. Then for $a \in \gamma_n(A)$, $b(a) = a^{n-1}$ is the greatest Boolean element below $a$.

Proof. Clearly $b(a) \leq a$, and if $b'$ is another Boolean element below $a$, $b' = (b')^{n-1} \leq a^{n-1} = b(a)$. The fact that $b(a)$ is Boolean is easily verified recalling that $\gamma_n(A)$ is an MV$_n$-algebra, so we can check this in MV$_n$-chains. If $a = 1$,

110
this is immediate, for $b(a) = 1$; while if $a < 1$, the equation $a^n = a^{n-1}$ implies that $b(a) = a^{n-1} = 0$.

We are now going to show that directly indecomposable $\text{MVR}_n$-algebras are indeed generalized $\delta$-rotations of residuated lattices.

**Notation 4.2.9.** Let $A \in \text{MVR}_n$, thus we set:

- $\mathcal{R}(A) = \gamma_n^{-1}([1])$,
- $\mathcal{M}(A) = \gamma_n(A)$,
- $\mathcal{C}(A) = \gamma_n^{-1}([0])$.

We will refer to $\mathcal{M}(A)$ as the MV-skeleton of $A$.

**Proposition 4.2.10.** For $A$ directly indecomposable $\text{MVR}_n$-algebra, the following holds:

(i) $\mathcal{R}(A)$ is a filter in $A$, therefore the universe of a residuated lattice $\mathcal{R}(A)$, $\mathcal{M}(A)$ is an $\text{MV}_n$-algebra and $\mathcal{C}(A)$ is closed under lattice operations and the product.

(ii) $\mathcal{M}(A) \cong L_s$ for some $s \leq n$ with $s - 1$ dividing $n - 1$.

(iii) $A = \mathcal{R}(A) \cup \mathcal{M}(A) \cup \mathcal{C}(A)$.

(iv) $\mathcal{C}(A)$ is involutive.

(v) Let $\delta : \mathcal{R}(A) \to \mathcal{R}(A)$ be the double negation $\delta(d) = \neg\neg d$, then $\delta$ is wdl-admissible as in Definition 2.1.1.

(vi) As a lattice, $\mathcal{C}(A)$ is dually isomorphic to $\delta(\mathcal{R}(A))$.

**Proof.** (i) Easily follows from the definition, recalling that $\gamma_n$ is a homomorphism.

(ii) Since $A$ has only two Boolean elements, it is directly indecomposable, and from Lemma 4.2.7, $\mathcal{M}(A)$ can only have two Boolean elements, therefore it is a directly indecomposable $\text{MV}_n$-algebra (Proposition 1.2.2), and this implies the claim (see Proposition 3.4.2 [31]).
(iii) The claim follows from the fact that, if \( x \in A \) with \( \gamma_n(x) \notin \{0, 1\} \), then since \( \Delta_n \) and \( \nabla_n \) are Boolean elements, we have that \( \Delta_n(x) \) cannot be 0 (therefore it is 1) and \( \nabla_n(x \lor \neg x) \) cannot be 1 (and therefore it is 0), hence:

\[
\gamma_n(x) = \Delta_n(x) \land (x \lor \nabla_n(x \lor \neg x)) = x \lor \nabla_n(x \lor \neg x) = x.
\]

(iv) Let \( z \in C_A \), therefore \( \gamma_n(z) = 0 \). Notice that hence \( \Delta_n(z) = 0 \), indeed otherwise if \( \Delta_n(z) = 1 \) we would have \( \gamma_n(z) = 1 \land (z \lor \nabla_n(z \lor \neg z)) = 0 \), which would imply \( z = 0 \). But if \( z = 0 \), then \( \Delta_n(z) = 0 \), a contradiction. Hence, \( \Delta_n(z) = 0 \), and then \( \neg \Delta_n(z) = 1 \) and therefore \( \neg \neg z = z \), as from (M4) it is \( \neg \Delta_n(z) \leq \neg \neg z \rightarrow z \).

(v) The proof is the same as the one given for Theorem 2.2.9.

(vi) Let \( f : C(A) \rightarrow \delta(\mathcal{R}(A)) \) be given by \( f(z) = \neg z \). It is easy to see that it is a bijection, as it is clearly well defined, onto \( \delta(\mathcal{R}(A)) \), and involutivity of \( C(A) \) ensures that \( f \) is injective. From De Morgan laws we obtain that it is a lattice homomorphism.

\[\square\]

**Theorem 4.2.11.** Let \( A \) be a directly indecomposable MVR\(_n\)-algebra, then \( A \) is isomorphic to \( \mathcal{R}_{\delta M}(R(A)) \), where \( \delta = \neg \).

**Proof.** From the facts proven in Proposition 4.2.10, we can define \( \phi : A \rightarrow \mathcal{R}_{\delta M}(R(A)) \) by:

\[
\phi(x) = \begin{cases} 
(1, x), & x \in \mathcal{R}(A) \\
(x, 1), & x \in M(A) \\
(0, \delta(-x)) = (0, -x), & x \in C(A)
\end{cases}
\]

and it is easy to see that it is a bijection. Indeed, injectivity follows from the definition of \( \phi \), plus Proposition 4.2.10 (iii) and (iv). Surjectivity is trivial, considering again Proposition 4.2.10 (iv). We will prove now that it is a homomorphism of MVR\(_n\)-algebras. It is immediate to see that it is a bounded
lattice homomorphism. To prove that it preserves product and implication, observe that:

\[
\phi(x \cdot y) = \begin{cases} 
(1, x \cdot y), & x, y \in \mathcal{R}(A) \\
(x \cdot y, 1) & x, y \in \mathcal{M}(A) \setminus \{0, 1\} \\
(0, -0) = (0, 1) & x, y \in \mathcal{C}(A) \\
y & x \in \mathcal{R}(A) y \in \mathcal{M}(A) \setminus \{0, 1\} \\
(0, -(x \cdot y)) = (0, \delta(x \rightarrow -y)) & x \in \mathcal{R}(A), y \in \mathcal{C}(A) \\
(0, -0) = (0, 1) & x \in \mathcal{M}(A) \setminus \{0, 1\}, y \in \mathcal{C}(A)
\end{cases}
\]

where \(-(x \cdot y) = \neg(\neg(x \rightarrow -y) = \delta(x \rightarrow -y) is proved using Proposition 1.2.1 (v) and (xiv), thus \(\phi(x \cdot y) coincides with \(\phi(x) \cdot_R \phi(y). Analogously, for the implication,

\[
\phi(x \rightarrow y) = \begin{cases} 
(1, x \rightarrow y), & x, y \in \mathcal{R}(A) \\
(x \rightarrow y, 1) & x, y \in \mathcal{M}(A) \setminus \{0, 1\} \\
(1, x \rightarrow y) = (1, -y \rightarrow -x) & x, y \in \mathcal{C}(A) \\
y & x \in \mathcal{R}(A), y \in \mathcal{M}(A) \setminus \{0, 1\} \\
(0, -(x \rightarrow y)) = (0, \delta(x \cdot -y)) & x \in \mathcal{R}(A), y \in \mathcal{C}(A) \\
(-x, 1) & x \in \mathcal{M}(A) \setminus \{0, 1\}, y \in \mathcal{C}(A) \\
(1, 1) & x \in \mathcal{M}(A) \setminus \{0, 1\}, y \in \mathcal{R}(A) \\
(1, 1) & x \in \mathcal{C}(A), y \in \mathcal{R}(A) \\
(1, 1) & x \in \mathcal{C}(A), y \in \mathcal{M}(A) \setminus \{0, 1\}
\end{cases}
\]

where we recall that for any element \(x \in \mathcal{C}(A), x = \neg x \) (Proposition 4.2.10 (iv)), and in particular \(x \rightarrow y = \neg x \rightarrow \neg y = \neg y \rightarrow \neg x = \neg y \rightarrow \neg x\) using Proposition 1.2.1 (xii), and \(\delta(x \cdot -y) = \neg(\neg(x \cdot -y) = \neg(x \rightarrow -y) = \neg(x \rightarrow y) via Proposition 1.2.1 (v). Hence \(\phi(x \rightarrow y) coincides with \(\phi(x) \rightarrow_R \phi(y) and the proof is settled.

Thus, we can easily see that the following results hold, as they follow from identities that are satisfied in any directly indecomposable MVR_n-algebra.

**Corollary 4.2.12.** Let \(A \in MVR_n\), then \(\mathcal{R}(A) = \gamma_n^{-1}(\{1\})\) is the radical of \(A\), and \(\mathcal{C}(A) = \gamma_n^{-1}(\{0\})\) is the coreadical. Moreover, the following are equivalent:

1. \(x \in \text{Rad}(A)\);
Chapter 4. Generalized $\delta$-rotations of residuated lattices

ii. $\gamma_n(x) = 1$;

iii. $\nabla_n(x) = 1$;

iv. $\beta_n(x) = x$.

Corollary 4.2.13. Let $A \in \text{MVR}_n$, then each element $x \in A$ can be written as:

$$x = (\gamma_n(x) \lor \neg \beta_n(x)) \land (\neg \gamma_n(x) \lor \beta_n(x)).$$

Remark 4.2.14. The representation of elements in Corollary 4.2.13 allows to write every element of an $\text{MVR}_n$-algebra in terms of an element in the MV-skeleton and an element in the radical, suggesting that any $\text{MVR}_n$-algebra $A$ can be decomposed in the pair $(\gamma_n(A), \mathcal{R}(A))$. Moreover, the representation applies to the examples of $\text{MVR}_n$-algebras given: for rDL-algebras it coincides with the representation given in Proposition 2.2.11 (ii), while for $A$ being either a Stonean residuated lattice or a BL$_n$-algebra, we have that for every $x \in A$:

$$x = \gamma_n(x) \land \beta_n(x) = \neg \neg x \land (\neg \neg x \to x).$$

We will now show some other technical results.

Lemma 4.2.15. Given $A \in \text{MVR}_n$, $a \in \mathcal{M}(A)$, the following properties about $b(a)$, the greatest Boolean element below $a$, hold:

- $b(a \cdot a') = b(a) \cdot b(a')$, for $b(a) \cdot b(a') = a^{n-1} \cdot (a')^{n-1} = (a \cdot a')^{n-1}$;
- $b(a \land a') = b(a) \land b(a')$, for $b(a) \land b(a') = a^{n-1} \land (a')^{n-1} = a^{n-1} \cdot (a')^{n-1} = (a \land a')^{n-1}$;
- $b(a \lor a') = b(a) \lor b(a')$.

Proof. The properties can be easily verified in directly indecomposable $\text{MVR}_n$-algebras.

Lemma 4.2.16. Given $A \in \text{MVR}_n$, $a \in \mathcal{M}(A)$ and $d, d' \in \mathcal{R}(A)$, the following are equivalent:

1. $a \cdot d = a \cdot d'$;
Chapter 4. Generalized $\delta$-rotations of residuated lattices

2. $b(a) \cdot d = b(a) \cdot d'$;

3. $\neg b(a) \lor d = \neg b(a) \lor d'$.

Proof. We will prove that (1) and (2) are equivalent, and then that (2) and (3) are equivalent.

For the first equivalence, observe that if $a \cdot d = a \cdot d'$, then $a^{n-1} \cdot d = a^{n-1} \cdot d'$ so $b(a) \cdot d = b(a) \cdot d'$. For the other implication, observe that

$$(\gamma_n(x)^{n-1} \cdot \beta_n(y) \rightarrow \gamma_n(x)^{n-1} \cdot \beta_n(z))$$

$$\rightarrow (\gamma_n(x) \cdot \beta_n(y) \rightarrow \gamma_n(x) \cdot \beta_n(z)) = 1$$

holds for directly indecomposable algebras, as if $\gamma_n(x) = 1$ then it is trivial, and if $\gamma_n(x) < 1$ then $(\gamma_n(x))^{n-1} = 0$ and $\gamma_n(x) \cdot \beta_n(y) = \gamma_n(x) \cdot \beta_n(z) = \gamma_n(x)$.

Now, for the other equivalence, if $b(a) \cdot d = b(a) \cdot d'$ then $b(a) \land d = b(a) \land d'$ (as they coincide) and

$$\neg b(a) \lor d = \neg b(a) \lor (b(a) \land d) = \neg b(a) \lor (b(a) \land d') = \neg b(a) \lor d'.$$

And if $\neg b(a) \lor d = \neg b(a) \lor d'$, then

$$b(a) \land d = b(a) \land (\neg b(a) \lor d) = b(a) \land (\neg b(a) \lor d') = b(a) \land d'.$$

4.3 Quadruple-representation for MVR$_n$-algebras

Following the same line of thought as in Chapter 2, from Sections 2.3 to 2.5, we are going to define for any fixed $n > 1$ a category of quadruples that will be equivalent to the algebraic category of MVR$_n$-algebras.

Definition 4.3.1. We say that $(M, R, \vee_e, \delta)$ is a $n$-quadruple if $M \in \text{MV}_n$, $R$ is a residuated lattice, $\vee_e : \mathcal{B}(M) \times R \rightarrow R$ is an external join as in Definition 2.3.1, and $\delta : R \rightarrow R$ is a wdl-admissible operator. Recall that $\nu_b(r) = b \vee_e r$.

If $(M, R, \vee_e, \delta)$ and $(M', R', \vee'_e, \delta')$ are $n$-quadruples, we say that $(h, k)$ is a good morphism pair between them, if $h : M \rightarrow M'$ is an MV-homomorphism
and $k : R \to R'$ a residuated lattice homomorphism satisfying $k \circ \delta = \delta' \circ k$.
Moreover, for every $b \in \mathcal{B}(M)$, $r \in R$, the following holds:

$$k(\nu_b(r)) = \nu_{h(b)}(k(r)).$$

For any subvariety $R$ of $\mathcal{RL}$, we call $\mathcal{Q}_R^n$ the category of $n$-quadruples $(M, R, \lor, \delta)$ such that $R \in R$, with good morphism pairs as morphisms.

**MVR$_n$-algebras from $n$-quadruples**

In order to prove that it is possible to reconstruct a MVR$_n$-algebra from an $n$-quadruple, we will follow the same lines of the proofs in Chapter 2, that actually will result in being particular cases of this construction. We first need some preliminary results. In particular, recalling that any MV$_n$-algebra is hyperarchimedean [39], we can use the following important fact.

**Lemma 4.3.2** (Theorem 6.3.2, [39]). In every hyperarchimedean MV-algebra, every prime filter is maximal.

A filter of an MV-algebra is called Stonean if it is generated by a filter the Boolean skeleton of the algebra. From [39, Theorem 6.3.2] we also have:

**Lemma 4.3.3.** Every filter of an hyperarchimedean MV-algebra is Stonean.

In what follows we assume that $(M, R, \lor, \delta) \in \mathcal{Q}_R^n$, for some subvariety $R$ of residuated lattices. From previous lemmas we obtain the following.

**Lemma 4.3.4.** For every $p \in \text{Max } \mathcal{B}(M)$, $M/\langle p \rangle$ is a directly indecomposable $MV_n$-algebra. Moreover, $M$ is a subdirect product of the family

$$\{M/\langle p \rangle \}_{p \in \text{Max } \mathcal{B}(M)}.$$

Now, for each $p \in \text{Max } \mathcal{B}(M)$, let

$$\Theta_p = \{(d, d') \in R \times R \mid \exists b \in p : \nu_{-\delta}(d) = \nu_{-\delta}(d')\}.$$ 

**Lemma 4.3.5.** $\Theta_p$ is a congruence of $R$ for each $p \in \text{Max } \mathcal{B}(M)$.

*Proof.* The proof is completely analogous to the one of Lemma 2.3.5. \qed
Chapter 4. Generalized $\delta$-rotations of residuated lattices

Lemma 4.3.6. \[ \bigcap_{p \in \text{Max } \mathcal{B}(M)} \Theta_p = \{1\}. \]

Proof. The proof is the same as in Lemma 2.3.6. \[ \square \]

To simplify notation, for each $p \in \text{Max } \mathcal{B}(M)$, let $M/p$ be $M/\langle p \rangle$ and let $R/p$ be the quotient of $R$ by the filter $\Theta_p$.

Lemma 4.3.7. $R$ is a subdirect product of the family $\{R/p\}_{p \in \text{Max } \mathcal{B}(M)}$.

Proof. From Lemma 4.3.5, Lemma 4.3.6. \[ \square \]

Before defining the algebra related to a quadruple, we state some other technical results which will be needed later. Again we are considering a quadruple $Q = (M, R, \lor, \delta)$ in $Q_n^R$.

For each $p \in \text{Max } \mathcal{B}(M)$, let $\lor_p : \mathcal{B}(M)/p \times R/p \to R/p$ and $\delta_p : R/p \to R/p$ be defined respectively, for each $c \in R$, by:

\[ 0/p \lor_p c/p = c/p, \quad 1/p \lor_p c/p = 1/p, \]
\[ \delta_p(c/p) = (\delta(c))/p \]

Lemma 4.3.8. For each $(b, c) \in \mathcal{B}(M) \times R$,

\[ b \lor \lor_p c = (b/p \lor_p c/p)_{p \in \text{Max } \mathcal{B}}. \]

Proof. Same as in Lemma 2.3.9. \[ \square \]

Thus we obtain the following.

Lemma 4.3.9. $(M/p, R/p, \lor_p, \delta_p)$ is a $n$-quadruple for each $p \in \text{Max } \mathcal{B}$. Moreover, $\lor_p$ is the only map $\lor : \mathcal{B}(M)/p \times R/p \to R/p$ such that $(M/p, R/p, \lor, \delta_p)$ is a quadruple.

Definition 4.3.10. For every $(a, d), (a', d') \in M \times R$, let us write

\[ (a, d) \sim (a', d') \text{ iff } a = a' \text{ and } \lor_{-\lor_n(a)}(d) \land \lor_{\Delta_n(a)}(\delta(d)) = \lor_{-\lor_n(a)}(d') \land \lor_{\Delta_n(a)}(\delta(d')) \]
Chapter 4. Generalized δ-rotations of residuated lattices

Remark 4.3.11. Notice that if $M$ is directly indecomposable:

$$\nu_{-\nabla_n(a)}(d) \land \nu_{\Delta_n(a)}(\delta(d)) = \begin{cases} 
  d & \text{if } a = 1; \\
  \delta(d) & \text{if } a = 0; \\
  1 & \text{if } a \in M \times \{0, 1\}.
\end{cases}$$

It is not difficult to show that $\sim$ is an equivalence relation on $M \times R$.

Definition 4.3.12. For each quadruple $Q = (M, R, \lor, \delta)$ let us consider the structure

$$M \otimes^\delta R = (M \times R/\sim, \odot, \cap, \cup, [0, 1], [1, 1])$$

with binary operations defined as follows:

1. $[a, d] \odot [a', d'] =
   \begin{align*}
   [a \cdot a', \nu_{b(a) \lor b(a')} (d' \to d) \land \nu_{-b(a) \lor b(a')} (d \to d') \land \\
   \nu_{-b(a) \lor b(a')} (d \cdot d')]
   \end{align*}$

2. $[a, d] \Rightarrow [a', d'] =
   \begin{align*}
   [a \to a', \nu_{b(a) \lor \Delta_n(a')} (\delta(d') \to \delta(d)) \land \\
   \nu_{-b(a) \lor \Delta_n(a')} (d \cdot d') \land \nu_{-b(a) \lor b(a')} (d \to d')]
   \end{align*}$

3. $[a, d] \cap [a', d'] =
   \begin{align*}
   [a \wedge a', \nu_{\Delta_n(a) \lor \Delta_n(a')} (d \lor d') \land \\
   \nu_{b(a) \lor \Delta_n(a')} (d) \land \nu_{-b(a) \lor \Delta_n(a')} (d) \land \\
   \nu_{-b(a) \lor b(a')} (d \land d')]
   \end{align*}$

4. $[a, d] \cup [a', d'] =
   \begin{align*}
   [a \lor a', \nu_{b(a) \lor b(a')} (d \lor d') \land \\
   \nu_{b(a) \lor \Delta_n(a')} (d') \land \\
   \nu_{b(a) \lor \Delta_n(a')} (d') \land \\
   \nu_{-b(a) \lor b(a')} (d) \land \\
   \nu_{-b(a) \lor b(a')} (d \lor d')]
   \end{align*}$

Remark 4.3.13. When $n = 2$, Definition 4.3.12 coincides with Definition 2.3.10.

We will now show that the operations of $M \otimes^\delta R$ are well defined with respect to the equivalence relation of Definition 4.3.10.

Lemma 4.3.14. For every quadruple $Q = (M, R, \lor, \delta)$, the operations of $M \otimes^\delta R$ are well-defined.

Proof. Let us check that the operations are well defined with respect of the equivalence relation $\sim$, the proof being similar to the one of Lemma 2.3.11. Let us suppose that $[a', d'] \sim [a'', d'']$. This means by definition that $a' = a''$ and $\nu_{-\nabla(a')} (d') \land \nu_{\Delta(a')} (\delta(d')) = \nu_{-\nabla(a')} (d'') \land \nu_{\Delta(a')} (\delta(d''))$. Via Lemma 4.3.7
Chapter 4. Generalized δ-rotations of residuated lattices

and Lemma 4.3.8, we can use the subdirect representation of R and A, and we obtain that when \( a' = 1 \), then \( d' = d'' \), while if \( a' = 0 \), \( \delta(d') = \delta(d'') \). We shall prove the following, the proofs for the lattice operations being similar and hence omitted:

1. \([a, d] \odot [a', d'] \sim [a, d] \odot [a'', d'']\);
2. \([a, d] \Rightarrow [a', d'] \sim [a, d] \Rightarrow [a'', d'']\);
3. \([a', d'] \Rightarrow [a, d] \sim [a'', d''] \Rightarrow [a, d]\).

Let us prove 1. The condition becomes \( a \cdot a' = a \cdot a'' \), which is clearly satisfied, and we can split the second equation in the following two:

\[
\forall_{\nabla (a \cdot a')}(\forall_{b(a) \land b(a')}(d' \rightarrow d) \land \forall_{b(a) \lor b(a')}(d \rightarrow d') \land \forall_{\neg b(a) \lor b(a')}(d \cdot d'))
\]

\[
\forall_{\nabla (a \cdot a')}(\forall_{b(a) \land b(a')}(d'' \rightarrow d) \land \forall_{b(a) \lor b(a')}(d \rightarrow d'') \land \forall_{\neg b(a) \lor b(a')}(d \cdot d''))
\]

\[
\forall_{\Delta (a \cdot a')}(\delta(\forall_{b(a) \land b(a')}(d' \rightarrow d) \land \forall_{b(a) \lor b(a')}(d \rightarrow d') \land \forall_{\neg b(a) \lor b(a')}(d \cdot d')))
\]

\[
\forall_{\Delta (a \cdot a')}(\delta(\forall_{b(a) \land b(a')}(d'' \rightarrow d) \land \forall_{b(a) \lor b(a')}(d \rightarrow d'') \land \forall_{\neg b(a) \lor b(a')}(d \cdot d'')))
\]

Again, via Lemma 4.3.8, we will consider the subdirect representation. Notice that if \( a' = 1 \), then \( d' = d'' \), thus the conditions clearly holds. Let us consider the case \( a' < 1 \). Notice than hence \( a \cdot a' < 1 \), thus Eq. (4.4) holds for any \( a \), being \( \nabla (a \cdot a') = 0 \). Now, if \( a < 1 \) then Eq. (4.5) becomes \( \forall_{\Delta (a \cdot a')}(c) = 1 \) for any \( c \in R \), and again Eq. (4.5) is satisfied. It remains to check the case in which \( a = 1, a' = 0 \). Eq. (4.5) reduces to \( \delta(d \rightarrow d') = \delta(d \rightarrow d'') \), which holds since \( \delta(d \rightarrow d') = d \rightarrow \delta(d') \), \( \delta(d \rightarrow d'') = d \rightarrow \delta(d'') \), and \( \delta(d') = \delta(d'') \) by hypothesis.
Chapter 4. Generalized δ-rotations of residuated lattices

Let us now prove 2. The condition becomes \( a → a' = a → a'' \), again satisfied, and the other two:

\[
\begin{align*}
\nu_{\neg \nu(\nu\rightarrow \nu)}(\nu_b(\nu\rightarrow \nu)(\delta(d') \rightarrow \delta(d)) \land \nu_{\neg \nu(\nu\rightarrow \nu)}(d \cdot d') \land \nu_{\neg \nu(\nu\rightarrow \nu)}(d \rightarrow d'))
\end{align*}
\]

\(= \nu_{\neg \nu(\nu\rightarrow \nu)}(\nu_b(\nu\rightarrow \nu)(\delta(d'') \rightarrow \delta(d)) \land \nu_{\neg \nu(\nu\rightarrow \nu)}(d \cdot d'') \land \nu_{\neg \nu(\nu\rightarrow \nu)}(d \rightarrow d''))
\]

(4.6)

\[
\begin{align*}
\nu_{\Delta(\nu\rightarrow \nu)}(\delta(d') \rightarrow \delta(d)) \land \nu_{\neg \nu(\nu\rightarrow \nu)}(d \cdot d') \land \nu_{\neg \nu(\nu\rightarrow \nu)}(d \rightarrow d'))
\end{align*}
\]

\(= \nu_{\Delta(\nu\rightarrow \nu)}(\delta(d'') \rightarrow \delta(d)) \land \nu_{\neg \nu(\nu\rightarrow \nu)}(d \cdot d'') \land \nu_{\neg \nu(\nu\rightarrow \nu)}(d \rightarrow d''))
\]

(4.7)

Reasoning as before, we shall check the case where \( a' < 1 \). Let us first assume that \( a' = 0 \). Then, if \( a \notin \{0,1\} \), \( a → a' = \neg a \notin \{0,1\} \), thus \( \nu_{\neg \nu(\nu\rightarrow \nu)}(c) = \nu_{\Delta(\nu\rightarrow \nu)}(c) = 1 \) for any \( c \in R \), thus Eq. (4.6) and (4.7) are satisfied. If \( a = 0 \) then \( a → a' = 1 \), thus Eq. (4.7) is trivially satisfied, while Eq. (4.6) becomes \( \delta(d') → \delta(d) = \delta(d'') → \delta(d) \), which holds since by hypothesis \( \delta(d') = \delta(d'') \).

If \( a = 1 \), then \( a → a' = 0 \), Eq. (4.6) trivially holds and Eq. (4.7) becomes \( \delta(d' \cdot d') = \delta(d' \cdot d'') \), that holds since \( \delta(d \cdot d') = \delta(d \cdot d'') = \delta(d') \cdot \delta(d'') \). We shall now check the case in which \( a' \notin \{0,1\} \). If \( a = 1 \), then \( a → a' \notin \{0,1\} \), hence the conditions become both 1 = 1. If \( a \notin \{0,1\} \) then Eq. (4.6) reduces to \( \nu_{\neg \nu(\nu\rightarrow \nu)}(1) = \nu_{\neg \nu(\nu\rightarrow \nu)}(1) \), and Eq. (4.7) to \( \nu_{\Delta(\nu\rightarrow \nu)}(1) = \nu_{\Delta(\nu\rightarrow \nu)}(1) \), thus they are satisfied. Finally, if \( a = 0 \), again both conditions reduce to 1 = 1.

We shall now prove 3, thus \( a' → a = a'' → a \), easily satisfied, and:

\[
\begin{align*}
\nu_{\neg \nu(\nu\rightarrow \nu)}(\nu_b(\nu\rightarrow \nu)(\delta(d) \rightarrow \delta(d')) \land \nu_{\neg \nu(\nu\rightarrow \nu)}(d' \cdot d') \land \nu_{\neg \nu(\nu\rightarrow \nu)}(d' \rightarrow d))
\end{align*}
\]

\(= \nu_{\neg \nu(\nu\rightarrow \nu)}(\nu_b(\nu\rightarrow \nu)(\delta(d'') \rightarrow \delta(d)) \land \nu_{\neg \nu(\nu\rightarrow \nu)}(d' \cdot d'') \land \nu_{\neg \nu(\nu\rightarrow \nu)}(d' \rightarrow d''))
\]

(4.8)

\[
\begin{align*}
\nu_{\Delta(\nu\rightarrow \nu)}(\delta(d) \rightarrow \delta(d')) \land \nu_{\neg \nu(\nu\rightarrow \nu)}(d' \cdot d') \land \nu_{\neg \nu(\nu\rightarrow \nu)}(d' \rightarrow d))
\end{align*}
\]

\(= \nu_{\Delta(\nu\rightarrow \nu)}(\delta(d'') \rightarrow \delta(d)) \land \nu_{\neg \nu(\nu\rightarrow \nu)}(d' \cdot d'') \land \nu_{\neg \nu(\nu\rightarrow \nu)}(d' \rightarrow d''))
\]

(4.9)
Again, we check for $a' < 1$, and we first check for $a' = 0$. Notice that hence $a' \to a = 1$, thus Eq. 4.9 is always satisfied. Now, if $a > 0$, Eq. 4.8 reduces to $\nu_\cdot\nu(a'\to a)(1) = \nu_\cdot\nu(a'\to a)(1)$. If $a = 0$, Eq. 4.8 reduces to $\delta(d) \to \delta(d') = \delta(d) \to \delta(d'')$, that holds since by hypothesis $\delta(d') = \delta(d'')$. Now we check for $a' \notin \{0,1\}$. If $a = 1$, $a' \to a = 1$, thus Eq. 4.9 clearly holds, while Eq. 4.8 reduces to $\nu_\cdot\nu(a'\to a)(1) = \nu_\cdot\nu(a'\to a)(1)$. If $a = 0$, $a' \to a = -a'$, thus both conditions hold. Finally, if $a \notin \{0,1\}$, Eq. 4.8 and 4.9 reduce to $\nu_\cdot\nu(a'\to a)(1) = \nu_\cdot\nu(a'\to a)(1)$ and $\nu_\Delta(a'\to a)(1) = \nu_\Delta(a'\to a)(1)$, and hence they both hold. Thus, the operations are well defined.

**Remark 4.3.15.** We can define in $M \otimes^\delta e R$ the negation as usual, and we obtain the following:

$$\neg[a,d] = [a,d] \Rightarrow [0,1] = [-a, \nu_{b(a)}(d) \land \nu_{-b(a)} \delta(d)].$$

Lemma 4.3.9 and Lemma 4.3.14 yield that for each $p \in \text{Max } \mathcal{B}(M)$ we have the corresponding algebra $M/p \otimes^\delta e R/p$. We shall see that $M/p \otimes^\delta e R/p$ is a directly indecomposable algebra in $\text{MVR}_n$. Explicitly we have:

**Lemma 4.3.16.** For each $p \in \text{Max } \mathcal{B}(M)$, $M/p \otimes^\delta e R/p \cong \mathcal{R}_{\lambda/p}(R/p)$. Thus, $M/p \otimes^\delta e R/p$ is a directly indecomposable algebra in $\text{MVR}_n$.

**Proof.** Notice that, if $m \notin \{0,1\}$, then $(m,d) \sim (m,d')$ for every $d,d' \in R/p$, and $(0,z) \sim (0,z')$ iff $\neg z = \neg z'$. We can then define $\alpha : M/p \otimes^\delta e R/p \to \mathcal{R}_{\lambda/p}(R/p)$, such that $\alpha[1,d] = (1,d)$, $\alpha[m,1] = (m,1)$ if $m \notin \{0,1\}$, and $\alpha[0,z] = (0,z)$. The fact that there is a bijection can be proved by easy calculations. Let us now show that there is an homomorphism. We first show that the monoidal operation is preserved, supposing $d,d' \in R/p$, $m,m' \in M/p \setminus \{0,1\}$.

1. $[1,d] \otimes [1,d'] = [1,1 \land \nu_0(d \cdot d')] = [1,d \cdot d']$;
2. $[0,d] \otimes [0,d'] = [0,1 \land 1] = [0,1]$;
3. $[0,d] \otimes [1,d'] = [0,\nu_0(d' \to d) \land 1] = [0,d' \to d]$;
4. $[1,d] \otimes [0,d'] = [0,d \to d']$;
5. \([m, 1] \odot [m', d'] = [m \cdot m', 1]\);
6. \([m, 1] \odot [0, d'] = [0, 1]\);
7. \([m, 1] \odot [1, d'] = [m, d' \rightarrow 1] = [m, 1]\).

Thus, it follows that \(\alpha([m, d] \odot [m, d']) = \alpha([m, d]) \cdot_R \alpha([m', d'])\). Let us now check the implication, with \(d, d' \in R/p\), \(m, m' \in M/p \setminus \{0, 1\}\):

1. \([1, d] \Rightarrow [1, d'] = [1, 1 \land \nu_0(d \rightarrow d') \land 1] = [1, d \rightarrow d']\);
2. \([0, d] \Rightarrow [0, d'] = [1, \nu_0(\delta(d') \rightarrow \delta(d)) \land 1 \land 1] = [1, d' \rightarrow d]\);
3. \([0, d] \Rightarrow [1, d'] = [1, 1 \land 1 \land 1] = [0, 1]\);
4. \([1, d] \Rightarrow [0, d'] = [0, \delta(d \cdot d')]\);
5. \([m, d] \Rightarrow [m', d'] = [m \rightarrow m', 1]\);
6. \([m, d] \Rightarrow [0, d'] = [\neg m, d' \rightarrow d] = [\neg m, 1]\);
7. \([m, d] \Rightarrow [1, d'] = [1, 1]\);
8. \([0, d] \Rightarrow [m, d'] = [1, 1]\);
9. \([1, d] \Rightarrow [m, d'] = [m, \delta(d \cdot d')] = [m, 1]\).

From which it follows that \(\alpha([m, d] \Rightarrow [m, d']) = \alpha([m, d]) \rightarrow_R \alpha([m', d'])\).

The proofs for the lattice operations are similar and hence omitted. \(\square\)

Now we are ready to prove that for each quadruple \(Q\) its associated algebra is in \(\text{MVR}_n\), and that its associated quadruple is isomorphic to \(Q\). First we observe that if \(a \in M\) then we set \(b(a)\) to be the greatest boolean element below \(a\), that exists because \(M\) is an \(\text{MV}_n\)-algebra. Since \(b(a) = a^{n-1}\) we have \(b(a)/p = a^{n-1}/p = (a/p)^{n-1}\) is the greatest boolean element below \(a/p\) for each \(p \in \text{Max}\ \mathcal{R}(M)\). Thus for each \(a \in A\) and each \(p \in \text{Max}\ \mathcal{R}(M)\) we have:

\[
b(a/p) = b(a)/p. \tag{4.10}\]
**Theorem 4.3.17.** $\mathbf{M} \otimes^\delta_e \mathbf{R}$ is an algebra in $\text{MVR}_n$. Moreover, the MV-skeleton of $\mathbf{M} \otimes^\delta_e \mathbf{R}$ is isomorphic to $\mathbf{M}$ and the radical of $\mathbf{M} \otimes^\delta_e \mathbf{R}$ is isomorphic to $\mathbf{R}$.

**Proof.** Let $h : \mathbf{M} \otimes^\delta_e \mathbf{R} \rightarrow \prod_{p \in \text{Max } \mathcal{R}(\mathbf{M})} \mathbf{M}/p \otimes^\delta_e \mathbf{R}/p$ be defined as

$$h(a, d) = \{(a/p, d/p)\}_{p \in \text{Max } \mathcal{R}(\mathbf{M})}.$$  

We will prove that $h$ is an injective homomorphism. This will mean that $\mathbf{M} \otimes^\delta_e \mathbf{R}$ is a subalgebra of a direct product of algebras in $\text{MVR}_n$ and hence an algebra in $\text{MVR}_n$ itself. Injectivity of $h$ follows from Lemma 4.3.4 and Lemma 4.3.7. The fact that $h$ is an homomorphism follows from the definition of the operations, Lemma 4.3.8 and Equation (4.10) plus the fact that since each $\Theta_p$ is a congruence, the quotient respects the operations.

It is left to prove that the MV-skeleton of $\mathbf{M} \otimes^\delta_e \mathbf{R}$ is isomorphic to $\mathbf{M}$ and the radical of $\mathbf{M} \otimes^\delta_e \mathbf{R}$ is isomorphic to $\mathbf{R}$. Let us recall that the MV-elements of $\mathbf{M} \otimes^\delta_e \mathbf{R}$ will be pairs $[a, d]$ such that $\gamma_n[a, d] = [a, d]$, where $\gamma_n$ is the one defined in Equation 4.2. We define the map $\lambda_M : M \rightarrow \mathcal{M}(\mathbf{M} \otimes^\delta_e \mathbf{R})$ as $\lambda_M(a) = [a, 1]$, and prove that it is an isomorphism. First of all, notice that for any $a \in A$, $[a, 1]$ is an MV-element, indeed $\gamma_n[a, 1] = \Delta_n[a, 1] \land ([a, 1] \lor \nabla_n([a, 1] \lor \neg[a, 1]) = [a, \nu(a \lor \neg a)\cdot 1)] = [a, 1]$. Now, if $[a, d]$ is an MV-element, we have that $\lambda_M(d) \sim (a, 1)$, thus $[a, d] = [a, 1]$. Indeed, via Lemma 4.3.16, for every $p \in \text{Max } \mathcal{R}(\mathbf{M})$, it is $(a/p, d/p) \sim (a/p, 1/p)$. This proves that $\lambda_M$ is surjective. Injectivity follows from the fact that if $a \neq a'$ then $[a, 1] \neq [a', 1]$, by the definition of the equivalence relation. It is also easy to verify that $\lambda_M$ is an homomorphism, indeed for example:

$$\lambda_M(a \cdot a') = [a \cdot a', 1] = [a \cdot a', \nu(b(a) \lor b(a'))(1 \rightarrow 1) \land \nu(b(a) \lor b(a'))(1 \rightarrow 1) \land \nu(b(a) \lor b(a'))(1 \rightarrow 1)] = [a, 1] \otimes [a', 1] = \lambda_M(a) \otimes \lambda_M(a'),$$  

and similarly $\lambda_M(a \rightarrow a') = [a \rightarrow a', 1] = [a, 1] \Rightarrow [a', 1] = \lambda_M(a) \Rightarrow \lambda_M(a')$.

Now we shall prove that the radical of $\mathbf{M} \otimes^\delta_e \mathbf{R}$ is isomorphic to $\mathbf{R}$. The elements of the radical will be pairs $[a, d]$ such that $\gamma_n[a, d] = [1, 1]$. Let us now define the map $\lambda_R : R \rightarrow \mathcal{R}(\mathbf{M} \otimes^\delta_e \mathbf{R})$ as $\lambda_R(d) = [1, d]$, and prove that it is an isomorphism. First, for any $d \in R$, $[1, d]$ is in the radical, indeed $\gamma_n[1, d] = [1, 1]$, as can be easily calculated in the components. Viceversa, if
Chapter 4. Generalized δ-rotations of residuated lattices

\([a, d]\) is in the radical, it follows from the subdirect representation that for every \(p \in \text{Max } \mathcal{B}(\text{M})\), it is \([a/p, d/p] = [1/p, 1/p]\), thus \(a = 1\). This proves surjectivity. Injectivity follows from the fact that \([1, d] \neq [1, d']\) iff \(d \neq d'\), by the definition of the equivalence relation. Moreover, we can prove that \(\lambda_R\) is an homomorphism. Indeed, any operation \(* \in \{\cdot, \lor, \land\}\) is preserved, and (as easily seen) top and bottom are preserved. Let us for example check the product: \(\lambda_R(d \cdot d') = [1, d \cdot d'] = [1, \nu_0(d \cdot d')] = [1, \nu_{1 \lor 1}(d' \to d) \land \nu_{1 \land 1}(d \to d')] = [1, d] \circ [1, d'] = \lambda_R(d) \odot \lambda_R(d')\). This settles the proof.

\[\square\]

**Corollary 4.3.18.** Let \((\text{M}, \text{R}, \nu_e, \delta)\) be a \(n\)-quadruple. Then the following hold:

1. \(\text{M}\) and \(\text{R}\) embed into \(\text{M} \otimes^\delta \text{R}\) via the maps \(\lambda_M\) and \(\lambda_R\) of Theorem 4.3.17,

2. for every \(b \in \mathcal{B}(\text{M})\) and every \(d \in \text{R}\), \(\lambda_R(\nu_b(d)) = \lambda_M(b) \lor \lambda_R(d)\),

3. the representation of the elements of \(\text{M} \otimes^\delta \text{R}\) is unique. Indeed, for any element \([a, d]\), \([a, d] = (\gamma_n[a, d] \lor \beta_n[a, d]) \land (\neg \gamma_n[a, d] \lor \beta_n[a, d]).\)

**Proof.** Claim (1) is trivial. As to prove (2), notice that we have \(\lambda_R(\nu_b(d)) = [1, \nu_b(d)] = [b \lor 1, \nu_b(d) \land \nu_{-\delta}(1)] = [b, 1] \cup [1, d] = \lambda_M(b) \lor \lambda_R(d).\) It is left to prove (3). Using the representation in components, we can prove by calculations that the following identities hold:

\[
\begin{align*}
\Delta_n[a, d] & = [(\neg (a)^n)^2, 1], \\
\nabla_n[a, d] & = [\neg (a)^n, 1], \\
\gamma_n[a, d] & = \Delta_n[a, d] \cap ([a, d] \cup \nabla_n([a, d] \lor \neg[a, d]) = [a, \nu_{(a \lor \neg a)^n}(d)], \\
\beta_n[a, d] & = ([a, d] \cup \neg[a, d]) \cup \nabla_n([a, d] \lor \neg[a, d]) = [1, \nu_{(a \lor \neg a)^n}(d)].
\end{align*}
\]

Thus we can obtain via further computations:

\[
\begin{align*}
\neg \gamma_n[a, d] & = [\neg a, \nu_{(a \lor \neg a)^n}(d)], \\
\neg \beta_n[a, d] & = [0, \delta(\nu_{(a \lor \neg a)^n}(d))], \\
\gamma_n[a, d] \lor \neg \beta_n[a, d] & = [a, \nu_{\Delta_n(a)}(\delta(d)) \land \nu_{(a \lor \neg a)^n}(d)], \\
\neg \gamma_n[a, d] \lor \beta_n[a, d] & = [1, \nu_{\nabla_n(a)}(d)].
\end{align*}
\]

124
And finally:

\[(\gamma_n[a, d] \lor \beta_n[a, d]) \land (\neg \gamma_n[a, d] \lor \beta_n[a, d]) = [a, \nu_{\Delta_n(a)}(\delta(d)) \land \nu_{(\alpha \lor \eta)} (d)] \land [1, \nu_{\neg \lambda_n(a)}(d)] = [a, \nu_{\beta(a)}(\nu_{\Delta_n(a)}(\delta(d)) \land \nu_{(\alpha \lor \eta)} (d)) \land \nu_{\lambda(a)}(\nu_{\Delta_n(a)}(\delta(d)) \land \nu_{(\alpha \lor \eta)} (d))]
\]

We shall now prove that \([a, \nu_{\Delta_n(a)}(d) \land \nu_{\Delta_n(a)}(\delta(d)))] = [a, d]\), that is to say

\[a = a,
\]

\[\nu_{\neg \lambda_n(a)}(d) = \nu_{\neg \lambda_n(a)}(\nu_{\Delta_n(a)}(d) \land \nu_{\Delta_n(a)}(\delta(d))),
\]

\[\nu_{\Delta_n(a)}(\delta(d)) = \nu_{\Delta_n(a)}(\nu_{\Delta_n(a)}(d) \land \nu_{\Delta_n(a)}(\delta(d))).
\]

The first one is clearly verified, and the other two can be easily seen in the components. Indeed if \(a = 1\), the identities become \(d = d\) and \(1 = 1\), while if \(a \in M \setminus \{1, 0\}\), both reduce to \(1 = 1\). Finally if \(a = 0\), we have \(1 = 1\) and \(\delta(d) = \delta(\delta(d))\), that holds because \(\delta\) is idempotent.

### 4.4 Categorical equivalences for \(\text{MVR}_n\)-algebras

In this section we will prove categorical equivalences results for subcategories of \(\text{MVR}_n\), following the same ideas of Section 2.5. In particular, we first define the functors.

**Notation 4.4.1.** For any subvariety of residuated lattices \(R\), let us write \(\text{MVR}_R^n\) for the class of \(\text{MVR}_n\)-algebras whose radical is in \(R\).

**Definition 4.4.2.** Let \(\hat{\Phi} : \text{MVR}_R^n \to Q_R^n\) be given by

\[\hat{\Phi}(A) = (\mathcal{M}(A), \mathcal{R}(A), \lor, \neg)
\]

\[\hat{\Phi}(f) = (\hat{f}_{|\mathcal{M}(A)}, \hat{f}_{|\mathcal{R}(A)}).
\]

We can prove the following.

**Theorem 4.4.3.** \(\hat{\Phi}\) is a functor.
Proof. Given any \(A \in \text{MVR}_n\), it is easy to see that \(\hat{\Phi}(A) = (\mathcal{M}(A), \mathcal{R}(A), \vee_e, \neg\neg)\) is in \(\mathcal{Q}_n\). Moreover, for any morphism \(f : A \to A'\), \(\hat{\Phi}(f) = (f_1, \mathcal{M}(A)), f_1, \mathcal{R}(A))\) is a morphism pair. Indeed, since \(\mathcal{M}(A)\) is an MV-subalgebra of \(A\), \(\mathcal{R}(A)\) is a sub-residuated lattice of the 0-free reduct of \(A\), and \(f\) is a homomorphism, it follows that \(f_1, \mathcal{M}(A)\) and \(f_1, \mathcal{R}(A)\) are respectively a MV-homomorphism from \(MV(A)\) to \(MV(A')\) and a residuated lattice homomorphism from \(\mathcal{R}(A)\) to \(\mathcal{R}(A')\). Clearly \(f \circ (\neg\neg) = (\neg' \neg') \circ f\). Moreover, being \(f\) a homomorphism, \(f_1, \mathcal{M}(A)(\vee_e(c)) = \vee'_f, \mathcal{M}(A)(b)\) \((f_1, \mathcal{R}(A)(c))\). Again, the fact that \(T\) preserves the identity map and composition of morphisms can be proven as in the proof of Theorem 2.4.6.

\[\hat{\Xi} : \mathcal{Q}_n \to \text{MVR}_n\] as follows:

\[\hat{\Xi}(M, R, \vee_e, \delta) = \mathbf{M} \otimes^\delta R,\]
\[\hat{\Xi}(h, k)[[a, d]] = [h(a), k(d)].\]

We will now prove that \(\hat{\Xi}\) is a functor, and that it establishes a categorical equivalence.

**Theorem 4.4.5.** The map \(\hat{\Xi}\) is a functor.

Proof. We have already proved in Theorem 4.3.17 that \(\mathbf{M} \otimes^\delta R\) is an algebra in \(\text{MVR}_n\). The proof of the fact that if \((h, k)\) is a morphism, \(\hat{\Xi}(h, k)\) is a morphism of \(\text{MVR}_n\), is analogous to the one given for Theorem 2.5.5.

**Lemma 4.4.6.** \(\hat{\Xi}\) is full and faithful.

Proof. We are following the same lines of the proofs for Theorem 2.4.7 and Theorem 2.5.6. Let \((\mathbf{M}, R, \vee_e, \delta)\) and \((\mathbf{M}', R', \vee'_e, \delta')\) be quadruples and let \(\tau : \mathbf{M} \otimes^\delta R \to \mathbf{M}' \otimes'^\delta R'\) be a morphism. Let \(h : \mathbf{M} \to \mathbf{M}'\) be defined as follows: for every \(a \in M\), \(h(a) = a'\), where by what we proved in Theorem 4.3.17, \(a'\) is the unique element of \(M'\) such that \(\tau([a, 1]) = [a', 1]\). Notice that, since \([a, 1] \in \mathcal{M}(M \otimes^\delta R)\) and \(\tau\) is a homomorphism, \(\tau([a, 1]) \in \mathcal{M}(M' \otimes'^\delta R').\)

Analogously, let \(k : R \to R'\) be defined as follows: for every \(d \in R\), \(k(d) = d'\), where \(d'\) is the unique element of \(R'\) such that \(\tau([1, d]) = [1, d']\). Notice that \(a'\) and \(d'\) are uniquely determined. Indeed as it can be easily seen by
the definition of the equivalence relation, \([a, 1] = [a', 1]\) iff \(a = a'\), and also \([1, d] = [1, d']\) iff \(d = d'\). Thus, \(h\) and \(k\) are well-defined. The maps \(h\) and \(k\) are homomorphisms since by Theorem 4.3.17 they are, respectively, the restriction of \(\tau\) to \(\mathcal{M}(M \otimes_e^R)\) and \(\mathcal{R}(M \otimes_e^R)\), that are respectively a subalgebra and a subreduct. Observing what we showed in Corollary 4.3.18, we have proven that \(\hat{\Xi}\) is full.

To prove that \(\hat{\Xi}\) is faithful, let \((h, k), (h', k') : (M, R, \phi, \delta) \rightarrow (M', R', \phi', \delta')\) and assume that, for every \([a, d] \in M \otimes_e R\):

\[
\hat{\Xi}((h, k))([a, d]) = \hat{\Xi}((h', k'))([a, d]),
\]

that is, \([h(a), k(d)] = [h'(a), k'(d)]\). This means that \(h(a) = h'(a)\) for every \(a\). Hence, in particular, if \(a = 1\), \([1, k(d)] = [1, k'(d)]\). Thus, again by the definition of the equivalence relation, \(k(d) = k'(d)\), whence \((h, k) = (h', k')\) and \(\hat{\Xi}\) is faithful.

**Lemma 4.4.7.** \(\hat{\Xi}\) is essentially surjective.

**Proof.** To prove that \(\hat{\Xi}\) is essentially surjective, we will follow again the proof of Theorem 2.5.6. Let \(X \in \text{MVR}_n\). We need to prove that

\[
X \cong \mathcal{M}(X) \otimes_{\mathcal{R}(X)} \mathcal{R}(X)
\]

For any \(x \in X\), \(x = (\gamma_n(x) \vee \neg \beta_n(x)) \land (\neg \gamma_n(x) \vee \beta_n(x))\). To simplify the notation, we will write \(\gamma\) for \(\gamma_n(x)\), and \(\beta\) for \(\beta_n(x)\), recalling that \(\gamma_n \in \mathcal{M}(X), \beta \in \mathcal{R}(X)\). Let \(\alpha : X \rightarrow \mathcal{M}(X) \otimes_{\mathcal{R}(X)} \mathcal{R}(X)\) be such that \(\alpha(x) = [\gamma, \beta]\). By Corollary 4.3.18, it follows that \(\alpha\) is well defined and injective. Surjectivity is trivial. We should then prove that \(\alpha\) is an order preserving residuated lattice homomorphism. Let \(X\) be represented as a subdirect product of \(\prod_{p \in \text{Spec}(X)} X/p\). Hence, every component \(x_p\) of \(x \in X\) is in the form \(((\gamma \land \neg \beta) \land (\neg \gamma \land \beta))/p = (\gamma/p \lor \neg \beta/p) \land (\neg \gamma/p \lor \beta/p)\) where now \(\gamma/p \in \text{L}_n\). We recall that \(\delta = \neg \gamma\). For the sake of a lighter notation, let us write \(\gamma_p\) instead of \(\gamma/p\) and \(\beta_p\) for \(\beta/p\). Let \(x, x' \in X\) with \(x \leq x'\). Then, \(x_p \leq x'_p\) for every \(p \in \text{Spec}(X)\) whence

\[
(\gamma_p \lor \neg \beta_p) \land (\neg \gamma_p \lor \beta_p) \leq (\gamma'_p \lor \neg \beta'_p) \land (\neg \gamma'_p \lor \beta'_p).
\]

(4.11)
Let us show that $\alpha(x) \leq \alpha(x')$ that is, $[\gamma, \beta] \cap [\gamma', \beta'] = [\gamma, \beta]$, that is to say, $[\gamma \land \gamma', \nu_{\Delta_n(\gamma)} \nu_{\Delta_n(\gamma')} (\beta \lor \beta')] \land \nu_{\beta(\gamma)} \nu_{\beta(\gamma')} (\beta' \land \beta') = [\gamma, \beta]$ and hence, componentwise,

$$\gamma_p \land \gamma'_p = \gamma_p$$  \hfill (4.12)

$$\nu_{\neg \gamma_n(\gamma_p)} (\nu_{\Delta_n(\gamma_p)} \nu_{\Delta_n(\gamma'_p)} (\beta_p \lor \beta'_p) \land \nu_{\beta(\gamma_p)} \nu_{\beta(\gamma'_p)} (\beta'_p) \land \nu_{\beta(\gamma_p)} \nu_{\beta(\gamma'_p)} (\beta' \land \beta'_p)) = \nu_{\neg \gamma_n(\gamma_p)} (\beta_p)$$ \hfill (4.13)

$$\nu_{\Delta_n(\gamma_p)} (\delta (\nu_{\Delta_n(\gamma_p)} \nu_{\Delta_n(\gamma'_p)} (\beta_p \lor \beta'_p) \land \nu_{\beta(\gamma_p)} \nu_{\beta(\gamma'_p)} (\beta'_p) \land \nu_{\beta(\gamma_p)} \nu_{\beta(\gamma'_p)} (\beta' \land \beta'_p)) = \nu_{\Delta_n(\gamma_p)} (\delta (\beta_p))$$ \hfill (4.14)

- For the cases in which $\gamma_p, \gamma'_p$ are both Boolean, the proof is the same as in Theorem 2.5.6.

- If $\gamma_p = 0$ and $\gamma'_p \in L_n \setminus \{0, 1\}$, Equation (4.12) reduces to $0 = 0$, Equation (4.13) to $1 = 1$ and Equation (4.14) $\delta (\beta_p) = \delta (\beta'_p)$.

- If $\gamma_p \in L_n \setminus \{0, 1\}$ and $\gamma'_p = 0$, then Equation (4.11) is false since it reduces to $\gamma_p \leq \gamma'_p$, that does not hold in the components.

- If $\gamma_p, \gamma'_p \in L_n \setminus \{0, 1\}$, Equation (4.11) reduces to $\gamma_p \leq \gamma'_p$. Thus, the other identities hold, since Equation (4.12) becomes $\gamma_p = \gamma_p$, and the other two reduces both to $1 = 1$.

- If $\gamma_p \in L_n \setminus \{0, 1\}$ and $\gamma'_p = 1$, Equation (4.12) reduces to $\gamma_p = \gamma_p$, Equation (4.13) and Equation (4.14) to $1 = 1$.

- If $\gamma_p = 1$ and $\gamma'_p \in L_n \setminus \{0, 1\}$, Equation (4.11) is false since it reduces to $\beta_p \leq \gamma'_p$, that does not hold in the components.

Hence $\alpha$ is order preserving. Let us now show that $\alpha$ is a homomorphism of monoids. First of all $\alpha(1) = [1, 1]$ by definition. Moreover, if $x, x' \in X$, let us prove that $\alpha(x \cdot x') = [\gamma, \beta] \cdot [\gamma', \beta']$. 

128
Claim 4.4.8. \( x \cdot x' = (a \lor -d) \land (-a \lor d) \) for \( a = \gamma \cdot \gamma' \) and \( d = \mathcal{V}_{b(\gamma)v-b(\gamma')}(\beta' \rightarrow \beta) \land \mathcal{V}_{-b(\gamma)v-b(\gamma')}(\beta \rightarrow \beta') \land \mathcal{V}_{-b(\gamma)v-b(\gamma')}(\beta \cdot \beta') \).

Proof. (of Claim 4.4.8). First of all \( x \cdot x' = ((\gamma \lor -\beta) \land (-\gamma \lor \beta)) \cdot ((\gamma' \lor -\beta') \land (-\gamma' \lor \beta')) \). Since \( x \cdot x' \in X \) there are \( a \in \mathcal{M}(A) \) and \( d \in \mathcal{R}(A) \) such that \( x \cdot x' = (a \lor -d) \land (-a \lor d) \). Let us prove that, indeed, \( a = \gamma \land \gamma' \) and \( d = \mathcal{V}_{b(\gamma)v-b(\gamma')}(\beta' \rightarrow \beta) \land \mathcal{V}_{-b(\gamma)v-b(\gamma')}(\beta \rightarrow \beta') \land \mathcal{V}_{-b(\gamma)v-b(\gamma')}(\beta \cdot \beta') \). Clearly \( a \in \mathcal{M}(A) \) and, since \( \mathcal{R}(A) \) is a residuated lattice, it is closed under \( \rightarrow \) and \( \cdot \). Moreover, \( \mathcal{R}(A) \) is a filter of \( A \) and hence \( \mathcal{V}_{-b(a)}(d) \in \mathcal{R}(A) \) for every \( a \in \mathcal{M}(A) \) and \( d \in \mathcal{R}(A) \). Thus, \( d \in \mathcal{R}(A) \). Let us prove the claim componentwise, that is

\[(\gamma_{p} \lor -\beta_{p}) \land (-\gamma_{p} \lor \beta_{p}) \cdot ((\gamma'_{p} \lor -\beta'_{p}) \land (-\gamma'_{p} \lor \beta'_{p})) = (a_{p} \lor -d_{p}) \land (-a_{p} \lor d_{p}) \]  

(4.15)

where, conventionally, \( a_{p} = \gamma_{p} \cdot \gamma'_{p} \) and \( d_{p} \) is defined analogously. We shall denote with (A) the left-hand side of Equation (6.4), and with (B) its right-hand side.

Again we enter a case distinction.

- For the cases in which \( \gamma_{p}, \gamma'_{p} \) are Boolean we refer again to the proof of Theorem 2.5.6.
- If \( \gamma_{p} = 0 \) and \( \gamma'_{p} \in L_{n} \setminus \{0,1\} \), \( a_{p} = 0 \) and \( d_{p} = 1 \). Thus Equation (6.4) becomes \( -\beta_{p} \cdot \gamma'_{p} \), which holds in directly indecomposable components.
- If \( \gamma_{p} \in L_{n} \setminus \{0,1\} \) and \( \gamma'_{p} = 0 \), \( a_{p} = 0 \) and \( d_{p} = 1 \). Thus Equation (6.4) becomes \( \gamma_{p} \cdot -\beta'_{p} = 0 \), which again holds in directly indecomposable components.
- If \( \gamma_{p}, \gamma'_{p} \in L_{n} \setminus \{0,1\} \), \( a_{p} = \gamma_{p} \cdot \gamma'_{p} \) and \( d_{p} = 1 \). Thus, both (A) and (B) reduce to \( \gamma_{p} \cdot \gamma'_{p} \).
- If \( \gamma_{p} \in L_{n} \setminus \{0,1\} \) and \( \gamma'_{p} = 1 \), \( a_{p} = \gamma_{p} \) and \( d_{p} = \beta'_{p} \rightarrow \beta_{p} \). Thus (B) becomes \( (\gamma_{p} \lor -((\beta'_{p} \rightarrow \beta_{p})) \land (-\gamma_{p} \lor (\beta'_{p} \rightarrow \beta_{p})) = \gamma_{p} \land (\beta'_{p} \rightarrow \beta_{p}) = \gamma_{p} \). While (A) reduces to \( \gamma_{p} \cdot \beta'_{p} = \gamma_{p} \).
Thus Claim 4.4.8 is completely proved.

Hence, $\alpha(x \cdot x') = [a, d] = [\gamma \cdot \gamma', \lor_{b(\gamma) \lor b(\gamma')} (\beta' \to \beta) \land \lor_{b(\gamma) \lor b(\gamma')} (\beta \to \beta') \land \lor_{a(\gamma) \lor a(\gamma')} (\beta' \to \beta')]$.

We now have to prove that $\alpha$ preserves the implication. That is to say, if $x, x' \in X$, let us prove that $\alpha(x \to x') = [\gamma, \beta] \to [\gamma', \beta']$.

Claim 4.4.9. $x \to x' = (a \lor d) \land (\lnot a \lor d)$ for $a = \gamma \to \gamma'$ and $d = \lor_{b(\gamma)} (\beta' \to \beta) \land \lor_{b(\gamma)} (\beta \to \beta')$.

Proof. (of Claim 4.4.9). We will follow the same reasoning of Claim 4.4.8. Clearly $x \to x' = ((\gamma \lor \lnot \beta) \land (\lnot \gamma \lor \beta)) \to ((\gamma' \lor \lnot \beta') \land (\lnot \gamma' \lor \beta'))$. Let us prove the claim on the components, that is to say for each $p \in \text{Spec}X$:

$((\gamma_p \lor \lnot \beta_p) \land (\lnot \gamma_p \lor \beta_p)) \to ((\gamma'_p \lor \lnot \beta'_p) \land (\lnot \gamma'_p \lor \beta'_p)) = (a_p \lor d_p) \land (\lnot a_p \lor d_p)$ (4.16)

We shall again denote with (A) the left-hand side of Equation (4.16), and with (B) its right-hand side.

Again we enter a case distinction.

- See the proof of Theorem 2.5.6 for the cases in which $\gamma_p, \gamma'_p$ are Boolean.

- If $\gamma_p = 0$ and $\gamma'_p \in L_n \setminus \{0, 1\}$, $a_p = \gamma'_p$ and $d_p = \beta_p \to \beta'_p$. Thus Equation (4.16) becomes $\lnot \beta_p \to \gamma'_p = 1$, which holds in directly indecomposable components.

- If $\gamma_p \in L_n \setminus \{0, 1\}$ and $\gamma'_p = 0$, $a_p = \lnot \gamma_p$ and $d_p = \beta'_p \to \beta_p$. Thus (B) becomes $(\lnot \gamma_p \lor (\beta'_p \to \beta_p)) \land (\lnot \gamma_p \lor (\beta'_p \to \beta_p)) = \lnot \gamma_p \land (\beta'_p \to \beta_p) = \lnot \gamma_p$. While (A) reduces to $\gamma_p \to \lnot \beta_p$, which equals $\lnot \gamma_p$ as can be seen by the very definition of $\to$.

- If $\gamma_p, \gamma'_p \in L_n \setminus \{0, 1\}$, $a_p = \gamma_p \to \gamma'_p$ and $d_p = 1$. Thus, both (A) and (B) reduce to $\gamma_p \to \gamma'_p$.

130
Chapter 4. Generalized $\delta$-rotations of residuated lattices

- If $\gamma_p \in L_n \setminus \{0,1\}$ and $\gamma_p' = 1$, $a_p = 1$ and $d_p = 1$. Thus (B) becomes 1, while (A) reduces to $\gamma_p \to \beta_p' = 1$, since $\gamma_p \leq \beta_p'$ being $\beta_p'$ in the radical.

- If $\gamma_p = 1$ and $\gamma_p' \in L_n \setminus \{0,1\}$, $a_p = \gamma_p'$ and $d_p = \delta(\beta_p \cdot \beta_p')$. Thus, (B) becomes $(\gamma_p' \lor -\delta(\beta_p \cdot \beta_p')) \land (-\gamma_p' \lor \delta(\beta_p \cdot \beta_p')) = \gamma_p' \land \delta(\beta_p \cdot \beta_p') = \gamma_p'$, and (A) reduces to $\beta_p \to \gamma_p' = \gamma_p'$.

Thus Claim 4.4.9 is completely proved. \hfill \Box

Hence, $\alpha(x \to x') = [a,d] = [\gamma \to \gamma', \nu_{b(\gamma) \lor \Delta_R(\gamma')}(\beta' \to \beta) \land \nu_{-b(\gamma) \lor \Delta_R(\gamma')}(\beta \to \beta')] = [\gamma,\beta] \to [\gamma',\beta']$, thus $\alpha$ is an isomorphism.

Hence, $\hat{\Xi}$ is essentially surjective and the proof is settled. \hfill \Box

Thus, via Lemma 4.4.6 and Lemma 4.4.7 we obtain the following characterization theorem.

**Theorem 4.4.10.** $MVR^n_R$ and $\mathcal{Q}_R^n$ are categorically equivalent, for every subvariety $\mathcal{R}$ of residuated lattices.

Finally, we can reason again as in the first chapter to obtain categorical equivalences between subcategories of $MVR^n_R$-algebras. Let $\mathcal{R}$ be any subclass of $RL$ and let $I(\mathcal{R})$ be the collection of the isomorphic copies, in $RL$, of the elements in $\mathcal{R}$. Let $\Delta_{\mathcal{R}}$ be an indexed family, on $I(\mathcal{R})$, of wdl-admissible operators such that, for every $\mathcal{R} \in I(\mathcal{R})$, $\delta_{\mathcal{R}} : R \to R$. Recall that $\Delta_{\mathcal{R}}$ is $\mathcal{R}$-compatible if for every homomorphism $g : R_1 \to R_2$, with $R_1, R_2 \in I(\mathcal{R})$,

$$g \circ \delta_{R_1} = \delta_{R_2} \circ g$$

Let hence $\mathcal{Q}^n(\Delta_{\mathcal{R}})$ be the subcategory of $\mathcal{Q}^n_R$ whose objects are quadruples of the form $(M, R, \phi, \delta_R)$ where $R \in I(\mathcal{R})$ and $\delta_R \in \Delta_R$ and morphisms are pairs

$$(f, g) : (M_1, R_1, \nu_{e_1}, \delta_{R_1}) \to (M_2, R_2, \nu_{e_2}, \delta_{R_2})$$

which are morphism pairs of $\mathcal{Q}^n_{RL}$.

**Proposition 4.4.11.** Let $\mathcal{R}$ be any subclass of $RL$ and let $\Delta_{\mathcal{R}} = \{\delta_{\mathcal{R}} \mid R \in I(\mathcal{R})\}$ and $\Delta'_{\mathcal{R}} = \{\delta'_{\mathcal{R}} \mid R \in I(\mathcal{R})\}$ be two indexed families of wdl-admissible operators on $I(\mathcal{R})$. The map $\hat{\Psi} : \mathcal{Q}^n(\Delta_{\mathcal{R}}) \to \mathcal{Q}^n(\Delta'_{\mathcal{R}})$ that on objects is defined as:

$$\hat{\Psi}(M, R, \nu_e, \delta_R) = (M, R, \nu_e, \delta'_R)$$

131
Chapter 4. Generalized \(\delta\)-rotations of residuated lattices

and it is the identity on morphisms, is an equivalence of the categories \(Q^n(\Delta_R)\) and \(Q^n(\Delta'_R)\) if \(\Delta_R\) and \(\Delta'_R\) are \(R\)-compatible.

\[\text{Proof.}\] The proof is completely analogous to the one of Proposition 2.5.9. \(\square\)

\[\text{Proposition 4.4.12.}\] Let \(R\) be any subclass of \(RL\). Let us consider \(\Delta^L_R = \{\delta^R_L \mid R \in I(R)\}\) and \(\Delta^D_R = \{\delta^D_R \mid R \in I(R)\}\) where \(\delta^R_L\) and \(\delta^R_D\) denote respectively the maps \(\delta_L\) and \(\delta_D\) over \(R\). Then \(Q^n(\Delta^L_R)\) and \(Q^n(\Delta^D_R)\) are equivalent categories.

\[\text{Proof.}\] Immediate, by Lemma 2.4.3 and Proposition 4.4.11. \(\square\)

\[\text{Remark 4.4.13.}\] Notice that each category of the kind \(Q^n(\Delta^D_R)\), \(Q^n(\Delta^L_R)\), is clearly equivalent to the category whose objects are the triples obtained by the quadruples in which the \(\delta\) is omitted, and the morphisms are the same(by Lemma 2.4.3).

Let now \(SMVR_{n,R}\) and \(IMVR_{n,R}\) be the full subcategories of \(SMVR_n\) and \(IMVR_n\) respectively, where the algebras have radical in \(R\). We can restrict functors \(\hat{\Phi}\) and \(\hat{\Xi}\) to respectively the full subcategories \(SMVR_{n,R}\) and \(Q^n(\Delta^L_R)\) first, and to \(IMVR_{n,R}\) and \(Q^n(\Delta^D_R)\) secondly, and easily adapt the proofs of Lemma 4.4.6 and Lemma 4.4.7, in the lines of Theorem 2.4.7, to obtain the following.

\[\text{Theorem 4.4.14.}\] For every \(n \geq 2\), it holds:

1. \(SMVR_{n,R}\) is categorically equivalent to \(Q^n(\Delta^L_R)\).
2. \(IMVR_{n,R}\) is categorically equivalent to \(Q^n(\Delta^D_R)\).

\[\text{Corollary 4.4.15.}\] For every \(n \geq 2\), \(SMVR_{n,R}\) is categorically equivalent to \(IMVR_{n,R}\). In particular, the following categories are equivalent:

1. For \(n=2\), all instances of Corollary 2.4.10.
2. \(NM\) and \(Q^3(\Delta^D_{GH})\).
3. The category of regular Nelson lattices and \(Q^3(\Delta^D_{GHA})\), where \(GHA\) is the variety of generalized Heyting algebras.
4. \( BL_n \) and \( Q^n(\Delta^L_R) \).

Notice that in Figure 4.1 the sublattice between \( rDL \) and \( BRL \) is isomorphic to the lattice of the varieties \( MV_n \), for \( n \geq 2 \).
Part II

Theory of states
Chapter 5

States of residuated structures

We are now going to deal with the theory of states, that is, a generalization of probability theory to the many-valued setting. In the context of t-norm based fuzzy logics as developed by Petr Hájek [74], states were first introduced by Daniele Mundici in 1995 [94], with the aim of capturing the notion of average degree of truth of a proposition in Lukasiewicz logic. We will approach the problem of defining a notion of state for many-valued structures in two different ways. For the case of product algebras, we will use the functional representation of the free $n$-generated algebra, in order to axiomatize states that actually characterize Lebesgue integral on such functions. As a second approach, we will take inspiration by the insights given by Mundici’s work, and see how the categorical equivalences shown in the first part of this thesis suggest that a notion of state can be suitably defined for some classes of rDL-algebras, in a way that will be made more precise later on.

5.1 Preliminaries on states: a probability theory for many-valued logics

The use of the term “state” can be traced back to quantum mechanics, where they correspond to functionals meant to measure the expected value for observables in a physical system, we refer to [71] for a more detailed exposure. Inside the theory of ordered structures, states of $\ell$-groups were first defined and studied by Goodearl in [71], in 1986, as a useful tool to investigate the
States of residuated structures

structure of the group, in particular studying the topological state space as its dual object.

States of ℓ-groups

We recall that an ℓ-group is an algebra \((G, +, -, \lor, \land, 0)\) where \((G, +, -)\) is an abelian group, \((G, \lor, \land)\) is a lattice, and the equation \(x + (y \lor z) = (x + y) \lor (x + z)\) holds. Let \(G^+ = \{ x \in G : x \geq 0 \}\) and \(G^- = \{ x \in G : x \leq 0 \}\).

A unital ℓ-group \((G, u)\) is an ℓ-group \(G\) with a strong order unit \(u\), such that for every \(x \in G\), there exists \(n \in \mathbb{N}\) such that \(nu \geq x\) (where \(nu = u + \ldots + u\), \(n\) times).

Definition 5.1.1. A state of an unital ℓ-group \((G, u)\) is a group homomorphism \(s : G \to \mathbb{R}\) such that \(s(a) \geq 0\), for every \(a \in G^+\), and \(s(u) = 1\).

The state space \(S(G, u)\) of \((G, u)\) is the set of all states of the unital ℓ-group. \(S(G, u)\) can be viewed as a subset of the real vector space \(\mathbb{R}^G\) of all real-valued functions on \(G\). \(\mathbb{R}^G\) can be equipped with the product topology, so that it becomes a linear topological space. We shall hence assume that \(S(G, u)\) is endowed with the relative topology from \(\mathbb{R}^G\). Let us briefly recall some needed basic notions of convex sets. Given any real linear space \(E\), a convex set in \(E\) is any subset \(K\) of \(E\) that is closed under convex combinations: if \(x_1, \ldots, x_n \in K\) and \(\alpha_i \geq 0\) with \(\sum_{i=1}^{n} \alpha_i = 1\), then \(\alpha_1 x_1 + \ldots + \alpha_n x_n \in K\). Given any set \(X \subseteq E\), the convex hull of \(X\) is the set \(\text{co}(X)\) of all convex combinations of elements in \(X\). Let us denote with \(\overline{\text{co}}(X)\) the topological closure of \(\text{co}(X)\). An extreme point of a convex set \(K\) is a point \(e \in K\) such that the set \(K \setminus \{ e \}\) remains convex, or, equivalently, it is such that whenever \(e\) can be expressed as a convex combination \(e = \alpha_1 x_1 + \alpha_2 x_2\), then it must be \(\alpha_1 = 1\) or \(\alpha_2 = 1\) or \(x_1 = x_2 = e\). Let us denote with \(\text{ext}(K)\) the set of extreme points of \(K\). Intuitively, an extreme point does not lie in the interior of any line segment with endpoints in \(K\).

The following theorem is a well-known characterization of compact convex sets by their extreme points.

136
Chapter 5. States of residuated structures

Figure 5.1: Example of a convex set with extreme points $e_1, e_2, e_3$.

**Theorem 5.1.2** (Krein-Milman). If $K$ is a compact convex subset of a locally convex Hausdorff space $E$, then $K = \overline{\text{co}}(\text{ext}(K))$.

Goodearl proves that Krein-Milman theorem can be applied to unital $\ell$-groups, indeed the following holds.

**Proposition 5.1.3** ([71], Proposition 6.2). Let $(G, u)$ be an $\ell$-group, then $R^G$ is a locally convex Hausdorff space, and $S(G, u)$ is a compact convex subset of $R^G$.

Thus, $S(G, u)$ coincides with the closure of the convex hull of what we shall call *extremal* states.

**States of MV-algebras**

We will now see states of MV-algebras, as defined by Mundici in [94], and how they relate to states of $\ell$-groups via Mundici’s well-known categorical equivalence. We will follow [60].

**Definition 5.1.4.** Let $A = (A, \wedge, \vee, \cdot, \to, 0, 1)$, be an MV-algebra, where we set as usual $x \oplus y = \neg(\neg x \cdot \neg y)$. A mapping $s : A \to [0, 1]$ is a state of $A$ whenever $s(1) = 1$ and for every $a, b \in A$ the following condition is satisfied:

$$
\text{if } a \cdot b = 0, \text{ then } s(a \oplus b) = s(a) + s(b). \quad (5.1)
$$

Condition 5.1 clearly means additivity with respect to Lukasiewicz sum $\oplus$. Indeed, the requirement $a \cdot b = 0$ is analogous to disjointness of a pair of elements in a Boolean algebra. Thus states can be thought of as generalizations of
Chapter 5. States of residuated structures

finitely additive probabilities: every finitely additive probability on a Boolean algebra is a state as a special case of the above definition. Moreover, every Borel probability measure is a state as well.

**Example 5.1.5** (Finitely-additive probabilities). Any Boolean algebra $B$ is an MV-algebra in which the MV-operations $\oplus$ and $\cdot$ coincide with the lattice operations $\lor$ and $\land$, respectively. Every state $s$ of $B$ is a finitely additive probability since the condition 5.1 reads as follows:

$$if \ a \land b = 0, \ then \ s(a \lor b) = s(a) + s(b).$$

**Example 5.1.6** (Homomorphisms). Every homomorphism $h$ of an MV-algebra $A$ into the standard MV-algebra $[0,1]_L$ is a state of $A$. In particular, whenever $A$ is a subalgebra of the MV-algebra $[0,1]^X$ of all functions $X \to [0,1]$. For any $x \in X$ the evaluation mapping $s_x : A \to [0,1]$ given by

$$s_x(f) = f(x), \ f \in A,$$

is a state of $A$.

Mundici’s well-known categorical equivalence between MV-algebras and unital $\ell$-groups extends to the framework of states of MV-algebras and states of $\ell$-groups, indeed states of corresponding structures result to be in bijection. Let us recall that if $(G,1)$ is an abelian unital $\ell$-group, then the order interval $\Gamma(G,1) = \{a \in G : 0 \leq a \leq 1\}$ becomes an MV-algebra with the induced operations $a \oplus b = (a + b) \land 1$, and $\neg a = 1 - a$, from which we can recover $\cdot$ and $\to$ in the following way: $a \cdot b = -(\neg a \oplus -b)$ and $a \to b = \neg a \oplus b$. The group operation $+$ of $(G,1)$ and the MV-algebraic operations of $\Gamma(G,1)$ are related as follows:

$$a + b = (a \oplus b) + (a \cdot b), \ for \ every \ a,b \in \Gamma(G,1).$$

For the inverse construction, that given an MV-algebra $A$ constructs the $\ell$-group $(G_A,1)$ such that $A$ is isomorphic to $\Gamma(G_A,1)$, see [93], where it is proved that $\Gamma$ indeed provides a categorical equivalence between the algebraic category of MV-algebras and that of unital $\ell$-groups with unit-preserving homomorphisms. Moreover, Mundici shows that $\Gamma$ also preserves states of corresponding structures.
Theorem 5.1.7 ([94], Theorem 2.4). Let \((G, 1)\) be a unital \(\ell\)-group, and let \(A = \Gamma(G, 1)\). Then for every state \(s\) of \((G, 1)\):

1. the restriction \(s'\) of \(s\) to \(A\) is a state on \(A\).

2. The map \(s \mapsto s'\) is a one-one correspondence between states of \((G, 1)\) and states of \(A\). Under this correspondence, the extremal states of \((G, 1)\) are mapped onto the extremal states of \(A\).

In analogy with the case of \(\ell\)-groups, the state space \(S(A)\) of an MV-algebra \(A\) can be viewed as a subspace of the locally convex space \(R^A\) endowed with product topology. Moreover, \(S(A)\) is a compact convex subset of \(R^A\) ([60], Proposition 3.17), thus again the state space is generated by extremal states, whose characterization is henceforth fundamental. Mundici proves that extremal states of an MV-algebra form a compact Hausdorff space that is homeomorphic to the spectrum of maximal ideals of the MV-algebra, endowed with spectral topology ([94], Theorem 2.5). Moreover, it can be proved that extremal states of an MV-algebra \(A\) coincide exactly with the homomorphisms from \(A\) to \([0, 1]_L\) ([60], Theorem 4.11). Notice that when we apply this result to the free \(n\)-generated MV-algebra, such homomorphisms are in 1-1 correspondence with the \([0, 1]\)-valuations of Lukasiewicz logic with \(n\) variables.

One of the most important results in the study of MV-algebraic states is certainly the so-called Kroupa-Panti theorem ([95], §10), a representation result showing that every state of an MV-algebra \(A\) is the Lebesgue integral with respect to a regular Borel probability measure over the maximal ideal space of \(A\). Moreover, the correspondence between states and regular Borel probability measures is one-one and turns out to have very strong geometrical and topological properties, establishing in particular an affine homeomorphism between \(S(A)\) and the space of measure (for more details, see for instance [60], §4). In order to be more precise, let \(A\) be an MV-algebra, then let \(M(Max(A))\) be the Bauer simplex of all regular Borel probability measures over \(Max(A)\), equipped with the weak* topology (see [60], §2.4 for more details). We shall mention that if an MV-algebra \(A\) is semisimple, then it is isomorphic to a separating MV-algebra \(A^*\) of \([0,1]\)-valued continuous
functions defined on the compact Hausdorff space \( \text{Max}(A) \) (see [36, Theorem 5.4.7]). Most importantly, given \( A \) any MV-algebra, its state space \( S(A) \) is affinely homeomorphic to the state space of \( A/\text{Rad}(A) \), which is semisimple. Hence, the study of MV-states can be reduced to the study of states of semisimple MV-algebras. The following theorem was independently proved by Kroupa [85][86], and Panti [100].

**Theorem 5.1.8.** Let \( A \) be an MV-algebra and \( M(\text{Max}(A)) \) be the set of all regular Borel probability measures on \( \text{Max}(A) \). Then there is an affine homeomorphism \( \Psi : S(A) \to M(\text{Max}(A)) \) such that, for every \( a \in A \),

\[
s(a) = \int_{\text{Max}(A)} a^*(M) \, d\mu_s(M),
\]

where \( \mu_s = \Psi(s) \).

**States of perfect MV-algebras**

Every MV-algebra admits at least one state. However, there are relevant examples of MV-algebras whose unique state is trivial, i.e., it only takes Boolean values, 0 and 1. This is the case, for instance, of perfect MV-algebras. Perfect MV-algebras, as we have seen, are disconnected rotations of cancellative hoops and the variety \( \text{DLMV} \) they generate is equivalent to the category whose objects are triplets made of a Boolean algebra, a cancellative hoop and an external join (Theorem 2.4.7). They can also be seen, up to isomorphisms, as those MV-algebras of the form \( \Gamma(Z \times G, (1,0)) \) where \( \Gamma \) is Mundici's functor, \( Z \) is the \( \ell \)-group of integers, \( G \) is any \( \ell \)-group, \( \times \) denotes the lexicographic product between \( \ell \)-groups (which is an \( \ell \)-group iff the first component is totally ordered [70, Example 3]), and \( (1,0) \in Z \times G \) is indeed a strong unit for \( Z \times G \). This construction lifts to a categorical equivalences shown by Di Nola and Lettieri between perfect MV-algebras and \( \ell \)-groups [51].

Summing up, as cancellative hoops coincide with negative cones of \( \ell \)-groups, every perfect MV-algebra has for domain the disjoint union \( G^+ \cup G^- \) for a unique \( \ell \)-group \( G \), where \( G^+ \) denotes the positive cone of \( G \), \( G^- = \{-x \mid x \in G^+\} \) and \( x > y \) for every \( x \in G^- \) and \( y \in G^+ \). Furthermore, in every perfect MV-algebra \( A \), displayed as above, \( x \oplus x = 1 \) for every \( x \in G^- \) and \( y \odot y = 0 \) for
Chapter 5. States of residuated structures

all $y \in G^+$. Therefore, every state $s$ of $A$ maps $G^-$ in 1 and $G^+$ in 0, whence $s$ is also unique.

In order to overcome this limitation, and noticing that $\ell$-groups have more than one trivial state, in [48] the authors introduced the notion of *lexicographic states* for a wide class of MV-algebras which includes perfect MV-algebras. For any algebra $A$ in the variety generated by perfect MV-algebras, a lexicographic state is any map of $A$ to the MV-algebra $\mathcal{L}(R) = \Gamma(R \times R, (1, 0))$ satisfying $s(1) = 1$, $s(x \oplus y) = s(x) + s(y)$ whenever $x \odot y = 0$ and such that the restriction of $s$ to its maximal semisimple quotient, is a state in its usual sense. [48, Corollary 6.7] shows that the class of lexicographic states of a perfect MV-algebra $\Gamma(Z \times G, (1, 0))$ is in one-one correspondence with the class of states of $G$. As we shall see in Section 7.2 the previous definition is a particular case of a more general construction that we will exhibit later on.

**States in other residuated structures**

Many attempts of defining suitable notions of state in different structures have been made (see for instance [60], §8 for a short survey). We shall mention the work in [36] (see also [37], [96]), were the authors study states and probability measures on Lukasiewicz tribes ($\sigma$-complete MV-algebras of fuzzy sets) as well as on other $t$-norm based tribes with continuous operations.

In [8], the authors provide a definition of state for the Lindenbaum algebra of Gödel logic that corresponds to the integral of the n-place truth-functions corresponding to Gödel formulas, with respect to Borel probability measures on the real unit cube $[0, 1]^n$.

**Theorem 5.1.9.** [8, Theorem 4.3]

1. If a map $s$ from the free $n$-generated Gödel algebra $\mathcal{F}_G(n)$ to $[0, 1]$ is a finitely additive probability measure, there exists a Borel probability measure $\mu$ on $[0, 1]^n$ such that:

$$\int_{[0,1]^n} f d\mu = s(f) \quad (A)$$

for every $f \in \mathcal{F}_G(n)$. 

141
2. For any Borel probability measure $\mu$ on $[0,1]^n$, the function $s : \mathcal{F}_G(n) \to [0,1]$ defined by (A) is a finitely additive probability measure.

Moreover, such states are shown to correspond to convex combinations of finitely many truth-value assignments. Subsequently, Aguzzoli and Gerla define a notion of state for the free $n$-generated NM$^{-}$-algebra, and again prove an integral representation theorem with respect to Borel probability measures.

In next chapter, we will see how it is possible to axiomatize a notion of state for Product logic.
Chapter 6

States of the free \( n \)-generated product algebra

In this chapter we will introduce and study states for product logic, the remaining fundamental fuzzy logic for which such a notion is still lacking, and we will prove that our axiomatization results in characterizing Lebesgue integrals of truth-functions of product logic formulas with respect to regular Borel probability measures. More precisely, our approach will focus on the Lindenbaum algebra of product logic over \( n \) variables, i.e. the free \( n \)-generated product algebra, and show that its states correspond, one-one, to regular Borel probability measures on \([0,1]^n\). Moreover, and quite surprisingly since in the axiomatization of states the product t-norm operation will only be indirectly involved via a condition concerning double negation, we prove that every state belongs to the convex closure of product logic valuations.

6.1 Product algebras and product functions

In this section we will recall the functional representation of the free \( n \)-generated product algebra \( \mathcal{F}_P(n) \), as presented in [4] (see also [45]). We will easily prove that such functions, although they are not continuous, are indeed Borel measurable. In particular, from that functional representation of product logic functions, it follows that the domain \([0,1]^n\) of each such a function can be partitioned in locally compact and Hausdorff subsets of \([0,1]^n\), named \( G_\varepsilon \) (with \( \varepsilon \) varying in a certain set \( \Sigma \), depending on the atoms of the boolean
Chapter 6. States of the free n-generated product algebra

skeleton of $\mathcal{F}_P(n)$. More precisely, each $G_\epsilon$ is a so-called $F_\sigma$ set (see [23, §6.1]) since, in fact, it is a countable union of a family $\{G^q_\epsilon\}_{q(0,1)\subseteq \mathbb{Q}}$ of nested closed subsets of $[0,1]^n$, and hence it is a $\sigma$-locally compact set (see [106, §1.11]). Over each $G_\epsilon$, the function is actually continuous. Moreover, any continuous function with domain one of the compact sets $G^q_\epsilon$ can be uniformly approximated by linear combinations of the functions of $\mathcal{F}_P(n)$ restricted to such subsets.

For every $n \in \mathbb{N}$, let $\mathcal{F}_P(n)$ be the free product algebra over $n$ free generators. Recall that $[0,1]_\Pi = ([0,1],\cdot_\Pi,\rightarrow_\Pi,\min,\max,0,1)$ (where $\cdot_\Pi$ is the product between reals) is generic for $P$, thus $\mathcal{F}_P(n)$ is isomorphic to the product subalgebra of $[0,1]^{[0,1]^n}$ generated by the projection maps (see Theorem 1.1.6 and [4]). Thus, every element of $\mathcal{F}_P(n)$ can be regarded as a function $[0,1]^n \rightarrow [0,1]$ that we shall call a product function.

**Remark 6.1.1.** It is known that $\mathcal{B}(\mathcal{F}_P(n))$, the Boolean skeleton of $\mathcal{F}_P(n)$, coincides with the free Boolean algebra over $n$ generators. In particular, $\mathcal{B}(\mathcal{F}_P(n))$ is finite and hence atomic.

Now, we can safely identify the set of atoms of $\mathcal{B}(\mathcal{F}_P(n))$, say $at(n)$, with the set $\Sigma = \{1,2\}^n$ of strings $\epsilon = (\epsilon_1,\ldots,\epsilon_n)$ of length $n$ over the binary set $\{1,2\}$ and adopt the same notation of [4] without danger of confusion:

$$at(n) = \{p_\epsilon \in \mathcal{F}_B(n) \mid \epsilon \in \Sigma\},$$

that is to say, for every $\epsilon = (\epsilon_1,\ldots,\epsilon_n) \in \Sigma$, $p_\epsilon = \land_{i=1}^n \neg^{\epsilon_i} x_i$, where $x_1,\ldots,x_n$ are the free generators of $\mathcal{F}_P(n)$, while $\neg^1 = \neg$ and $\neg^2 = \neg\neg$. Thus, for every $\epsilon = (\epsilon_1,\ldots,\epsilon_n)$, we define

$G_\epsilon = \{(t_1,\ldots,t_n) \in [0,1]^n \mid t_i > 0 \text{ if } \epsilon_i = 2 \text{ and } t_i = 0 \text{ if } \epsilon_i = 1\}$.

The set $\{G_\epsilon \mid \epsilon \in \Sigma\}$ is then a partition of $[0,1]^n$ (cf. [4]).

**Example 6.1.2.** For instance, for $n = 2$, we have $p_{(1,1)} = \neg x_1 \land \neg x_2$, $p_{(1,2)} = \neg x_1 \land \neg\neg x_2$, $p_{(2,1)} = \neg\neg x_1 \land \neg x_2$, $p_{(2,2)} = \neg\neg x_1 \land \neg\neg x_2$, while the following Figure 6.1 shows how $[0,1]^2$ is partitioned by $G_{(1,1)}$, $G_{(1,2)}$, $G_{(2,1)}$ and $G_{(2,2)}$. 

144
Chapter 6. States of the free \( n \)-generated product algebra

\[
\begin{array}{c}
\text{Figure 6.1: The partition of } [0,1]^2 \text{ into } G_{(1,1)}, G_{(1,2)}, G_{(2,1)} \text{ and } G_{(2,2)}. \\
\end{array}
\]

**Notation 6.1.3.** In what follows, for every function \( f : [0,1]^n \rightarrow [0,1] \) and for every \( \epsilon \in \Sigma \), we will denote by \( f_{\epsilon} \) the restriction of \( f \) to \( G_{\epsilon} \), i.e. \( f_{\epsilon} = f|_{G_{\epsilon}} \).

**Definition 6.1.4.** Let \( n \in \mathbb{N} \) and \( \mathcal{P}(n) \) be the set of functions \( f : [0,1]^n \rightarrow [0,1] \) such that, for every \( \epsilon \in \Sigma \), either \( f_{\epsilon} = 0 \) or, if \( f_{\epsilon} > 0 \) pointwise, it is continuous and piecewise monomial. The pointwise application of the operations \( \cdot, \rightarrow, \wedge \) and \( \lor \), together with the functions constantly 0 and 1 make \( \mathcal{P}(n) \) into a product algebra that we still denote by \( \mathcal{P}(n) \) without danger of confusion.

The functional representation theorem for free, finitely generated, product algebras then reads as follows:

**Theorem 6.1.5 ([4]).** For every \( n \in \mathbb{N} \), \( \mathcal{F}(n) \) is isomorphic to \( \mathcal{P}(n) \). Thus a function \( f : [0,1]^n \rightarrow [0,1] \) is a product function iff \( f \) is such that, for every \( \epsilon \in \Sigma \), either \( f_{\epsilon} = 0 \) or \( f_{\epsilon} > 0 \) and it is continuous and piecewise monomial.

In the rest of this section we shall provide preparatory results about the sets \( G_{\epsilon} \).

**Lemma 6.1.6.** For every \( \epsilon \in \Sigma \), \( G_{\epsilon} \) is a Borel subset of \([0,1]^n\), locally compact and Hausdorff.

*Proof.* First of all, by definition,

\[
G_{\epsilon} = \prod_{i=1}^{n} A_i,
\]

145
where $A_i = (0, 1]$ if $\epsilon_i = 2$ and $A_i = \{0\}$ if $\epsilon_i = 1$. Hence, $G_\epsilon$ can also be expressed as the following countable union of closed sets in the product topology of $[0, 1]^n$,

$$G_\epsilon = \bigcup_{q \in Q \cap (0,1]} \prod_{i=1}^n B_i^q,$$

(6.1)

where $B_i^q = [q, 1]$ if $\epsilon_i = 2$ and $B_i^q = \{0\}$ if $\epsilon_i = 1$. Therefore $G_\epsilon$ is a Borel subset of $[0, 1]^n$. It also easily follows that each $G_\epsilon$ is locally compact and Hausdorff.

In the proof of Lemma 6.1.6 above, we showed that every $G_\epsilon$ is a countable union of compact subsets of $[0, 1]^n$, through (6.1). For the sake of a later use and a lighter notation, let us introduce the following.

**Notation 6.1.7.** For every $\epsilon \in \Sigma$ and for every $q \in (0, 1] \cap Q$,

$$G_\epsilon^q = \prod_{i=1}^n B_i^q.$$

**Remark 6.1.8.** For every $\epsilon \in \Sigma$, the set $\{G_\epsilon^q \mid q \in (0, 1] \cap Q\}$ is a countable family of compact subsets of $G_\epsilon$ and

$$G_\epsilon = \bigcup_{q \in (0,1] \cap Q} G_\epsilon^q.$$

Therefore, each $G_\epsilon$ is $\sigma$-locally compact and Hausdorff (see [115] for further details about $\sigma$-compact spaces). Moreover, for each $q_1, q_2 \in (0, 1] \cap Q$, $G_\epsilon^{q_1} \subset G_\epsilon^{q_2}$ iff $q_1 > q_2$.

Given the previous result, we can easily prove what follows.

**Theorem 6.1.9.** Every $f \in \mathcal{P}(n)$ is measurable.

**Proof.** We can write each $f \in \mathcal{P}(n)$ as $f = \bigvee_\epsilon (f \wedge p_\epsilon)$, where the restriction of each $f \wedge p_\epsilon$ to $G_\epsilon$ is either 0 or is a piecewise monomial function. By Lemma 6.1.6, each $G_\epsilon$ is a Borel set. Thus, each $f \wedge p_\epsilon$ is continuous on a Borel set, and 0 outside, hence it is Borel measurable. The supremum of measurable functions is measurable, thus the claim follows.

**Notation 6.1.10.** For every $n \in \mathbb{N}$ and for every $\epsilon \in \Sigma$, let:
In other words, \( \mathcal{P}_e(n) \) is obtained by restricting each function of \( \mathcal{P}(n) \) to \( G_e \), while \( \mathcal{L}_e(n) \) is in fact the linear span of \( \mathcal{P}_e(n) \) with nonzero coefficients.

**Proposition 6.1.11.** For every \( \epsilon \in \Sigma \) and every \( g_\epsilon \in \mathcal{L}_e(n) \), either \( g_\epsilon = 0 \) or \( g_\epsilon \) is a piecewise polynomial function. Moreover, in the latter case, \( g_\epsilon \) is represented in a unique way as a linear combination of pairwise distinct \( f_{1,\epsilon}, \ldots, f_{k,\epsilon} \in \mathcal{P}_e(n) \) with non-zero coefficients.

**Proof.** Assume \( g_\epsilon \neq 0 \). Then there is a \( k \in \mathbb{N} \), \( f_{1,\epsilon}, \ldots, f_{k,\epsilon} \in \mathcal{P}_e(n) \) and \( \lambda_1, \ldots, \lambda_k \in \mathbb{R} \setminus \{0\} \) such that \( g_\epsilon = \sum_{i=1}^k \lambda_i f_{i,\epsilon} \). Each \( f_{i,\epsilon} \) is piecewise monomial, meaning that there is a partition \( \mathcal{P}_i = \{ P_{1,i}, \ldots, P_{m,i} \} \) of \( G_e \) such that the restricted function \( f_{i,\epsilon} \restriction_{P_{j,i}} \) is monomial. Let \( \{ Q_1, \ldots, Q_m \} \) be the refined partition of \( G_e \) obtained by taking all possible non-void intersections of elements in the \( \mathcal{P}_i \)'s. Obviously, \( f_{i,\epsilon} \restriction_{Q_j} \) is monomial for all \( Q_j \). Moreover,

\[
g_\epsilon \restriction_{Q_j} = \left( \sum_i \lambda_i f_{i,\epsilon} \right) \restriction_{Q_j} = \sum_i \lambda_i (f_{i,\epsilon} \restriction_{Q_j}),
\]

whence \( g_\epsilon \restriction_{Q_j} \) is polynomial. Thus, the claim follows since a finite intersection of semialgebraic sets is semialgebraic, where a semialgebraic set is a set defined by Boolean combination of equalities and inequalities of real polynomials, and hence each \( Q_j \) is semialgebraic. This shows that \( g_\epsilon \) is piecewise polynomial. As to prove that \( g_\epsilon \) is uniquely determined, assume by way of contradiction that there are two different sets \( \{ f_{1,\epsilon}, \ldots, f_{k,\epsilon} \} \) and \( \{ f'_{1,\epsilon}, \ldots, f'_{k',\epsilon} \} \) such that \( g_\epsilon = \sum_i \lambda_i f_{i,\epsilon} = \sum_j \lambda'_j f'_{j,\epsilon} \). Let \( P_1, \ldots, P_m \) and \( P'_1, \ldots, P'_{m'} \) be the semialgebraic sets on which, respectively, the \( f_{i,\epsilon} \)'s and the \( f'_{j,\epsilon} \)'s are monomial. Let \( Y \) be a semialgebraic set contained in \( P_i \cap P'_j \) for some \( i, j \). Thus \( g_\epsilon \) restricted to \( Y \) is polynomial, but \( g_\epsilon \) is not uniquely displayed on \( Y \), since \( g_\epsilon \restriction_Y = \sum_i \lambda_i f_{i,\epsilon} \restriction_Y = \sum_j \lambda'_j f'_{j,\epsilon} \restriction_Y \). But this is contradictory, since polynomial functions have a unique representation on semialgebraic sets. Thus, on each intersection \( P_i \cap P'_j \), \( g_\epsilon \) has a unique representation, and thus it is uniquely determined. \( \square \)
For every $n \in \mathbb{N}$, every $\epsilon \in \Sigma$ and every $q \in (0,1] \cap \mathbb{Q}$, we denote by $\mathcal{L}_\epsilon^q(n)$ the set of functions obtained by restricting those in $\mathcal{L}_\epsilon(n)$ to $G^q_\epsilon$. Further, for every subset $X$ of $[0,1]^n$, we denote by $\mathcal{C}(X)$ the set of all continuous functions from $X$ to $\mathbb{R}$.

**Proposition 6.1.12.** For every $\epsilon$, for every $q \in (0,1] \cap \mathbb{Q}$ and for every $c \in \mathcal{C}(G^q_\epsilon)$, there is a sequence $g_1, g_2, \ldots \in \mathcal{L}_\epsilon^q(n)$ such that $g_i \leq c$ for every $i$, and \( \{g_i\} \) uniformly converges to $c$.

**Proof.** Since every $c$ is continuous and defined on a compact set, we can get the claim by the Stone-Weierstrass Theorem [52, §VIII] if we show that $\mathcal{L}_\epsilon^q(n)$ is a subalgebra of $\mathcal{C}(G^q_\epsilon)$, i.e., it is a vector subspace of $\mathcal{C}(G^q_\epsilon)$ that is closed under multiplication of functions, $\mathcal{L}_\epsilon^q(n)$ contains a non-zero constant function and it separates the points. The first two claims are trivial. Thus, let us show that $\mathcal{L}_\epsilon^q(n)$ separates the points, i.e., for every $x, y \in G^q_\epsilon$, if $x \neq y$, then $g(x) \neq g(y)$ for some $g \in \mathcal{L}_\epsilon^q(n)$. Indeed, each monomial function $m$ defined on a subset of $[0,1]^n$ is strictly increasing and hence $m(x) \neq m(y)$ if $x \neq y$, whence the claim is settled. 

## 6.2 States of free product algebras

Now we will axiomatize our notion of state of $\mathcal{F}_P(n)$, and show its properties together with some examples. In particular, we will investigate states of the 1-generated free product algebra, and see how this analysis reflects into its spectral space.

**Definition 6.2.1.** A state of $\mathcal{F}_P(n)$ is a map $s: \mathcal{F}_P(n) \to [0,1]$ satisfying the following conditions:

S1. $s(1) = 1$ and $s(0) = 0$,

S2. $s(f \land g) + s(f \lor g) = s(f) + s(g)$,

S3. If $f \leq g$, then $s(f) \leq s(g)$,

S4. If $f \neq 0$, then $s(f) = 0$ implies $s(\neg\neg f) = 0$. 

148
Chapter 6. States of the free $n$-generated product algebra

The following proposition shows some basic facts about states of free product algebras. Their proofs are straightforward and hence omitted.

**Proposition 6.2.2.** For any state $s : \mathcal{F}_P(n) \to [0,1]$ the following hold:

(i) $s$ restricted to $\mathcal{B}(\mathcal{F}_P(n))$ is a finitely additive probability measure;

(ii) if $f \land g = 0$, $s(f \lor g) = s(f) + s(g)$. Thus, $s(f \lor \neg f) = s(f) + s(\neg f)$;

(iii) if $f \lor g = 1$, $s(f \land g) = s(f) + s(g) - 1$. Thus, $s(f \leftrightarrow g) = s(f \rightarrow g) + s(g \rightarrow f) - 1$;

(iv) $s(\neg f) + s(\neg \neg f) = 1$.

**Remark 6.2.3.** It is worth pointing out that states of a free product algebra are lattice valuations (axioms S1–S3) as introduced by Birkhoff in [15]. However, if we compare Definition 6.2.1 with the axiomatization of states of an MV-algebra [94, 60], it is clear that, while for the MV-case the monoidal operations are directly involved in the axiomatization of states, in our case the unique axiom that involves the multiplicative connectives of product logic is S4.

In the following Proposition 6.2.4 we will prove that S4 can be equivalently substituted by the condition

(S4′) For every $\epsilon \in \Sigma$ and $f \in \mathcal{F}_P(n)$, if $f \land p_\epsilon \neq 0$, then $s(f \land p_\epsilon) = 0$ implies $s(p_\epsilon) = 0$,

which involves the atoms of $\mathcal{B}(\mathcal{F}_P(n))$ and does not make use of the negation connective $\neg$. It is also worth noticing that the condition (S4′) quite closely resembles the condition (C4) of [7] where the authors axiomatized the integral on functions of free Gödel algebras $\mathcal{F}_G(n)$. To be more precise, the condition (C4) (see [7, §2.2]) says the following: for every $x, y, z \in \mathcal{F}_G(n)$ which are either join-irreducible or 0, if $x < y < z$ and $s(x) = s(y)$, then $s(y) = s(z)$. Turning back to (S4′), if we take $0 < f \land p_\epsilon < p_\epsilon$, then we get something similar to (C4). Indeed, if $0 = s(0) = s(f \land p_\epsilon)$, we have that $s(p_\epsilon) = 0$ as well, whence $s(f \land p_\epsilon) = s(p_\epsilon)$.
Proposition 6.2.4. The axiom $S_4$ is equivalent to $S_4'$. For every $\epsilon \in \Sigma$, if $f \land p_\epsilon \neq 0$, then $s(f \land p_\epsilon) = 0$ implies $s(p_\epsilon) = 0$.

Proof. ($S_4 \Rightarrow S_4'$). If $f \land p_\epsilon \neq 0$, then $s(f \land p_\epsilon) = 0$ implies, by $S_4$, $s(\neg \neg (f \land p_\epsilon)) = 0$ as well. But $\neg \neg (f \land p_\epsilon) = \neg \neg f \land \neg \neg p_\epsilon = \neg \neg f \land p_\epsilon$. Since $f \land p_\epsilon \neq 0$, it turns out that $f_\epsilon \neq 0$, and by Theorem 6.1.5, $f_\epsilon > 0$ on $G_\epsilon$, and hence $\neg \neg f_\epsilon = 1$, thus $\neg \neg f \land p_\epsilon = p_\epsilon$.

($S_4' \Rightarrow S_4$). Let $f \neq 0$. Since $f = \bigvee_{\epsilon \in \Sigma} f \land p_\epsilon$, then $s(f) = \sum_{\epsilon \in \Sigma} s(f \land p_\epsilon)$. If $s(f) = 0$ then we have $s(f \land p_\epsilon) = 0$ for each $\epsilon$. Thus, by $S_4'$, if $f \land p_\epsilon \neq 0$ we have $0 = s(p_\epsilon) = s(\neg \neg f \land p_\epsilon)$; otherwise, if $f \land p_\epsilon = 0$, then $\neg \neg f \land p_\epsilon = 0$ as well. Therefore, $s(\neg \neg f) = \sum_{\epsilon} s(\neg \neg f \land p_\epsilon) = 0$. 

In the next subsection we will investigate, as an example, the states of the free 1-generated product algebra $F_P(1)$ with the aim of exhibiting a first representation for these functional in terms of measures on the dual side.

States of the free 1-generated product algebra

The free 1-generated product algebra $F_P(1) \subset [0,1]^{[0,1]}$ consists of one-variable functions $f$ of the form $f(x) = t(x)$, where the term $t(x)$ can be either 1 (the constant function equal to 1), 0 (the constant function equal to 0), $x$, $\neg x$, $\neg \neg x$, or it belongs to the following set:

$$\{x^n \mid n \in \mathbb{N}\} \cup \{x^n \lor \neg x \mid n \in \mathbb{N}\}.$$ 

The lattice structure of $F_P(1)$ is depicted in Figure 6.2.

As we recalled in Section 6.1, the Boolean skeleton $\mathcal{B}(F_P(1))$ of $F_P(1)$ coincides with the free Boolean algebra over 1 generator. Thus, in this case, $\Sigma = \{\epsilon_1, \epsilon_2\}$ and the two atoms of $\mathcal{B}(F_P(1))$ will be denoted by $p_1$ and $p_2$. Therefore, identifying terms with functions, the elements of $\mathcal{B}(F_P(1))$ are 1, 0, $\neg x = p_1$ and $\neg \neg x = p_2$ and the partition $\{G_1, G_2\}$ of $[0,1]$ is given by $G_1 = \{0\}$ and $G_2 = (0,1]$. 150
Chapter 6. States of the free \( n \)-generated product algebra

Then, as it is easy to check, any map \( s : \mathcal{F}_P(1) \to [0,1] \) satisfying the following conditions is a state:

- \( s(1) = 1, \ s(0) = 0, \)
- \( s(\neg x) + s(\neg\neg x) = 1, \)
- either \( s(\neg x) = s(x) = s(x^n) = 0 \) for all \( n \), or all of them are positive,
- \( s(x^n) \leq s(x^m) \) whenever \( n \geq m, \)
- \( s(x^n \lor \neg x) = s(x^n) + s(\neg x). \)

For every \( y \in \mathcal{F}_P(1) \), let \( \langle y \rangle \) denote the principal lattice filter generated by \( y \). The spectrum, denoted by \( \mathcal{F}_P(1)_s \), of prime \( \ell \)-filters of \( \mathcal{F}_P(1) \), ordered by reverse inclusion, is as in Figure 6.3.

Notice that \( \mathcal{F}_P(1)_s \) is partially ordered as follows: \( \langle \neg x \rangle \) is incompatible with any other element of \( \mathcal{F}_P(1)_s \); \( \langle \neg \neg x \rangle \geq \langle x \rangle \geq \langle x^2 \rangle \geq \langle x^3 \rangle \geq \ldots \). Priestley duality for bounded distributive lattices [102], provides us with a lattice isomorphism \( R(\cdot) \) between the lattice subreduct of \( \mathcal{F}_P(1) \) and the lattice of those downsets of \( \mathcal{F}_P(1)_s \), which are clopen with respect to the usual spectral

Figure 6.2: The lattice of the free product algebra with one generator \( \mathcal{F}_P(1) \)
Chapter 6. States of the free n-generated product algebra

Figure 6.3: The spectral space \( \mathcal{F}_P(1)_* \) of the lattice subreduct of the free product algebra with one generator \( \mathcal{F}_P(1) \).

topology. However, since every downset of \( \mathcal{F}_P(1)_* \) is clopen, \( R(\cdot) \) is onto the whole lattice of downsets of \( \mathcal{F}_P(1)_* \). For every \( x \in \mathcal{F}_P(1) \), it is:

\[
R_x = \{ \{y\} \in \mathcal{P}|x \in \{y\} \}.
\]

In particular, we have that for instance, \( R_{\neg x} = \{ \{y\} \in \mathcal{P}|\langle y \rangle \leq_\mathcal{P} \langle \neg x \rangle \} \) and \( R_{\neg x \lor z} = \{ \langle y_1 \rangle \in \mathcal{P}|\langle y_1 \rangle \leq_\mathcal{P} \langle \neg x \rangle \} \cup \{ \langle y_2 \rangle \in \mathcal{P}|\langle y_2 \rangle \leq_\mathcal{P} \langle x^n \rangle \} \). (Fig. 6.4 provides examples aimed at clarifying this correspondence). The following fact clearly holds:

**Fact 1.** For every \( z \in \mathcal{F}_P(1) \), if \( z = z_1 \lor z_2 \), then \( R_z = R_{z_1} \cup R_{z_2} \) and \( R_{z_1 \land z_2} = R_{z_1} \cap R_{z_2} \).

Figure 6.4: This figure shows the downsets \( R_{\neg x} \) (dashed parabola) and \( R_{\neg x \lor z^n} \) (continuous parabola)

152
With such a representation in mind, let $s$ be a state of $\mathcal{F}_P(1)$ and let us define a $[0, 1]$-valued function $d_s$ on $\mathcal{F}_P(1)_*$ in the following way:

i. $d_s(\neg x) = s(\neg x)$,

ii. $d_s(\neg \neg x) = s(\neg x) - s(x)$,

iii. for every $n \in \mathbb{N}$, $d_s(\langle x^n \rangle) = s(x^n) - s(x^{n+1})$.

First of all, let us show that $d_s$ is a (discrete probability) distribution function on $\mathcal{F}_P(1)_*$, indeed:

\[
\sum_{y \in \mathcal{P}} d_s(\langle y \rangle) = d_s(\langle \neg x \rangle) + d_s(\langle \neg \neg x \rangle) + d_s(\langle x \rangle) + \sum_{n>1} d_s(\langle x^n \rangle)
\]

\[
= s(\neg x) + s(\neg \neg x) - s(x) + s(x) - s(x^2) + \sum_{n>1} d_s(\langle x^n \rangle)
\]

\[
= s(\neg x) + s(\neg \neg x) - s(x^2) + s(x^2) - s(x^3) + \sum_{n>2} d_s(\langle x^n \rangle)
\]

\[
= \cdots
\]

\[
= s(\neg x) + s(\neg \neg x)
\]

\[
= s(x \lor \neg \neg x)
\]

\[
= s(1)
\]

\[
= 1.
\]

Hence, every state of $\mathcal{F}_P(1)$ determines a distribution function $d_s$ on $\mathcal{F}_P(1)_*$. Moreover, notice that the condition (S4) of Definition 6.2.1 forces $d_s$ to satisfy the following further condition:

(D) If $d_s(\langle y \rangle) = 0$, then $d_s(\langle y' \rangle) = 0$ for every $\langle y' \rangle \geq_\mathcal{P} \langle y \rangle$.

Conversely, let $d : \mathcal{P} \to [0, 1]$ be a distribution satisfying (D) and define $s_d : \mathcal{F}_P(1) \to [0, 1]$ by the following stipulation: for every $z \in \mathcal{F}_P(1)$,

\[
s_d(z) = \sum_{\langle y \rangle \in R_z} d(\langle y \rangle).
\] (6.2)

Let us show that $s_d$ is a state of $\mathcal{F}_P(1)$. Obviously $s_d(1) = 1$ and $s_d(0) = 0$. As to prove additivity, let $z_1, z_2$ in $\mathcal{F}_P(1)$. From Fact 1, $R_{z_1 \lor z_2} = R_{z_1} \cup R_{z_2}$ and
Chapter 6. States of the free \( n \)-generated product algebra

\[ R_{z_1 \wedge z_2} = R_{z_1} \cap R_{z_2}. \]  Thus,

\[
s_d(z_1 \vee z_2) = \sum_{y \in R_{z_1} \cup R_{z_2}} d(\{y\}) = \sum_{y' \in R_{z_1}} d(\{y\}) + \sum_{y'' \in R_{z_2}} d(\{y''\}) - \sum_{y'' \in R_{z_1} \cap R_{z_2}} d(\{y''\}) = \sum_{y' \in R_{z_1}} d(\{y\}) + \sum_{y'' \in R_{z_2}} d(\{y''\}) - \sum_{y'' \in R_{z_1} \cap R_{z_2}} d(\{y''\}) = s_d(z_1) + s_d(z_2) - s_d(z_1 \wedge z_2).
\]

The monotonicity of \( s_d \) can be proved in a similar manner observing that \( z_1 \leq z_2 \iff R_{z_1} \subseteq R_{z_2} \).

Let us finally prove that (S4) is satisfied. The two atoms of \( \mathcal{B}(\mathcal{F}_P(1)) \) are \( \neg x \) and \( \neg \neg x \) and for every \( y \in \mathcal{F}_P(1) \), either \( y \wedge \neg x = \neg x \) if \( y = \neg x \), or \( y \wedge \neg x = 0 \) and in this case (S4) is trivially satisfied. As for \( \neg \neg x \), let \( y \in \mathcal{F}_P(1) \) such that \( y \wedge \neg x \neq 0 \). Then, as it is evident from Figure 6.2 (and skipping the trivial cases of \( y = 1 \) and \( y = \neg x \)) either \( y = x^n \) for \( n \geq 1 \), or \( y = \neg x \vee x^n \), for \( n \geq 1 \). In both cases of \( y = x^n \) or \( y = \neg x \vee x^n \), \( \neg x \wedge y = x^n \). Thus, if \( s_d(y \wedge \neg x) = s(x^n) = \sum_{t \in R_{x^n}} d(\{t\}) = 0 \), \( d(\{t\}) = 0 \) for all \( t \in R_{x^n} \). Notice that for any other \( \langle a \rangle \in R_{\neg x} \), one has \( \langle a \rangle \geq_{\mathcal{F}} \{t\} \) for each \( t \in R_{x^n} \), whence by (D), \( d(\{a\}) = 0 \) ensuring that \( s_d(\neg x) = \sum_{\langle a \rangle \in R_{\neg x}} d(\{a\}) = 0 \).

Thus, the following holds.

**Proposition 6.2.5.** There is a one-one correspondence between the set of states of \( \mathcal{F}_P(1) \) and the set of distribution functions on \( \mathcal{F}_P(1)_* \) that satisfy (D).

**Proof.** From what we showed above, it is clear that the map associating states to distributions is well-defined. Thus, it is sufficient to prove that it is one-one. To this end, and since surjectivity is obvious from (6.2), let \( s_1 \neq s_2 \) be two states of \( \mathcal{F}_P(1) \). Thus, there is a \( y \in \mathcal{F}_P(1) \) such that \( s_1(y) \neq s_2(y) \). Now, if \( y \) is one among \( \{-x, \neg x\} \cup \{x^n \mid n \in \mathbb{N}\} \), it is clear that \( d_{s_1}(\{y\}) \neq d_{s_2}(\{y\}) \). If \( y \) is of the kind \( x^n \vee x \), for \( n \in \mathbb{N} \), since \( \neg x \wedge x^n = 0 \), using (S2) we obtain that \( s_1(\neg x) + s_1(x^n) \neq s_2(\neg x) + s_2(x^n) \), thus either \( s_1(\neg x) \neq s_2(\neg x) \), and thus \( d_{s_1}(\neg x) \neq d_{s_2}(\neg x) \), or \( s_1(x^n) \neq s_2(x^n) \), and thus \( d_{s_1}(\{x^n\}) \neq d_{s_2}(\{x^n\}) \), which settles the proof. \( \square \)
6.3 Integral representation

In this section we will prove the main result of this chapter, that is to say, for every state \( s \) of \( \mathcal{F}_P(n) \) there is a unique Borel probability measure \( \mu \) on \([0,1]^n\) such that \( s \) is the Lebesgue integral with respect to \( \mu \), and vice versa, every such an integral operator is a state in our sense.

As we recalled in Section 6.1, product functions are not continuous, thus, unlike the case of (free) MV-algebras, an integral representation for states cannot be obtained by directly applying Riesz representation theorem for linear and monotone functionals, that we recall since it will be useful in the construction:

**Theorem 6.3.1** (Theorem 2.14, [106]). If \( X \) is a locally compact Hausdorff space, then for any positive linear functional \( I \) on the space \( \mathcal{C}_C(X) \) of continuous functions with compact support on \( X \), there is a unique regular Borel measure \( \mu \) on \( X \) such that \( I(f) = \int f \, d\mu \).

However, the finite partition \( \{ G_\epsilon \mid \epsilon \in \Sigma \} \) of \([0,1]^n\) is made of \( \sigma \)-locally compact sets (Remark 6.1.8) upon which the restriction \( f_\epsilon \) of each product function \( f \) is continuous. In this setting, we will suitably extend states to real-valued, positive, monotone and linear operators acting on all continuous functions on the restricted compact domain. This will allow us to apply Reisz representation theorem to obtain Borel measures over each \( G_\epsilon \), in such a way that the Lebesgue integral with respect to these measures will act exactly like our functionals.

Finally, we will suitably extend the measures obtained by Reisz theorem first to measures on every \( G_\epsilon \), and secondly to a measure \( \mu \) on the Borel subsets of real unit cube \([0,1]^n\). We will hence prove that the Lebesgue integral with respect to \( \mu \) behaves like the state \( s \) over the functions of \( \mathcal{F}_P(n) \).

**Definition 6.3.2.** Given a state \( s \) of \( \mathcal{F}_P(n) \), let us define, for every \( \epsilon \in \Sigma \), the map \( \tau_\epsilon : \mathcal{L}_\epsilon(n) \to \mathbb{R} \) as follows: by Proposition 6.1.11, every \( g \in \mathcal{L}_\epsilon(n) \) is represented in a unique way as a linear combination \( \Sigma_{i=1}^k \lambda_i \cdot f_{i,\epsilon} \) for uniquely determined, non-zero parameters \( \lambda_1, \ldots, \lambda_k \) and distinct \( f_{1,\epsilon}, \ldots, f_{k,\epsilon} \in \mathcal{P}_\epsilon(n) \). Thus, we can define by Proposition 6.1.11:
Proposition 6.3.3. For every state \( \tau \) is ensured by Proposition 6.1.11. As for the monotonicity of \( \tau \), Linearity follows by the very definition of \( \tau \) is well-defined, linear and monotone.

Proposition 6.3.4. For every state \( s \) of \( \mathcal{F}_P(n) \), and for every \( \epsilon \), the map \( \tau_\epsilon \) is well-defined, linear and monotone.

Proof. Linearity follows by the very definition of \( \tau_\epsilon \). That \( \tau_\epsilon \) is well-defined is ensured by Proposition 6.1.11. As for the monotonicity of \( \tau_\epsilon \) let \( 0 < g \leq g' \) with \( g = \sum_{i=1}^{\lambda_i-1} \lambda_i \cdot f_{i,\epsilon} \) and \( g' = \sum_{i=1}^{\lambda_i'} \lambda_i' \cdot f'_{i,\epsilon} \) as given by Proposition 6.1.11. Then, \( \tau_\epsilon(g) > 0 \) implies \( \tau_\epsilon(g') \geq \tau_\epsilon(g) \) by definition and the monotonicity of \( s \). On the other hand, if \( \tau_\epsilon(g) = 0 \), then \( s(f_{i,\epsilon} \land p_\epsilon) = 0 \) for every \( i \) whence, by (S4'), \( s(p_\epsilon) = 0 \). Therefore, \( s(f_{i,\epsilon} \land p_\epsilon) = 0 \) by monotonicity of \( s \) thus, \( \tau_\epsilon(g') = 0 \).  

Definition 6.3.4. For every \( q \in (0,1] \cap \mathbb{Q} \), let \( \tau^q_\epsilon : \mathcal{L}^q(n) \rightarrow \mathbb{R} \) be defined as follows: for every \( g \in \mathcal{L}^q(n) \),
\[
\tau^q_\epsilon(g) = \inf \{ \tau_\epsilon(g') \mid g' \in \mathcal{L}(n), g'_{|G_q} = g \}.
\]

Proposition 6.3.5. \( \tau^q_\epsilon \) is a well-defined, linear and monotone functional over \( \mathcal{L}^q(n) \). Moreover, if \( q_2 \leq q_1 \), \( g \in \mathcal{L}^{q_1}(n) \), and \( g' \in \mathcal{L}^{q_2}(n) \) extends \( g \), then \( \tau^q_\epsilon(g) \leq \tau^{q_2}_\epsilon(g') \).

Proof. The good definition of \( \tau^q_\epsilon \) straightly follows from its very definition. Thus, let us start showing that \( \tau^q_\epsilon \) is monotone.

Let \( g, g' \in \mathcal{L}^q(n) \), with \( g \leq g' \). In order to prove the monotonicity of \( \tau^q_\epsilon \), we will show that for each \( h \in \mathcal{L}(n) \) such that \( h_{|G^q} = g' \) we can find \( k \in \mathcal{L}(n) \) such that
\[
k_{|G^q} = g \quad \text{and} \quad k \leq h.
\]

Thus the claim will follow from the definition of \( \tau^q_\epsilon \) and the monotonicity of \( \tau \). Let hence \( h_{|G^q} = g' \) and let \( l \in \mathcal{L}(n) \) which extends \( g \). Thus, let \( k = h \land l \). Clearly \( k \in \mathcal{L}(n) \) and (6.4) holds.

Now, we prove the linearity of \( \tau^q_\epsilon \). First, we prove that if \( h \in \mathcal{L}(n) \) is such that \( h_{|G^q} = g + g' \), for some \( g, g' \in \mathcal{L}^q(n) \), then there are \( z, z' \in \mathcal{L}(n) \)
\[
\tau^q_\epsilon(h) = \tau^q_\epsilon(g + g').
\]
Chapter 6. States of the free $n$-generated product algebra

such that $z_{1|G^q_2} = g, z'_{1|G^q_2} = g'$, and $h = z + z'$. Since $g \leq h_{1|G^q_2}$, from the previous point, we know there is a $z \leq h$ and extending $g$. Thus, let $z' = h - z$. Hence, 
\[
\tau^g_e(g + g') = \inf\{\tau_e(h) \mid h_{1|G^q_2} = g + g'\} = \inf\{\tau_e(z + z') \mid z_{1|G^q_2} = g, z'_{1|G^q_2} = g'\} = \inf\{\tau_e(z) + \tau_e(z') \mid z_{1|G^q_2} = g, z'_{1|G^q_2} = g'\},
\]
where the last equality follows by the linearity of $\tau_e$.

Thus, we shall now prove the following: $\inf\{\tau_e(z) + \tau_e(z') \mid z_{1|G^q_2} = g, z'_{1|G^q_2} = g'\} = \inf\{\tau_e(z) \mid z_{1|G^q_2} = g\} + \inf\{\tau_e(z') \mid z'_{1|G^q_2} = g'\}$. Since one inequality is obviously valid, we are left to prove that
\[
\inf\{\tau_e(z) + \tau_e(z') \mid z_{1|G^q_2} = g, z'_{1|G^q_2} = g'\} \leq \inf\{\tau_e(z) \mid z_{1|G^q_2} = g\} + \inf\{\tau_e(z') \mid z'_{1|G^q_2} = g'\}. 
\]

As to prove this claim, it suffices to notice that for any $z$ such that $z_{1|G^q_2} = g$ and $z'$ such that $z'_{1|G^q_2} = g'$ it is always possible to find a $\hat{z} = \hat{z} + \hat{z}'$ where $\hat{z} \leq z, \hat{z}' \leq z'$, with $\hat{z}_{1|G^q_2} = g, \hat{z}'_{1|G^q_2} = g'$. Thus, being $\tau_e$ monotone, $\tau_e(\hat{z}) + \tau_e(\hat{z}') \leq \tau_e(z) + \tau_e(z')$, and the claim is settled.

In a very similar way, we can show that $\tau^q_e(\lambda z) = \lambda \tau^q_e(z)$. Thus, $\tau^q_e$ is linear.

Finally, in order to conclude the proof, let $q_2 \leq q_1$ and let $g' \in \mathcal{L}^{q_2}(n)$ extending $g \in \mathcal{L}^{q_1}(n)$. Then, $\tau_{q_1}^{q_2}(g) \leq \tau_{q_2}^{q_2}(g')$ from the very definition of $\tau^q_e$.

Now, we want to extend $\tau^q_e$ to a linear and monotone functional on the set $\mathcal{C}(G^q_2)$ of real-valued continuous functions over $G^q_2$. For every $c \in \mathcal{C}(G^q_2)$, let Seq($c$) be the set of countable increasing sequences $\overline{g} = \{g_i\}_{i \in \mathbb{N}}$ of elements in $\mathcal{L}^{q}(n)$ uniformly converging to $c$, in symbols, $\overline{g} \rhd c$.

**Definition 6.3.6.** For every $c \in \mathcal{C}(G^q_2)$ and for every $\overline{g} \in \text{Seq}(c)$ we define
\[
\sigma_{\overline{g}}(c) = \bigvee_{i \in \mathbb{N}} \tau^q_e(g_i)
\]
and finally
\[
\sigma^q_e(c) = \bigvee_{\overline{g} \in \text{Seq}(c)} \sigma_{\overline{g}}(c). \quad (6.5)
\]

**Lemma 6.3.7.** For every $c \in \Sigma$ and for every $q \in (0, 1] \cap \mathbb{Q}$, $\sigma^q_e$ is well-defined, positive, monotone and linear. Moreover $\sigma^q_e$ extends $\tau^q_e$ on $\mathcal{L}^{q}(n)$.
Proof. The fact that $\sigma^q_i$ is well-defined and positive follows by the very definition of such functional. In order to prove that $\sigma^q_i$ is monotone, let $c, c' \in \mathcal{C}(G^q_i)$ and assume $c \leq c'$. The following holds:

Fact 2. For each sequence $\{g_1, g_2, \ldots \} \succ c$, there is a sequence $\{g'_1, g'_2, \ldots \} \succ c'$ and and index $i_0$ such that, for every $i \geq i_0$, $g'_i \geq g_i$.

Proof. (of Fact 2). Let $\{g_1, g_2, \ldots \} \succ c$, and $\{g'_1, g'_2, \ldots \} \succ c'$. Define, for every $i$, $g'_i = g_i \vee g''_i$ and easily check that this settles the claim.

Thus, we prove that $\sigma^q_i(c) \leq \sigma^q_i(c')$. Indeed,

$$\sigma^q_i(c) = \bigvee_{\overline{g} \in \text{Seq}(c)} \sigma^q_{\overline{g}}(c) = \bigvee_{\overline{g} \in \text{Seq}(c)} \left( \bigvee_{i \in \mathbb{N}} \tau^q_{i}(g_i) \right)$$

Fact 2 ensures that, given a $\{g_i\} \succ c$, there is a $\{r_i\} \succ c'$ and, for every $i \geq i_0$, $g_i \leq r_i$, and hence, since $\tau^q_i$ is monotone, $\tau^q_i(g_i) \leq \tau^q_i(r_i)$. Whence, for every $\overline{g} \in \text{Seq}(c)$ there is $\overline{r} \in \text{Seq}(c')$ such that $\sigma^q_{\overline{g}}(c) \leq \sigma^q_{\overline{r}}(c')$. Therefore

$$\sigma^q_i(c) = \bigvee_{\overline{g} \in \text{Seq}(c)} \sigma^q_{\overline{g}}(c) \leq \bigvee_{\overline{r} \in \text{Seq}(c')} \sigma^q_{\overline{r}}(c') = \sigma^q_i(c')$$

showing that $\sigma^q_i$ is monotone.

Now, it is left to show that $\sigma^q_i$ is linear. To this end let us begin with the following claims:

Fact 3. For every $c, c' \in \mathcal{C}(G^q_i)$ and for every $\lambda \in \mathbb{R}$, the following hold

(1) For each $\{t_1, t_2, \ldots \} \succ c + c'$, there are $\{a_1, a_2, \ldots \} \succ c$ and $\{a'_1, a'_2, \ldots \} \succ c'$ such that, for every $i$, $t_i = a_i + a'_i$.

(2) For each $\{t_1, t_2, \ldots \} \succ \lambda c$, there is $\{a_1, a_2, \ldots \} \succ c$ such that, for every $i$, $t_i = \lambda a_i$.

Proof. (of Fact 3). (1) Let $\{t_1, t_2, \ldots \}$ be as given by hypothesis and let $\{a_1, a_2, \ldots \}$ be any sequence converging to $c$. Then let, for every $i$, $a'_i = t_i - a_i \in \mathcal{L}^q(n)$. Thus, $\{a'_1, a'_2, \ldots \} \succ (c + c') - c$, that is $\{a'_1, a'_2, \ldots \} \succ c'$ and this settles the claim.

(2) Let $\{t_1, t_2, \ldots \}$ as in the hypothesis and since $\lambda \neq 0$, put, for every $i$, $a_i = t_i / \lambda$. Thus, $\{a_1, a_2, \ldots \} \succ \lambda c / \lambda$, that is, $\{a_1, a_2, \ldots \} \succ c$. 

158
Now we prove that \( \sigma^q_i \) is linear. Let \( c, c' \in \mathcal{C}(G^q_i) \). Then

\[
\sigma^q_i(c + c') = \bigvee_{t \in \text{Seq}(c + c')} \sigma_T(c + c').
\]

Fact 3(1) shows that, for each \( \{t_1, t_2, \ldots \} \sim (c + c') \), we can find \( \{a_1, a_2, \ldots \} \sim c \) and \( \{b_1, b_2, \ldots \} \sim c' \) such that, for every \( i, t_i = a_i + b_i \). Thus, \( \sigma_T(c + c') = \bigvee_{i \in \mathbb{N}} \tau^q_i(a_i + b_i) \) and since \( \tau^q_i \) is linear, \( \tau^q_i(a_i + b_i) = \tau^q_i(a_i) + \tau^q_i(b_i) \). Thus,

\[
\sigma_T(c + c') = \bigvee_{i \in \mathbb{N}} \tau^q_i(a_i) + \tau^q_i(b_i) = \lim_{i \in \mathbb{N}} \tau^q_i(a_i) + \lim_{i \in \mathbb{N}} \tau^q_i(b_i) = \sigma_q(c) + \sigma_q(c'),
\]

where the previous limits exist because every sequence \( \{a_i\} \) and \( \{a'_i\} \) is bounded by \( c \) and \( c' \) which are continuous functions and \( \{a_i\} \) and \( \{a'_i\} \) converge to \( c \) and \( c' \) on the compact set \( G^q_i \). In a similar way, we can prove that

\[
\sigma^q_i(c + c') = \bigvee \{\sigma_T(c + c') \mid \tilde{t} \in \text{Seq}(c + c')\}
\]

Finally, using a similar argument, but using Fact 3(2), \( \sigma^q_i(\lambda c) = \lambda \sigma^q_i(c) \) so proving that \( \sigma^q_i \) is linear.

In order to conclude the proof, notice that, for each \( g \in \mathcal{L}(n) \), the constant sequence \( \{g\} \) belongs to \( \text{Seq}(g) \), and for any other sequence \( \tilde{t} = \{t_1, t_2, \ldots \} \sim g \) we have \( t_i \leq g \), whence

\[
\sigma^q_i(g) = \bigvee_{\tilde{t} \in \text{Seq}(g)} \bigvee_{i \in \mathbb{N}} \tau^q_i(t_i) = \bigvee_{i \in \mathbb{N}} \tau^q_i(g) = \tau^q_i(g).
\]

The previous Lemma 6.3.7 has the following immediate consequence.
Theorem 6.3.8. For every $\epsilon \in \Sigma$, for every rational $q$, there is a unique regular Borel measure $\mu^q_\epsilon$ such that, for any $c \in \mathcal{C}(G^q_\epsilon)$,

$$\sigma^q_\epsilon(c) = \int_{G^q_\epsilon} c \, d\mu^q_\epsilon.$$ 

In particular, for all $g \in \mathcal{L}^q_\epsilon(n)$,

$$\tau^q_\epsilon(g) = \int_{G^q_\epsilon} g \, d\mu^q_\epsilon.$$ 

Proof. From Lemma 6.3.7, $\sigma^q_\epsilon$ is a (positive) linear functional over $\mathcal{C}(G^q_\epsilon)$, with $G^q_\epsilon$ being compact, thus the first part of the claim follows from Riesz representation theorem [106, Theorem 2.14]. The last part of the claim, finally follows from the last part of Lemma 6.3.7.

With respect to the notation used in the previous Theorem 6.3.8, the following lemma holds.

Lemma 6.3.9. For every Borel subset $B$ of $G_\epsilon$ and for every $q \in (0,1] \cap \mathbb{Q}$, if $q' \leq q$, $\mu^q_\epsilon(B \cap G^q_\epsilon) = \mu^{q'}_\epsilon(B \cap G^{q'}_\epsilon)$.

Proof. Let us write $Z = B \cap G^q_\epsilon$. First of all, notice that $\mu^q_\epsilon(Z) = \inf \{ \sigma^q_\epsilon(c) \mid c \in \mathcal{C}(G^q_\epsilon), c \geq \chi_Z \}$ and $\mu^{q'}_\epsilon(Z) = \inf \{ \sigma^{q'}_\epsilon(c') \mid c' \in \mathcal{C}(G^{q'}_\epsilon), c' \geq \chi_Z \}$ (where $\chi_Z$ denotes the characteristic function of $Z$).

Since for each $r \in (0,1] \cap \mathbb{Q}$, $\mathcal{L}^r_\epsilon(n)$ is dense in $\mathcal{C}(G^r_\epsilon)$, we can safely write, for $q^*$ either being $q$ or $q'$,

$$\mu^{q^*}_\epsilon(Z) = \inf \{ \sigma^{q^*}_\epsilon(g) \mid g \in \mathcal{L}^{q^*}_\epsilon(n), g \geq \chi_Z \},$$

whence, from Lemma 6.3.7,

$$\mu^{q^*}_\epsilon(Z) = \inf \{ \tau^{q^*}_\epsilon(g) \mid g \in \mathcal{L}^{q^*}_\epsilon(n), g \geq \chi_Z \}.$$ 

Let us now define

$$\Delta = \{ g \in \mathcal{L}^q_\epsilon(n), g \geq \chi_Z \} \text{ and } \Delta' = \{ g' \in \mathcal{L}^{q'}_\epsilon(n), g' \geq \chi_Z \}$$

and

$$I = \{ f \in \mathcal{L}_\epsilon(n) \mid f |_{G^q_\epsilon} \in \Delta \} \text{ and } I' = \{ h \in \mathcal{L}_\epsilon(n) \mid h |_{G^{q'}_\epsilon} \in \Delta' \}.$$
Clearly \( I = I' \). Indeed, if \( f \in I \), then \( f \in \Delta(\epsilon) \), whence \( f|_{G^q_{\epsilon'}} \in L^q_{\epsilon'}(n) \) and \( f|_{G^q_{\epsilon'}} \geq \chi_Z \) because \( q' \leq q \), whence \( Z = B \cap G^q_{\epsilon} \subseteq B \cap G^q_{\epsilon'} \). Conversely, if \( h \in I' \), \( h|_{G^q_{\epsilon}} \geq \chi_Z \) again because \( q' \leq q \). Then, for every \( g \in \Delta \), there is a \( g' \in \Delta' \) such that \( \tau^q_{\epsilon}(g) = \tau^{q'}_{\epsilon}(g') \) and vice versa. Thus, by the very definition of \( \tau^q_{\epsilon} \), the claim is settled. \( \square \)

Now, recalling Remark 6.1.8, for every \( \epsilon \) and for \( q_1 \geq q_2 \), we have \( G^q_{\epsilon 1} \subseteq G^q_{\epsilon 2} \). Thus, the following is an immediate consequence of the above result.

**Corollary 6.3.10.** If \( B \) is a Borel subset of \( G^q_{\epsilon} \) for some \( q \), then for all \( q' \leq q \), \( \mu^q_{\epsilon}(B) = \mu^{q'}_{\epsilon}(B) \).

We can now establish an integral representation for the linear and monotone functionals \( \tau_{\epsilon} \) on \( L_{\epsilon}(n) \).

**Lemma 6.3.11.** For every \( \epsilon \in \Sigma \), there is a Borel probability measure \( \mu_{\epsilon} \) on the Borel subsets of \( G_{\epsilon} \) such that, for every \( g \in L_{\epsilon}(n) \),

\[
\tau_{\epsilon}(g) = \int_{G_{\epsilon}} g \, d\mu_{\epsilon}.
\]

**Proof.** Let, for every \( q \in Q \), \( \mu^q_{\epsilon} \) be a Borel measure that provides an integral representation of \( \tau^q_{\epsilon} \) (Theorem 6.3.8). Let us define for each \( \mu^q_{\epsilon} \), the map \( \hat{\mu}^q_{\epsilon} \) over the Borel subset of \( G_{\epsilon} \) in the following way:

\[
\hat{\mu}^q_{\epsilon}(B) = \mu^q_{\epsilon}(B \cap G^q_{\epsilon}).
\]

From Proposition 6.3.5 the sequence \( \{\hat{\mu}^q_{\epsilon}\} \) is increasing and clearly bounded. Thus, by the Vitali-Hahn-Saks Theorem [52, §III.10], it converges to a \( \sigma \)-additive measure \( \mu_{\epsilon} \). Further notice that, by Corollary 6.3.10, for every Borel subset \( X \) of \( G^q_{\epsilon} \),

\[
\mu_{\epsilon}(X) = \mu^q_{\epsilon}(X) = \hat{\mu}^q_{\epsilon}(X).
\]

Now, let us define for each \( g \in L_{\epsilon}(n) \), the function \( g_\epsilon : G_{\epsilon} \to [0,1] \) which equals \( g|_{G^q_{\epsilon}} \) over \( G^q_{\epsilon} \) and takes 0 outside. Observe that each \( g_\epsilon \) is not continuous but it is measurable. Clearly, each sequence \( \{g_q\}_{q \in Q} \) is non-decreasing and it
converges pointwise to \( g \): \( \lim_{q} g_q(x) = g(x) \), for every \( x \in G_\epsilon \). Then, by Levi’s Theorem (cf. [83, §30, Theorem 2]),

\[
\lim_{q} \int_{G_\epsilon} g_q \, d\mu_\epsilon = \int_{G_\epsilon} g \, d\mu_\epsilon.
\]

Finally, observe that

\[
\tau_\epsilon^q(g_{[G_\epsilon]}) = \int_{G_\epsilon^q} g_{[G_\epsilon]} \, d\hat{\mu}_\epsilon^q = \int_{G_\epsilon} g_q \, d\hat{\mu}_\epsilon^q,
\]

and also, by the definition of \( \tau_\epsilon^q \) and Proposition 6.3.5, \( \lim_{q} \tau_\epsilon^q(g_{[G_\epsilon]}) = \tau_\epsilon(g) \). Thus,

\[
\tau_\epsilon(h) = \lim_{q} \tau_\epsilon^q(g_{[G_\epsilon]}) = \lim_{q} \int_{G_\epsilon} g_q \, d\hat{\mu}_\epsilon^q = \lim_{q} \int_{G_\epsilon} g_q \, d\mu_\epsilon = \int_{G_\epsilon} g \, d\mu_\epsilon,
\]

where the third equality follows from (6.7) recalling that \( g_q(y) = 0 \) for each \( y \in G_\epsilon \cap G_\epsilon^q \).

Finally, based on the previous results, next theorem provides an integral representation for states of product logic functions. In the following statement we shall denote by \( \mathcal{M}(n) \) the set of regular Borel probability measures on \([0,1]^n\).

**Theorem 6.3.12** (Integral representation). For every state \( s \) of \( \mathcal{F}_P(n) \) there is a unique regular Borel probability measure \( \mu \in \mathcal{M}(n) \) such that

\[
s(f) = \int_{[0,1]^n} f \, d\mu.
\]

**Proof.** For every \( f \in \mathcal{F}_P(n) \) and for every \( \epsilon \in \Sigma, \)

\[
f = \bigvee_{\epsilon \in \Sigma} (f_\epsilon \wedge p_\epsilon) = \sum_{\epsilon \in \Sigma} (f_\epsilon \wedge p_\epsilon).
\]

Moreover, for distinct \( \epsilon_1, \epsilon_2, (f_{\epsilon_1} \wedge p_{\epsilon_1}) \wedge (f_{\epsilon_2} \wedge p_{\epsilon_2}) = 0 \) (since \( G_{\epsilon_1} \cap G_{\epsilon_2} = \emptyset \)), whence \( s((f_{\epsilon_1} \wedge p_{\epsilon_1}) \wedge (f_{\epsilon_2} \wedge p_{\epsilon_2})) = 0 \). Thus, by axiom S2

\[
s(f) = s\left( \bigvee_{\epsilon \in \Sigma} (f_\epsilon \wedge p_\epsilon) \right) = \sum_{\epsilon \in \Sigma} s(f_\epsilon \wedge p_\epsilon).
\]
Chapter 6. States of the free \( n \)-generated product algebra

Now, from the definition of \( \tau_c \), Proposition 6.3.3 and Lemma 6.3.11 it follows that

\[
s(f \land p_c) = s(p_c) \cdot \tau_c(f_c) = s(p_c) \cdot \int_{G_c} f_c \, d\mu_c \tag{6.9}
\]

for a Borel measures \( \mu_c \) on the Borel subsets of \( G_c \).

Let hence define \( \mu \) on the Borel subsets of \( [0,1]^n \) by the following stipulation: for every \( X \) Borel subset of \( [0,1]^n \),

\[
\mu(X) = \sum_{c \in \Sigma} s(p_c) \cdot \mu_c(X \cap G_c).
\]

Since \( \sum_{c \in \Sigma} s(p_c) = s(\bigvee_{c \in \Sigma} p_c) = s(1) = 1 \), \( \mu \) is a convex combination of the \( \mu_c \)'s. Moreover \( \mu \) is defined for every \( X \) since \( G_c \) is a Borel subset of \( [0,1]^n \) (recall Lemma 6.1.6), whence \( G_c \cap X \) is Borel as well. Thus, from (6.8) and (6.9),

\[
s(f) = \sum_{c \in \Sigma} s(p_c) \cdot \tau_c(f_c)
\]

\[
= \sum_{c \in \Sigma} \left( s(p_c) \cdot \int_{G_c} f_c \, d\mu_c \right)
\]

\[
= \int_{\bigcup_{c \in \Sigma} G_c} \sum_{c \in \Sigma} f_c \, d(s(p_c) \cdot \mu_c)
\]

\[
= \int_{[0,1]^n} f \, d\mu.
\]

Now, it is left to show that \( \mu \) is unique. Suppose, by way of contradiction, that for a state \( s \) there are two distinct Borel measures \( \mu_1 \) and \( \mu_2 \) such that, for every \( f \in \mathcal{F}_p(n) \), \( s(f) = \int_{[0,1]^n} f \, d\mu_1 = \int_{[0,1]^n} f \, d\mu_2 \).

If \( \mu_1 \neq \mu_2 \), then there must exist a \( c \in \mathcal{C}([0,1]^n) \) such that \( \int_{[0,1]^n} c \, d\mu_1 \neq \int_{[0,1]^n} c \, d\mu_2 \). Since \( [0,1]^n = \bigcup_{c \in \Sigma} G_c \), there is an \( c \) such that \( \int_{G_c} c \, d\mu_1 \neq \int_{G_c} c \, d\mu_2 \). Now, since \( G_c = \bigcup_{q \in \mathbb{N}} G_c^q \), we have \( \lim_{q \to 0} \int_{G_c^q} c \, d\mu_1 \neq \lim_{q \to 0} \int_{G_c^q} c \, d\mu_2 \).

Without loss of generality, assume \( \lim_{q \to 0} \int_{G_c^q} c \, d\mu_1 < \lim_{q \to 0} \int_{G_c^q} c \, d\mu_2 \). Hence, there is \( q \) such that for every \( q' \), \( \int_{G_c^{q'}} c \, d\mu_1 < \int_{G_c^{q}} c \, d\mu_2 \). In particular, \( \int_{G_c^q} c \, d\mu_1 < \int_{G_c^{q}} c \, d\mu_2 \).

Now, over \( G_c^q \), \( c \) is the limit of an increasing sequence \( \{g_k\}_{k \in \mathbb{N}} \in \mathcal{L}_c^q(n) \), and by the continuity of the integral,

\[
\sup_k \int_{G_c^q} g_k \, d\mu_1 < \sup_k \int_{G_c^q} g_k \, d\mu_2
\]
So there is \( k \) such that, for every \( k' \),
\[
\int_{G_q} g_{k'} \, d\mu_1 < \int_{G_q} g_k \, d\mu_2
\]
in particular,
\[
\int_{G_q} g_k \, d\mu_1 < \int_{G_q} g_k \, d\mu_2
\]
But \( g_k \) is the restriction of a function \( g \in \mathcal{L}_\omega(n) \) on \( G_q \) and hence \( g \) is of the form \( \sum_i \lambda_i f_i \), with \( f_i \in \mathcal{P}_\omega(n) \). Thus, we can apply Theorem 6.3.8 and, since the \( \tau \)'s are uniquely determined from the state \( s \), we get that
\[
\tau_q(g_k) = \int_{G_q} g_k \, d\mu_1 \quad \text{and} \quad \tau_q(g_k) = \int_{G_q} g_k \, d\mu_2,
\]
whence:
\[
\tau_q(g_k) < \tau_q(g_k),
\]
which is a contradiction. \( \square \)

We shall now see that the converse also holds.

**Theorem 6.3.13.** For every Borel probability measure \( \mu : \mathcal{B}([0,1]^n) \to [0,1] \), the function \( s : \mathcal{F}_\mathcal{P}(n) \to [0,1] \) defined as
\[
s(f) = \int_{[0,1]^n} f \, d\mu.
\]
is a state of \( \mathcal{F}_\mathcal{P}(n) \).

**Proof.** First we observe that for each \( f \in \mathcal{F}_\mathcal{P}(n) \), \( \int_{[0,1]^n} f \, d\mu \in [0,1] \), since \( \mu \) is normalized to 1 and the functions of the free product algebra take values in \([0,1]\). In order to prove that \( s \) is a state, we need to show that the integral of product functions satisfy the properties S1-S4:

1. \( \int_{[0,1]^n} 0 \, d\mu = 0 \) and \( \int_{[0,1]^n} 1 \, d\mu = 1 \), where \( 0 \) and \( 1 \) are respectively the functions constantly equal to 0 and 1.

2. \( \int_{[0,1]^n} (f \land g) \, d\mu + \int_{[0,1]^n} (f \lor g) \, d\mu = \int_{[0,1]^n} f \, d\mu + \int_{[0,1]^n} g \, d\mu, \) for each \( f, g \in \mathcal{F}_\mathcal{P}(n) \).

3. If \( f, g \in \mathcal{F}_\mathcal{P}(n) \) are such that \( f \leq g \), then \( \int_{[0,1]^n} f \, d\mu \leq \int_{[0,1]^n} g \, d\mu \).
Chapter 6. States of the free $n$-generated product algebra

(S4) For every $\epsilon \in \Sigma$, for any $f \in \mathcal{F}_P(n)$ such that $f \land p_\epsilon$ is non-zero, $\int_{[0,1]^n} f \land p_\epsilon \, d\mu = 0$ implies $\int_{[0,1]^n} p_\epsilon \, d\mu = 0$.

Properties (S1) and (S3) are well-known properties of the integral, with respect to probability measures. About property (S2), it is not difficult to realize that, since the operations are defined pointwise, it holds that $f + g = \min(f, g) + \max(f, g)$, which settles the proof. In order to prove (S4), we shall observe that

$$
\int_{[0,1]^n} (f \land p_\epsilon) \, d\mu = \int_{[0,1]^n \setminus G_\epsilon} (f \land p_\epsilon) \, d\mu + \int_{G_\epsilon} (f \land p_\epsilon) \, d\mu.
$$

The first integral is 0 since the function $f \land p_\epsilon$ is 0 outside $G_\epsilon$. Thus, if $\int_{[0,1]^n} (f \land p_\epsilon) \, d\mu = 0$ the second one must be 0 as well, and since $f \land p_\epsilon$ is strictly positive over $G_\epsilon$ (if it is 0 in one point, it is 0 in the whole $G_\epsilon$, [4, Lemma 3.2.3]) then it must be $\mu(G_\epsilon) = 0$, whence $\int_{[0,1]^n} p_\epsilon \, d\mu = 0$.

Therefore, our result can be stated in the following concise way.

**Corollary 6.3.14.** For every $n \in \mathbb{N}$, and for every map $s : \mathcal{F}_P(n) \to [0,1]$ the following are equivalent:

1. $s$ is a state,
2. there is a unique regular Borel measure $\mu \in \mathcal{M}(n)$ such that, for every $f \in \mathcal{F}_P(n)$,

$$
s(f) = \int_{[0,1]^n} f \, d\mu.
$$

6.4 The state space and its extremal points

In this section we shall prove that states of $\mathcal{F}_P(n)$ are actually convex combinations of product logic valuations. The idea is to show first that the state space is convex and compact and hence, by Krein-Milman Theorem, every state is in the closure of convex combination of extremal ones, and second, to show that the extremal states coincide with product logic valuations, i.e. homomorphisms of $\mathcal{F}_P(n)$ into $[0,1]_\Pi$.

Let $n$ be any positive integer. Let us denote by $\mathcal{H}(n)$ the set of homomorphisms of $\mathcal{F}_P(n)$ to the product algebra $[0,1]_\Pi$; $S(n)$ stands for the set of
states of $\mathcal{F}_P(n)$, while $\mathcal{M}(n)$ denotes, as in Section 6.3, the set of regular Borel probability measures on $\mathcal{B}([0,1]^n)$, the $\sigma$-algebra of Borel subsets of $[0,1]^n$.

**Proposition 6.4.1.** For every $n \in \mathbb{N}$, there is a bijection between $\mathcal{S}(n)$ and $[0,1]^n$.

*Proof.* Let $\varphi : [0,1]^n \to [0,1]^{\mathcal{F}_P(n)}$ be the map that associates to every $x \in [0,1]^n$ the function $\varphi_x : f \mapsto f(x)$, for every $f \in \mathcal{F}_P(n)$. Clearly, $\varphi_x$ is a homomorphism, and it is easy to see that if $x_1 \neq x_2$ then $\varphi_{x_1} \neq \varphi_{x_2}$. Moreover, every homomorphism $h$ is such that $h = \varphi_x$, for some $x \in [0,1]^n$. Indeed, let $x = (h(\pi_1),\ldots,h(\pi_n))$, where $\pi_i$ denotes the $i$-th projection. Moreover, for every $f \in \mathcal{F}_P(n)$ there is a term $t_f$ such that $f = t_f[\pi_1,\ldots,\pi_n]$. Thus $h(f) = h(t_f[\pi_1,\ldots,\pi_n]) = t_f[h(\pi_1),\ldots,h(\pi_n)] = f(x) = \varphi_x(f)$.

It is quite obvious that $\mathcal{S}(n)$ and $\mathcal{M}(n)$ are convex subsets of $[0,1]^{\mathcal{F}_P(n)}$ and $[0,1]^{\mathcal{B}([0,1]^n)}$ respectively. Furthermore, $\mathcal{M}(n)$ is clearly compact with respect to the subspace product topology. As for $\mathcal{S}(n)$, let us prove that it is closed, whence compact.

**Proposition 6.4.2.** $\mathcal{S}(n)$ is closed in the Tychonoff cube $[0,1]^{\mathcal{F}_P(n)}$.

*Proof.* Let $\{s_i\}_{i \geq 0}$ be a sequence of states of $\mathcal{F}_P(n)$ such that $\lim_{i \in \mathbb{N}} s_i = s$ exists, and let us prove that such $s$ is a state. Condition S1 of Definition 6.2.1 is clearly verified. Let us show that $s$ respects condition S2. We need to prove that $s(f \vee g) = s(f) + s(g) - s(f \wedge g)$. Being each $s_n$ a state, we have that:

$$\lim_{i \in \mathbb{N}} s_n(f \vee g) = \lim_{i \in \mathbb{N}} (s_n(f) + s_n(g) - s_n(f \wedge g))$$

and also, it clearly holds that:

$$\lim_{i \in \mathbb{N}} (s_n(f) + s_n(g) - s_n(f \wedge g)) = \lim_{i \in \mathbb{N}} s_n(f) + \lim_{i \in \mathbb{N}} s_n(g) - \lim_{i \in \mathbb{N}} s_n(f \wedge g),$$

thus the claim directly follows. It is easy to prove condition S3, since given $f, g \in \mathcal{F}_P(n)$, if $f \leq g$ then $s_n(f) \leq s_n(g)$ for every $n \in \mathbb{N}$. Thus, it follows that:

$$s(f) = \lim_{i \in \mathbb{N}} s_n(f) \leq \lim_{i \in \mathbb{N}} s_n(g) = s(g).$$
Let us finally prove S4. Let $f \in \mathcal{F}_P(n), f \neq 0$, such that $s(f) = 0$. We shall prove that $s(-f) = 0$. Let $\text{supp}(f) = \{ x \in [0,1]^n \mid f(x) > 0 \}$. Then $\text{supp}(f)$ is a union of $G_\epsilon$'s, whence it is a Borel subset of $[0,1]^n$. This observation, with Corollary 6.3.14, imply that:

$$s(f) = \lim_i \int_{[0,1]^n} f \, d\mu_i = \lim_i \int_{\text{supp}(f)} f \, d\mu_i = 0.$$ 

**Fact 4.** If $\lim_i \int_{\text{supp}(f)} f \, d\mu_i = 0$ then $\lim_i \mu_i(\text{supp}(f)) = 0$.

**Proof.** (of Fact 4) As we already noticed, $\text{supp}(f) = \bigcup_{\epsilon \in \Sigma^*} G_\epsilon$, for some $\Sigma^* \subseteq \Sigma$. Thus, if $\lim_{i \in \mathbb{N}} \int_{\text{supp}(f)} f \, d\mu_i = 0$, and since the $G_\epsilon$'s are disjoint, the following holds:

$$\lim_{i \in \mathbb{N}} \int_{\text{supp}(f)} f \, d\mu_i = \lim_{i \in \mathbb{N}} \int_{\bigcup_{\epsilon \in \Sigma^*} G_\epsilon} f \, d\mu_i = \lim_{i \in \mathbb{N}} \left( \sum_{\epsilon \in \Sigma^*} \int_{G_\epsilon} f \, d\mu_i \right) = \sum_{\epsilon \in \Sigma^*} \left( \lim_{i \in \mathbb{N}} \int_{G_\epsilon} f \, d\mu_i \right).$$

Therefore, $\sum_{\epsilon \in \Sigma^*} \left( \lim_{i \in \mathbb{N}} \int_{G_\epsilon} f \, d\mu_i \right) = 0$, whence $\lim_{i \in \mathbb{N}} \int_{G_\epsilon} f \, d\mu_i = 0$ for all $\epsilon \in \Sigma^*$. Now, $G_\epsilon = \bigcup_{q \in \mathbb{Q} \cap (0,1]} G_\epsilon^q$, and hence

$$\int_{G_\epsilon} f \, d\mu_i = \sup_{q \in \mathbb{Q} \cap (0,1]} \left( \int_{G_\epsilon^q} f \, d\mu_i \right).$$

Therefore:

$$0 = \lim_i \int_{G_\epsilon} f \, d\mu_i = \lim_i \sup_{q \in \mathbb{Q} \cap (0,1]} \left( \int_{G_\epsilon^q} f \, d\mu_i \right) = \lim_i \left( \int_{G_\epsilon^q} f \, d\mu_i \right).$$

Hence, for all $q \in \mathbb{Q} \cap (0,1]$, it follows that $\lim_i \left( \int_{G_\epsilon^q} f \, d\mu_i \right) = 0$. But since $G_\epsilon^q$ is compact, and $f$ is strictly positive on it, $\lim_i \mu_i(G_\epsilon^q) = 0$. Indeed, let $r = \min \{ f(x) \mid x \in G_\epsilon^q \}$, thus $\lim_{i \in \mathbb{N}} \left( \int_{G_\epsilon^q} r \, d\mu_i \right) \leq \lim_{i \in \mathbb{N}} \left( \int_{G_\epsilon^q} f \, d\mu_i \right) = 0$. Thus,

$$0 = \lim_{i \in \mathbb{N}} \left( \int_{G_\epsilon^q} r \, d\mu_i \right) = r \cdot \lim_{i \in \mathbb{N}} \mu_i(G_\epsilon^q),$$

that is, $\lim_{i \in \mathbb{N}} \mu_i(G_\epsilon^q) = 0$. Therefore,

$$\lim_{i \in \mathbb{N}} \mu_i(G_\epsilon) = \lim_{i \in \mathbb{N}} \left( \bigcup_{q \in \mathbb{Q} \cap (0,1]} G_\epsilon^q \right) \leq \lim_{i \in \mathbb{N}} \sum_{q \in \mathbb{Q} \cap (0,1]} \mu_i(G_\epsilon^q) = \sum_{q \in \mathbb{Q} \cap (0,1]} \lim_{i \in \mathbb{N}} \mu_i(G_\epsilon^q) = 0$$

that is to say, $\lim_{i \in \mathbb{N}} \mu_i(G_\epsilon) = 0$ for all $\epsilon \in \Sigma^*$. Hence, finally,

$$\lim_{i \in \mathbb{N}} \mu_i(\text{supp}(f)) = \lim_{i \in \mathbb{N}} \sum_{\epsilon \in \Sigma^*} \mu_i(G_\epsilon) = \sum_{\epsilon \in \Sigma^*} \lim_{i \in \mathbb{N}} \mu_i(G_\epsilon) = 0. \qed
Now, since \( \text{supp}(f) = \text{supp}(\neg\neg f) \), and \((\neg\neg f)(x) = 1\) for every \( x \in \text{supp}(f) \),

\[
s(\neg\neg f) = \lim_{i \in \mathbb{N}} s_i(\neg\neg f) = \lim_{i \in \mathbb{N}} \int_{[0,1]^n} \neg\neg f \, d\mu_i = \lim_{i \in \mathbb{N}} \int_{\text{supp}(\neg\neg f)} \neg\neg f \, d\mu_i = \lim_{i \in \mathbb{N}} \int_{\text{supp}(f)} 1 \, d\mu_i = \lim_{i \in \mathbb{N}} \mu_i(\text{supp}(f)) = 0,
\]

where the last equality clearly follows from Fact 4 above. \( \Box \)

Thus, by Krein-Milman Theorem, \( S(n) \) and \( M(n) \) are generated by their extremal points. It is well-known that the extremal points of \( M(n) \) are Dirac measures, i.e. those maps \( \delta_x : B([0,1]^n) \to \{0,1\} \), for each \( x \in [0,1]^n \), such that \( \delta_x(B) = 1 \) iff \( x \in B \) and \( \delta_x(B) = 0 \) otherwise.

Let us consider the map

\[
\delta : S(n) \to M(n)
\]

which associates, to each state \( s \in S(n) \) the unique regular Borel measure \( \mu \in M(n) \) provided by Theorem 6.3.12, such that for every \( f \in \mathcal{F}_P(n) \), \( s(f) = \int_{[0,1]^n} f \, d\mu \). Thus, \( \delta \) is well-defined. Furthermore, the following holds:

**Proposition 6.4.3.** For every \( n \in \mathbb{N} \), the map \( \delta : S(n) \to M(n) \) defined as above is bijective and affine.

**Proof.** Injectivity follows from Theorem 6.3.12, and surjectivity from Theorem 6.3.13. In order to prove that \( \delta \) is affine, let us suppose that \( s = \lambda s_1 + (1 - \lambda) s_2 \), with \( \lambda \in [0,1] \). Then we have, for every \( f \in \mathcal{F}_P(n) \),

\[
s(f) = \lambda s_1(f) + (1 - \lambda) s_2(f)
\]

\[
= \lambda \int_{[0,1]^n} f \, d\mu_1 + (1 - \lambda) \int_{[0,1]^n} f \, d\mu_2
\]

\[
= \int_{[0,1]^n} f \, d(\lambda \mu_1) + \int_{[0,1]^n} f \, d((1 - \lambda)\mu_2)
\]

\[
= \int_{[0,1]^n} f \, d[(\lambda \mu_1) + (1 - \lambda)\mu_2].
\]

168
Chapter 6. States of the free $n$-generated product algebra

Thus, $\delta(s) = \delta(\lambda s_1 + (1 - \lambda)s_2) = \lambda \delta(s_1) + (1 - \lambda)\delta(s_2)$, which proves that $\delta$ is affine.

Before showing the main result of this section (Theorem 6.4.4 below), let us point out an immediate but interesting consequence of Proposition 6.4.3 above which reveals a remarkable analogy between states of MV-algebras and states of product algebras. Indeed, an immediate consequence of the Kroupa-Panti Theorem, shows that for every positive integer $n$, the state space $S_{MV}(n)$ of the free MV-algebra over $n$-free generators is affinely isomorphic to $M(n)$. Thus, in particular, $S(n)$ and $S_{MV}(n)$ are affinely isomorphic via an isomorphism which is defined in the obvious way.

The main result of this section hence reads as follows.

**Theorem 6.4.4.** The following are equivalent for a state $s : F_P(n) \to [0,1]$

1. $s$ is extremal;

2. $\delta(s)$ is a Dirac measure;

3. $s \in \mathcal{H}(F_P(n))$.

**Proof.** (1) $\Rightarrow$ (2). If $s$ is extremal then its corresponding measure $\delta(s)$ is extremal in the space of Borel probability measures on $[0,1]$, since by Proposition 6.4.3 $\delta$ is affine, and hence it preserves extremality. Extremal Borel measures on $[0,1]$ are exactly Dirac measures (see for instance [95, Corollary 10.6]), thus $\delta(s) = \delta_x$ for some $x \in [0,1]^n$.

(2) $\Rightarrow$ (1). If $s$ is such that $\delta(s)$ is a Dirac measure, then it is extremal. Indeed, by way of contradiction, let us suppose that $s$ can be expressed as a convex combination of two states $s_1, s_2$, that is, $s = \lambda s_1 + (1 - \lambda)s_2$, $\lambda \in (0,1)$, but this would mean that $\delta(s) = \delta(\lambda s_1 + (1 - \lambda)s_2) = \lambda \delta(s_1) + (1 - \lambda)\delta(s_2)$, which contradicts the extremality of $\delta(s)$.

Hence, we proved that (1) $\iff$ (2).

(3) $\Rightarrow$ (2). Follows from Proposition 6.4.1.
(2) $\Rightarrow$ (3). Let us suppose that $\delta(s)$ is a Dirac measure $\delta(s) = \delta_x$, and let us prove that $s$ is a homomorphism. By Theorem 6.3.12, for every $f \in \mathcal{F}_P(n)$,

$$s(f) = \int_{[0,1]^n} f \, d\delta_x = f(x),$$

thus clearly $s$ is a homomorphism to $[0,1]$.

Hence we proved $(2) \iff (3)$, which settles the proof.

Thus, via Krein-Milman Theorem, we obtain the following:

**Corollary 6.4.5.** The state space $\mathcal{S}(n)$ is the convex closure of the set of product homomorphisms from $\mathcal{F}_P(n)$ into $[0,1]$.

**Remark 6.4.6.** For every $n$, the set of extremal states of the free MV-algebra $\mathcal{F}_{MV}(n)$, with the topology inherited by restriction from the product space $[0,1]^{\mathcal{F}_{MV}(n)}$, constitute a compact Hausdorff space $\text{ext}(\mathcal{S}_{MV}(n))$ which is homeomorphic to $[0,1]^n$ (see [94, Theorem 2.5] and [95, Corollary 10.6]). Thus, $\text{ext}(\mathcal{S}_{MV}(n))$ is closed. A similar result for extremal product states is false. Indeed, as a consequence of [87, Theorem 4.6], $\mathcal{H}(n) = \text{ext}(\mathcal{S}(n))$ is not closed in the Tychonoff cube $[0,1]^{\mathcal{F}_P(n)}$. Thus it cannot be homeomorphic to $[0,1]^n$. However, Theorem 6.4.4 still provides us with a bijection among $\text{ext}(\mathcal{S}_{MV}(n))$, $\text{ext}(\mathcal{S}(n))$ and $[0,1]^n$. 


Chapter 7

Towards a notion of states for sIDL-algebras

As we have seen, states of MV-algebras can be regarded as those mappings that arise applying Mundici’s functor $\Gamma$ to states of unital $\ell$-groups. Hence, mimicking the same insights, we shall see how a notion of state (more precisely, hyperstate) of perfect MV-algebras arises applying Di Nola and Lettieri’s functor. More generally, if we consider the variety generated by perfect MV-algebras $DLMV$, and we drop divisibility assumption, we get the variety of sIDL-algebras, and we reason in a similar way. Indeed, we know that the variety of sIDL-algebras is categorically equivalent to a category of triples made of a Boolean algebra, a GMTL-algebra and an external join (Theorem 2.4.7). Thus, we will show how a notion of state of sIDL-algebras can again be inspired by this structural decomposition, going through a suitable definition of states of GMTL-algebras.

7.1 From $\ell$-monoids to $\ell$-groups and hoops

In this section we will prove and recall some basic results we shall need in what follows. As to begin with, let us recall that a lattice-ordered monoid ($\ell$-monoid for short) is a structure $M = (M, +, \wedge, \vee, 0)$ such that $(M, +, 0)$ is a commutative monoid, $(M, \wedge, \vee)$ is a lattice, and the following distribution laws hold for all $x, y, z \in M$: 

171
(D1) \( x + (y \land z) = (x + y) \land (x + z) \),

(D2) \( x + (y \lor z) = (x + y) \lor (x + z) \).

The following result, that we need to reprove completely, is an extension of the usual Grothendieck group construction to the case of lattice-ordered monoids.

**Theorem 7.1.1.** Let \( M = (\ell, +, \land, \lor, 0) \) be a lattice-ordered monoid. Then, there is an abelian \( \ell \)-group \( K(M) \) and a \( \ell \)-monoid homomorphism \( h : M \to K(M) \) which is injective iff \( M \) is cancellative.

**Proof.** Starting from a commutative monoid \( M \) it is possible to define the Grothendieck group of \( M \), namely \( K(M) \) (see [114, Chapter II]), by means of the following construction. Consider the equivalence relation on the cartesian product \( M \times M \) given by \( (x, y) \sim (x', y') \) if there exists \( z \in M \) such that \( z + x + y' = z + x' + y \). Let \( K(M) = M \times M / \sim \), and for every \( [x, y], [x', y'] \in K(M) \), let

\[
[x, y] \hat{\ast} [x', y'] = [x + x', y + y'] .
\]

Then let the identity be defined by \([0, 0] \) and let the inverse of \([x, y] \) be \([-x, y] = [y, x] \). Then \( K(M) = (K(M), \hat{\ast}, -,[0, 0]) \) is an abelian group, and it satisfies the universal property: there exist a monoid homomorphism \( k : M \to G \) into an abelian group \( G \) such that for any other monoid homomorphism \( k : M \to G \) there exists a unique group homomorphism \( l : K(M) \to G \) such that \( k = l \circ h \). In particular, let \( h(x) = [x + x, x] \) for every \( x \in M \). Moreover, it is also known that such \( h \) is injective iff \( M \) is cancellative (cf. for instance [16]).

Now, we are going to prove that if \( M \) is lattice-ordered, it is possible to define on \( K(M) \) an \( \ell \)-group structure such that the claim holds. First, denoting with \( \leq_M \) the lattice order on \( M \), let \([x_1, y_1] \leq [x_2, y_2] \) if \( \exists z : z + x_1 + y_2 \leq_M z + y_1 + x_2 \). It is easy to see that it is a partial order on \( K(M) \), for instance let us prove transitivity. If \([x_1, y_1] \leq [x_2, y_2], [x_2, y_2] \leq [x_3, y_3] \), then by definition \( \exists z : z + x_1 + y_2 \leq_M z + y_1 + x_2 \) and \( \exists z' : z' + x_2 + y_3 \leq_M z' + y_2 + x_3 \). By monotonicity, \( z + x_1 + y_2 + z' + x_2 + y_3 \leq_M z + y_1 + x_2 + z' + y_2 + x_3 \) and putting \( z'' = z + z' + x_2 + y_2 \in M \), we have that \( z'' + x_1 + y_3 \leq_M z'' + y_1 + x_3 \), thus \([x_1, y_1] \leq [x_3, y_3] \) and \( \leq \) is transitive.

Let us now define lattice operations with respect to the order \( \leq \):

\[
[x_1, y_1] \cup [x_2, y_2] = [x_1 + x_2, (x_1 + y_2) \land (x_2 + y_1)] ,
\]
Chapter 7. Towards a notion of states for sIDL-algebras

\[ [x_1, y_1] \cap [x_2, y_2] = [(x_1 + y_2) \land (x_2 + y_1), y_1 + y_2]. \]

We prove that \( \lor \) is the join, the proof that \( \cap \) is the meet being similar. First we prove that it is an upper bound, that is, \([x_1, y_1], [x_2, y_2] \leq [x_1 + x_2, (x_1 + y_2) \land (x_2 + y_1)]\), indeed: \( x_1 + (x_1 + y_2) \land (x_2 + y_1) \leq x_1 + (x_2 + y_1) \) and similarly \( x_2 + (x_1 + y_2) \land (x_2 + y_1) \leq x_2 + (x_1 + y_2) \). Now we shall prove that it is the least upper bound, that is, for any \([x_3, y_3] \in K(M)\), if \([x_1, y_1], [x_2, y_2] \leq [x_3, y_3]\), then \([x_1 + x_2, (x_1 + y_2) \land (x_2 + y_1)] \leq [x_3, y_3]\). By hypothesis, \( \exists z: z + x_1 + y_3 \leq M \)

\[ z + y_1 + x_3 \text{ and } \exists z': z + x_2 + y_3 \leq M \]

Thus we get

\[ z + z' + x_1 + x_2 + y_3 \leq M \]

\[ z + z' + x_1 + x_2 + y_3 \leq M \]

Hence \( (z + z' + x_1 + x_2 + y_3) \land (z + z' + x_1 + x_2 + y_3) \leq M \) \( (z + z' + x_3 + y_1 + x_2) \land (z + z' + x_3 + y_2 + x_1) \) and since \( \land \) distributes over \( + \), we obtain that \( z + z' + (x_1 + x_2) + y_3 \leq M \) \( z + z' + x_3 + (y_1 + x_2) \land (y_2 + x_1) \), which means exactly that \([x_1 + x_2, (x_1 + y_2) \land (x_2 + y_1)] \leq [x_3, y_3]\).

In order to prove that \( K(M) \) with \( \lor, \cap \) is an \( \ell \)-group, we need to show that \( \hat{\lor} \) distributes over \( \lor \), that is:

\[ [x_1, y_1] \hat{\lor} ([x_2, y_2] \cup [x_3, y_3]) = ([x_1, y_1] \hat{\lor} [x_2, y_2]) \cup ([x_1, y_1] \hat{\lor} [x_3, y_3]) \]

Now, \([x_1, y_1] \hat{\lor} ([x_2, y_2] \cup [x_3, y_3]) = [x_1, y_1] \hat{\lor} [x_2 + x_3, (y_2 + x_3) \land (y_3 + x_2)] = [x_1 + x_2 + x_3, y_1 + (y_2 + x_3) \land (y_3 + x_2)] \), while \(([x_1, y_1] \hat{\lor} [x_2, y_2]) \cup ([x_1, y_1] \hat{\lor} [x_3, y_3]) = [x_1 + x_2, y_1 + y_2] \cup [x_1 + x_3, y_1 + y_3] = [x_1 + x_2 + x_1 + x_3, (y_1 + y_2 + x_1 + x_3) \land (y_1 + y_3 + x_1 + x_2)]\). It is easy to see that \([x_1 + x_2 + x_1 + x_3, (y_1 + y_2 + x_1 + x_3) \land (y_1 + y_3 + x_1 + x_2)] = [x_1 + x_2 + x_3, y_1 + (y_2 + x_3) \land (y_3 + x_2)] \), since \( x_1 + x_2 + x_3 + x_1 + (y_1 + y_2 + x_1 + x_3) \land (y_1 + y_3 + x_1 + x_2) \), which settles the claim.

In order to conclude the proof we need to prove that the monoid homomorphism \( h: M \to K(M) \), \( h(x) = [x + x, x] \), is also a lattice homomorphism. Let us prove that it respects the meet operation, the proof for the join being similar. We need to show that \( h(x \land y) = h(x) \cap h(y) \), that is to say,

\[ [(x \land y) + (x \land y), x \land y] = [x + x, x] \cap [y + y, y]. \]
Chapter 7. Towards a notion of states for sIDL-algebras

Now, $[x+x, x] \cap [y+y, y] = [(x+x+y) \land (y+y+x), x+y]$. Since $(x+y) + (x \land y) = (x+y+x) \land (y+y+y)$, we have that $(x \land y) + (x \land y) + x + y = (x+y+x) \land (y+y+y) + (x \land y)$, which proves the claim. □

The following result is a direct consequence of Theorem 7.1.1 plus the observation that the reduct $\hat{H} = (H, \cdot, \land, \lor, 1)$ of a GMTL-algebra $H$ is an $\ell$-monoid (written in multiplicative form). In the following result and in the rest of this paper, if $h : M \to K(M)$ is an $\ell$-monoid homomorphism, we shall denote by $J_{h[M]}$ the $\ell$-subgroup of $K(M)$ generated by $h[M] = \{h(a) | a \in M\}$.

**Corollary 7.1.2.** For every GMTL-algebra $H$ there is an abelian $\ell$-group $K(\hat{H})$ and a $\ell$-monoid homomorphism $h : H \to K(\hat{H})$ which is injective iff $H$ is cancellative. Furthermore, for every $x \in K(\hat{H})$, there exists a $y \in J_{h[H]}$ such that $y \leq x$. Consequently, for every $x \in K(\hat{H})$, there exists a $y' \in J_{h[H]}$ such that $y' \geq x$.

**Proof.** The first part directly follows from Theorem 7.1.1. As for the second part, let $[a, b]$ be a generic element of $K(\hat{H})$ and let, for every $x \in \hat{H}$, $h(x) = [x \cdot x, x]$. Thus, $a \geq a \land b$ and hence $a \cdot (a \land b) \geq (a \land b) \cdot (a \land b) \geq (a \land b) \cdot (a \land b) \cdot b$, since in every GMTL-algebra $z \geq z \cdot k$. Thus, by definition of $\leq$ in $K(\hat{H})$, $[a, b] \geq [(a \land b) \cdot (a \land b), a \land b] = h(a \land b) \in J_{h[H]}$. Obviously, since every element of $J_{h[H]}$ can be equivalently displayed as $-[c, d]$ for some $[c, d] \in J_{h[H]}$ and since $-$ reverses the order, there is a $y \in J_{h[H]}$ such that $[c, d] \geq y'$, whence $-[c, d] \leq -y \in J_{h[H]}$. □

### 7.2 States of GMTL-algebras

In this section we will introduce states of GMTL-algebras and we will show some basic properties. The following definition naturally arises by following the same lines that inspired the axiomatization of state of an MV-algebra.

**Definition 7.2.1.** A state of a GMTL-algebra $H = (H, \cdot, \to, \land, \lor, 1)$ is a map $w : H \to \mathbb{R}^+$ satisfying the following conditions:

\[(v1)\] $w(1) = 0,$
(v2) \( w(x \cdot y) = w(x) + w(y) \),

(v3) if \( x \leq y \), then \( w(x) \leq w(y) \).

Given any GMTL-algebra \( H \) we denote by \( W(H) \) the set of its states. Notice that \( W(H) \) is not empty. Indeed, letting \( x \odot y = \min\{0, x - y\} \) on \( R^- \), \( R^- = (R^-, +, \varnothing, min, max, 0) \) is a GMTL-algebra, and any homomorphism of \( H \) to \( R^- \) is a state.

**Proposition 7.2.2.** For every GMTL-algebra \( H \) and for every \( w \in W(H) \), the following holds:

1. \( w(x \land y) + w(x \lor y) = w(x) + w(y) \),

2. if \( H \) is a basic hoop, then \( w(x) + w(x \rightarrow y) = w(y) + w(y \rightarrow x) \),

3. if \( H \) is divisible, then (v3) is redundant.

**Proof.** (1). The variety GMTL is generated by its linearly ordered members, and in every totally ordered GMTL-algebra \( x \cdot y = (x \land y) \cdot (x \lor y) \). Thus, the latter equation holds in every \( H \in \text{GMTL} \). Therefore, for every \( w \in W(H) \), \( w(x) + w(y) = w(x \cdot y) = w((x \land y) \cdot (x \lor y)) = w(x \land y) + w(x \lor y) \) where the last equality follows from (v2).

(2). Immediate from (Div) and (v2).

(3). Let \( H \) be divisible and assume that \( x \leq y \). Then, \( x = x \land y = y \cdot (y \rightarrow x) \) and hence \( w(x) = w(y \cdot (y \rightarrow x)) = w(y) + w(y \rightarrow x) \leq w(y) \) where the last inequality holds since, by definition, \( w(y), w(y \rightarrow x) \in R^- \).

**Remark 7.2.3.** In any GMTL-algebra \((H, \cdot, \rightarrow, \land, \lor, 1)\), the reduct \((H, \land, \lor)\) is a distributive lattice and indeed Proposition 7.2.2 (1) above shows that states of GMTL-algebras are valuations on their lattice reduct as defined by Birkhoff in [16].

Furthermore, if \( H \) is a basic hoop, then Proposition 7.2.2 (2) shows that \( w \) satisfies Bosbach equation \( w(x) + w(x \rightarrow y) = w(y) + w(y \rightarrow x) \). Thus, in that case, it can be seen as a Bosbach state in the sense of [77].
For the next result, recall how the \( \ell \)-group \( K(\hat{H}) \) and the \( \ell \)-monoid homomorphism \( h \) are defined in Theorem 7.1.1 and Corollary 7.1.2.

**Proposition 7.2.4.** For every GMTL-algebra \( H \) and every \( w \in W(H) \), there is a state \( \sigma \) of the abelian \( \ell \)-group \( K(\hat{H}) \) such that \( w = \sigma \circ h \). Conversely, if \( \sigma \) is a state of \( K(\hat{H}) \), then the composition map \( w = \sigma \circ h \) is a state of \( H \).

**Proof.** Let \( h \) and \( K(\hat{H}) \) as in Theorem 7.1.1 and Corollary 7.1.2 and let \( J_{h[H]} \) be the \( \ell \)-subgroup of \( K(\hat{H}) \) generated by \( h[H] = \{ h(x) \mid x \in H \} \). For every element \([x, y] \in J_{h[H]}\), let \( \hat{\sigma}([x, y]) = w(y) - w(x) \in \mathbb{R} \).

**Claim 7.2.5.** The map \( \hat{\sigma} : J_{h[H]} \to \mathbb{R} \) is a state of \( J_{h[H]} \).

**Proof.** (of the Claim). The neutral element of \( K(\hat{H}) \), which obviously coincide with the neutral element of \( J_{h[H]} \), is \([1, 1]\). Thus, since \( w(1) = 0 \), \( \hat{\sigma}([1, 1]) = 0 \). Moreover, for every positive element \([x, 1] \) of \( J_{h[H]} \), \( \hat{\sigma}([x, 1]) = -w(x) \in \mathbb{R}^+ \), whence \( \sigma \) is positive.

It is left to show that \( \hat{\sigma} \) is a group homomorphism. Let \([x_1, y_1], [x_2, y_2] \in J_{h[H]} \). Then, \( \hat{\sigma}([x_1, y_1] + [x_2, y_2]) = \hat{\sigma}([x_1 \cdot x_2, y_1 \cdot y_2]) = w(y_1) + w(y_2) - w(x_1) - w(x_2) = w(y_1) - w(x_1) + w(y_2) - w(x_2) = \hat{\sigma}([x_1, y_1]) + \hat{\sigma}(x_2, y_2) \). \( \square \)

Turning back to the proof of Proposition 7.2.4, let \( \sigma : K(\hat{H}) \to \mathbb{R} \) be a state of \( K(\hat{H}) \) obtained by extending \( \hat{\sigma} \) from the \( \ell \)-subgroup \( J_{h[H]} \) of \( K(\hat{H}) \). The existence of \( \sigma \) is hence guaranteed by [71, Proposition 4.2] plus the observation that every element of \( K(\hat{H}) \) is bounded above by an element of \( J_{h[H]} \) (second part of Corollary 7.1.2).

Now, since every element \( x \) of \( H \) is represented in \( K(\hat{H}) \) as \([1, x] \), \( w(x) = w(x) - w(1) = \hat{\sigma}([1, x]) = \sigma([1, x]) \).

Conversely, let \( \sigma : K(\hat{H}) \to \mathbb{R} \) be a state. Then \( w(1) = \sigma(h(1)) = \sigma(0) = 0 \). Moreover, for every \( x, y \in H \), \( w(x \cdot y) = \sigma(h(x \cdot y)) = \sigma(h(x) + h(y)) = \sigma(h(x)) + \sigma(h(y)) = w(x) + w(y) \). Finally, the monotonicity of \( w \) comes from the monotonicity of \( \sigma \) and \( h \). \( \square \)
7.3 States of sIDL-algebras and their representation

In this section we will provide a notion of hyperstate of sIDL-algebras.

Let $^*[0,1]$ be a nontrivial ultraproduct of the real unit interval and let $\varepsilon$ be an infinitesimal in $^*[0,1]$. The MV-algebra $L(R) = \Gamma(R \times R, (1,0))$ discussed in Section 5.1 is, up to isomorphisms, the MV-subalgebra of $^*[0,1]_{MV}$ generated by $[0,1] \cup \{r\varepsilon | r \in \mathbb{R}\}$ (see [48, Example 6.1]). Therefore, every element of $L(R)$ can be uniquely displayed as $r + s\varepsilon$ for $r \in [0,1]$ and $s \in \mathbb{R}$. In what follows we will adopt the following notation: for every $x \in L(R)$, $x^\circ$ and $x^\ast$ denote those unique elements of $[0,1]$ and $\mathbb{R}$ respectively, such that $x = x^\circ + \varepsilon x^\ast$.

**Definition 7.3.1.** For any sIDL-algebra $A$, we define a hyperstate of $A$ as a map $s : A \rightarrow L(R)$ such that:

- (s1) $s(1) = 1$ and $s(0) = 0$,
- (s2) $s(x \oplus y) + s(x \cdot y) = s(x) + s(y)$,
- (s3) If $x \lor \neg x = 1$, then $s(x) \in [0,1]$.

**Proposition 7.3.2.** The following properties hold for hyperstates of sIDL-algebras:

- (i) $s(\neg x) = 1 - s(x)$,
- (ii) if $x \leq y$, then $s(x) \leq s(y)$,
- (iii) if $x \cdot y = 0$, $s(x \oplus y) = s(x) + s(y)$,
- (iv) if $x \oplus y = 1$, $s(x \cdot y) = s(x) \cdot s(y)$,
- (v) $s(x \land y) + s(x \lor y) = s(x) + s(y)$,
- (vi) the restriction $p$ of $s$ to $\mathcal{B}(A)$ is a $[0,1]$-valued and finitely additive probability measure.
(vii) if \( x \in \coRad(A) \), then \( s(x) \in \coRad(L(R)) \). If \( x \in \rad(A) \), \( s(x) \in \rad(L(R)) \).

(viii) the map \( w : R(A) \to R \) defined as

\[
w(x) = \frac{s(x) - 1}{\varepsilon}
\]

is a state in the sense of Definition 7.2.1.

Proof. (i). In any MTL-algebra, \( x \cdot \neg x = 0 \) and \( x \oplus \neg x = 1 \), thus (s2) and (s1) imply \( s(x) + s(-x) = s(x \oplus -x) + s(x \cdot -x) = s(1) + s(0) = 1 + 0 = 1 \), whence \( s(-x) = 1 - s(x) \).

(ii). If \( x \leq y \), then \( x \cdot -y = 0 \). Thus from (s2), \( s(x \oplus -y) = s(x) + s(-y) = s(x) + 1 - s(y) \leq 1 \). Thus \( s(x) \leq s(y) \).

(iii) and (iv) are direct consequences of (s1) and (s2).

(v). As we already observed in the proof of Proposition 7.3.2, in every MTL-algebra \( x \cdot y = (x \land y) \cdot (x \lor y) \). Analogously, in every sIDL-algebra, \( x \oplus y = (x \land y) \oplus (x \lor y) \). Thus, from (s2), \( s(x) + s(y) = s((x \land y) \cdot (x \lor y)) + s((x \land y) \oplus (x \lor y)) = s((x \land y) \cdot (x \lor y)) + s(x \land y) + s(x \lor y) - s((x \land y) \cdot (x \lor y)) = s(x \land y) + s(x \lor y) \).

(vi). That the restriction \( p \) of \( s \) to \( R(A) \) satisfies \( p(1) = 1 \) and \( p(x \land y) + p(x \lor y) = p(x) + p(y) \) is ensured by (s1), (s2) together with the fact that, for all \( x, y \in R(A) \), \( x \cdot y = x \land y \) and \( x \oplus y = x \lor y \). Finally, that for every \( x \in R(A) \), \( p(x) \in [0,1] \) is exactly (s3).

(vii). Let \( x \in \coRad(A) \). Then, for every \( n \in N \), \( n \cdot x \leq \neg x \) and, from (ii), \( s(n \cdot x) \leq s(\neg x) \). Now, \( x \cdot m \cdot x = 0 \) for every \( m \in N \), whence, in particular, \( s(n \cdot x) = n \cdot s(x) \). Thus, \( n \cdot s(x) \leq 1 - s(x) \) for every \( n \in N \), i.e., \( s(x) \in \coRad(L(R)) \). The second part of the claim now easily follows since \( x \in \rad(A) \) iff \( \neg x \in \coRad(A) \) and \( \alpha \in \rad(L(R)) \) iff \( \neg \alpha \in \coRad(L(R)) \) because both \( A \) and \( L(R) \) are strongly perfect MTL-algebras.

(viii). As we already recalled in Section 7.1, \( R(A) = \rad(A) \). Thus, if \( x \in R(A) \), from (vii), \( s(x) \in \rad(L(R)) \), whence there is \( r_x \in R^* \) such that \( s(x) = 1 - \varepsilon r_x \). Therefore, \( w(x) = s(x)/\varepsilon - 1/\varepsilon = -r_x \in R^* \). It is left to prove that
Thus, the claim is settled.

Therefore, from (7.1), we get

\[ \text{Now, since } \neg \neg (a \land b) \land (\neg b \land c) \land (b \land \neg c) = 0, \text{ it is true that} \]

The next result shows that each hyperstate of an sIDL-algebra \( A \) decomposes in a probability measure on its Boolean skeleton and a state on the radical of \( A \).

**Theorem 7.3.3.** For every sIDL-algebra \( A \) and every hyperstate \( s : A \to \mathcal{L}(R) \) there are a probability measure \( p : \mathcal{R}(A) \to [0, 1] \), a state \( w \in W(\mathcal{R}(A)) \) and an infinitesimal \( \varepsilon > 0 \) such that, for every \( a \in A \),

\[ s(a) = p(b_a) + \varepsilon(w(-b_a \lor c_a) - w(b_a \lor c_a)). \]

**Proof.** Let \( p \) and \( w \) respectively be as in Proposition 7.3.2 (vi) and (viii). Let \( a \in A \). Then, by Proposition 2.2.11 (i), for some \( b_a \in \mathcal{R}(A), c_a \in \mathcal{R}(A) \), it is true that \( a = (b_a \lor \neg c_a) \land (\neg b_a \lor c_a) \), which equals \((b_a \land c_a) \lor (\neg b_a \land \neg c_a)\). Thus, since \((b_a \land c_a) \land (\neg b_a \land \neg c_a) = 0\),

\[
\begin{align*}
  s(a) &= s((b_a \land c_a) \lor (\neg b_a \land \neg c_a)) \\
  &= s(b_a \land c_a) + s(\neg b_a \land \neg c_a) \\
  &= s(b_a \land c_a) + s(\neg b_a \lor c_a) \\
  &= s(b_a \land c_a) + 1 - s(b_a \lor c_a) \\
  &= s(b_a \land c_a) + s(b_a) + s(\neg b_a) - s(b_a \lor c_a) \\
\end{align*}
\]

Now, since \( -b_a \land (b_a \land c_a) = 0 \), \( s(-b_a) + s(b_a \land c_a) = s(-b_a \lor (b_a \land c_a)) = s(-b_a \lor c_a) \). Therefore, from (7.1), we get

\[
\begin{align*}
  s(a) &= s(b_a) + (s(-b_a \lor c_a) - s(b_a \lor c_a)) \\
  &= p(b_a) + \varepsilon w(-b_a \lor c_a) + 1 - \varepsilon w(b_a \lor c_a) - 1 \\
  &= p(b_a) + \varepsilon(w(-b_a \lor c_a) - w(b_a \lor c_a)). \\
\end{align*}
\]

Thus, the claim is settled.

The following result is hence a direct consequence of Theorem 7.3.3 and [60, Corollary 4.0.5].

179
Corollary 7.3.4. For every sIDL-algebra $A$ and every hyperstate $s : A \to \mathcal{L}(R)$ there are a regular Borel measure $\mu_s$ on the Stone space $\text{Max}(\mathcal{B}(A))$ of $\mathcal{B}(A)$, a state $w \in \mathcal{W}(\mathcal{B}(A))$ and an infinitesimal $\varepsilon > 0$ such that, for every $a \in A$,

$$s(a) = \int_{\text{Max}(\mathcal{B}(A))} (b_a)^* \, d\mu_s + \varepsilon(w(\neg b_a \lor c_a) - w(b_a \lor c_a)),$$

where $(b_a)^*$ denotes the characteristic function of the clopen subset of $\text{Max}(\mathcal{B}(A))$ corresponding to $b_a$ via Stone duality.

Now, let $A$ be a sIDL-algebra such that $\mathcal{B}(A)$ is cancellative (i.e., $A$ belongs to the variety of MV-algebras generated by perfect MV-algebras). Then, from Corollary 7.1.2 (see also [16]), $\mathcal{B}(A)$ embeds into $K(\mathcal{B}(A))$. Therefore, the following easily holds.

Corollary 7.3.5. Let $A$ be a sIDL-algebra such that $\mathcal{B}(A)$ is cancellative. Then, for a hyperstate $s : A \to \mathcal{L}(R)$ there are a probability measure $p : \mathcal{B}(A) \to [0,1]$ and an $\ell$-group state $\sigma : K(\mathcal{B}(A)) \to R$ such that, for every $a \in A$,

$$s(a) = p(b_a) + \varepsilon \cdot \sigma([\neg b_a \lor c_a, b_a \lor c_a])$$
Conclusions and future work

In the first part of this thesis we introduced, with two levels of generality, new ways of constructing bounded residuated lattices that generalize the notion of rotation, as studied in [79, 99]. In particular, disconnected $\delta$-rotations result in generating varieties of bounded residuated lattices having a Boolean retraction term, satisfying De Morgan laws, and whose coradical is involutive. While with generalized $\delta$-rotations we generate varieties again satisfying De Morgan laws, whose coradical is involutive, and having an MV-retraction term that we exhibit. Many known varieties of algebraic semantics of substructural logics result to be generated in this way, for instance: product algebras, Gödel algebras, DLMV-algebras, NM-algebras, $n$-potent BL-algebras, Stonean residuated lattices, pseudocomplemented MTL-algebras, sIDL-algebras. Moreover, starting from any variety of residuated lattices, we generate new varieties with the introduced techniques, hence the class of structures characterized by our construction is quite large. Furthermore, for every variety we generate with our construction, we prove a categorical equivalence with respect to a category of quadruples $(M, R, \lor_e, \delta)$, made of an MV$_n$-algebra $M$ (or a Boolean algebra as a special case), a residuated lattice $R$, an external join $\lor_e : M \times R \to R$, and a wdl-admissible operator $\delta : R \to R$. In particular, fixing as $\delta$ either the identity or the map constantly equal to 1, we obtain that the category of triplets $(M, R, \lor_e)$ where $R$ belongs to a subvariety $R$ of RL, is categorically equivalent from one side to subvarieties of SMVR$_n$ (generated by $n$-liftings of RLs) whose algebras have the radical in $R$, and from the other side to the category of IMVR$_n$-algebras (generated by disconnected $n$-rotations of RLs) whose radical is in $R$. Hence, SMVR$_n$ and IMVR$_n$ are equivalent categories. Such results specialize for instance to the equivalences of: Stonean residuated lattices
Conclusions

and wIDL-algebras, SMTL-algebras and sIDL-algebras, product algebras and DLMV-algebras, Gödel algebras and NM*-algebras, Stonean Heyting algebras and regular Nelson lattices without negation fixpoint. Then, for the varieties with a Boolean retraction term, we focused on filters. First, we obtained a (weak) Boolean product decomposition directly from our construction. Then we studied the poset of prime lattice filters, and proved an order isomorphism with respect to a structure of pairs made of a Boolean ultrafilter and a prime lattice filter of the radical, respecting an “external primality” condition.

The future work in this direction comprehends the following problems.

- Referring to Cignoli and Torrens [43], there is an hierarchy of varieties, that we may index by \( n \), of bounded residuated lattices admitting a Boolean retraction term. Our construction characterizes the algebras corresponding to \( n = 2 \) and whose coradical is involutive. It is hence interesting to understand how to characterize the directly indecomposable algebras of higher varieties in the hierarchy, and how to approach the case in which the coradical cannot be recovered from the radical.

- Again in the shade of the work of Cignoli and Torrens [43], there is most likely going to be a hierarchy of varieties with an MV-retract, that we plan to axiomatize and investigate.

- The construction of generalized \( n \)-rotation introduces a multitude of not yet studied varieties of residuated lattices, corresponding to the different possible choices of an \( n \geq 2 \), a wdl-admissible operator and a variety of residuated lattices. Moreover, notice that each of such varieties corresponds to a substructural logic. Some of them may result in having interesting properties that are worth to be examined.

- As a key for further generalization of the triple construction, in the present work we considered a class of MV-algebras, namely \( \text{MV}_n \)-algebras, that are hyperarchimedean, in order to generalize from the Boolean to the MV case. Thus, their filters are generated by the ones of the Boolean skeleton, which allowed us to approach the construction and the proofs.
in a similar way. What indeed seems necessary for the triple construction to work in this manner, is to have from one side an algebra $A$ (playing the role of the Boolean algebra) whose space of prime filters is compact, and in which De Morgan laws hold; from the other side, the integrality of the residuated lattice seems necessary. Regarding the wdl-admissible operator, it is likely that the construction could be extended with just nucleus operators (not necessarily preserving lattice operations), that would result in structures in which De Morgan laws do not necessarily hold.

- In [92], we use the decomposition of a product algebra into a Boolean algebra and a cancellative hoop to obtain a result on the side of the logic. In particular we derive the Co-NP containment of product logic from the Co-NP containment of classical logic and of cancellative hoop logic. Hence, a study from the side of the corresponding logics of the algebraic decompositions proven, certainly deserves to be addressed.

In the second part of the thesis, we focused on the theory of states for some of the structures studied in the first part. In particular, we introduced and studied states of the free $n$-generated product algebra, the Lindenbaum algebra of product logic with $n$-variables. We prove that our axiomatization results in characterizing Lebesgue integral of product logic formulas with respect to regular Borel probability measures on $[0,1]^n$, and that the relation between our states and such measures is one-one. Moreover, we prove that product logic valuations are extremal in the space of states, and hence every state belongs to the closure of convex combinations of product logic valuations.

Then we introduced a notion of states for GMTL-algebras, after showing that for any $\ell$-monoid we can extend Grothendieck construction and obtain an homomorphism to an $\ell$-group. Consequently, taking inspiration from the decomposition theorems studied in the first part of the thesis, we settled a notion of hyperstate for sIDL-algebras, that we prove to be decomposable in a probability map on the Boolean skeleton, and a state on the radical, which is a GMTL-algebra.
Conclusions

The future work in this direction comprehends the following problems.

- A first future direction concerns with the generalization to the frame of product logic of a coherence (no-Dutch-book) criterion à la de Finetti (see [60, §5]). In this regard, the non-finiteness of free product algebras and the discontinuity of product implication, makes the problem of generalizing de Finettis theorem to this setting non-trivial and hence particularly challenging. However, it is worth pointing out that the results concerning the state space pave the way for a first step in this direction.

- As Łukasiewicz, Gödel and product logics are the building blocks of Hájek logic BL, it is reasonable to think that the integral representation theorem for states of free product algebras, together with its analogous results for MV and Gödel algebras, and the remarkable functional representation theorem for free BL-algebras [3], are the necessary ingredients to shed a light on the problem of providing an appropriate axiomatization for states of free BL-algebras.

- We plan to deepen the methodologies applied to the work on states of sIDL-algebras to both extend hyperstates to other classes of (not necessarily involutive) MTL-algebras which satisfy the equation $2x^2 = (2x)^2$ as a first step, and more general structures as a further development. In particular, we plan to provide an integral representation for our hyperstates. Moreover, we plan to apply a similar reasoning to structures with an MV-skeleton, that should result in having hyperstates decomposing in an MV-state on the skeleton and a state associated to the radical.
Bibliography


Bibliography


